

LIMIT CYCLES FOR A GENERALIZED KUKLES POLYNOMIAL DIFFERENTIAL SYSTEMS

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ABSTRACT. We study the limit cycles of a generalized Kukles polynomial differential systems using the averaging theory of first and second order.

1. INTRODUCTION

One of the main problems in the qualitative theory of real planar differential equations is the determination of limit cycles. Limit cycles of planar vector fields were defined by Poincaré [14]. At the end of the 1920s van der Pol [15], Liénard [13] and Andronov [1] proved that a closed orbit of a self-sustained oscillation occurring in a vacuum tube circuit was a limit cycle as considered by Poincaré. After these works, the non-existence, existence, uniqueness and other properties of limit cycles were studied extensively by mathematicians and physicists, and more recently also by chemists, biologists, economists, etc. (see for instance the books [7, 17]).

The second part of the sixteen Hilbert’s problem [11] is related with the least upper bound on the number of limit cycles of polynomial vector fields having a fixed degree. This problem together with the Riemann conjecture are the unique two problems of the list of Hilbert which has not been solved. Here we consider a very particular case of the sixteen Hilbert’s problem, we want to study the upper bound of the generalized Kukles polynomial differential system

$$(1) \quad \ddot{x} = -y, \quad \dot{y} = Q(x, y),$$

where $Q(x, y)$ is a polynomial with real coefficients of degree n . This system was introduced by I.S. Kukles in [12] who gives necessary and sufficient conditions in order that the system

$$(2) \quad \ddot{x} = -y, \quad \dot{y} = x + a_0y + a_1x^2 + a_2xy + a_3y^2 + a_4x^3 + a_5x^2y + a_6xy^2 + a_7y^3,$$

has a center at the origin.

Recently the question about the number of limit cycles of these systems has had an increasing interest. In [16] A.P. Sadovskii solves the center-focus problem for system (2) with $a_2a_7 \neq 0$ and proves that systems (2) can have seven limit cycles.

H. Zong et al. in [18] study the number and distribution of limit cycles for a class of reduced Kukles systems under cubic perturbations. Using the techniques of bifurcation theory and qualitative analysis, they obtained three different distributions of five limit cycles for the considered systems.

In [6] J. Chavarriga et al. study the maximum number of small amplitude limit cycles for Kukles systems which can coexist with some invariant algebraic curves.

2000 *Mathematics Subject Classification.* 34C29, 34C25, 47H11.

Key words and phrases. limit cycle, Kukles systems, averaging theory.

In this work we study the maximum number of limit cycles given by the averaging theory of first and second order, which can bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, perturbed inside the following class of generalized Kukles polynomial differential systems

$$(3) \quad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x - \sum_{k \geq 1} \varepsilon^k (f_{n_1}^k(x) + g_{n_2}^k(x)y + h_{n_3}^k(x)y^2 + d_0^k y^3), \end{aligned}$$

where for every k the polynomials $f_{n_1}^k(x)$, $g_{n_2}^k(x)$ and $h_{n_3}^k(x)$ have degree n_1 , n_2 and n_3 respectively, $d_0^k \neq 0$ is a real number and ε is a small parameter. More precisely our main result is the following.

Theorem 1. *Assume that for $k = 1, 2$ the polynomials $f_{n_1}^k(x)$, $g_{n_2}^k(x)$ and $h_{n_3}^k(x)$ have degree n_1 , n_2 and n_3 respectively, with $n_1, n_2, n_3 \geq 1$, and $d_0^k \neq 0$ is a real number. Then for $|\varepsilon|$ sufficiently small the maximum number of limit cycles of the Kukles polynomial differential systems (3) bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, using the averaging theory*

- (a) of first order is $\max \left\{ \left[\frac{n_2}{2} \right], 1 \right\}$;
- (b) of second order is $\max \left\{ \left[\frac{n_1+1}{2} \right], \left[\frac{n_1}{2} \right] + \left[\frac{n_2-1}{2} \right], \left[\frac{n_2}{2} \right] + \left[\frac{n_1-1}{2} \right], \left[\frac{n_2+1}{2} \right] + \left[\frac{n_3}{2} \right], \left[\frac{n_3+1}{2} \right] + \left[\frac{n_2}{2} \right], \left[\frac{n_3+3}{2} \right], \left[\frac{n_2}{2} \right], 1 \right\}$.

The proof of this theorem is based on the averaging method. We will present the averaging method in section 2. The proofs of statement (a) and statement (b) of Theorem 1 are given in sections 3 and 4, respectively.

There are six methods for the analysis of the number of bifurcated limit cycles that bifurcate from the periodic orbits of a center of a planar polynomial differential system. The first one is based on the Poincaré return map (see for instance [4, 5]), the second on the Poincaré–Melnikov integral method (see section 6 of chapter 4 of [10]), the third on the abelian integral method (see section 5 of chapter 6 of [2]), the fourth on the integrating factor (see section 6 of [9]), the fifth is called the Françoise method and uses the language of differential forms (see [8]), and the last on the averaging theory (see [3] and the references quoted there). In fact in the plane and for polynomial differential systems all these methods are essentially equivalent because all of them are related with the displacement function. The averaging method as the mentioned methods only detect limit cycles bifurcating from the periodic orbits of the period annulus associated to a center, but it does not detect the possible limit cycles bifurcating from the boundary of that period annulus.

2. THE AVERAGING THEORY OF FIRST AND SECOND ORDER

In this section we summarize the main results on the theory of averaging that we will apply to systems (3). For a proof of the next two theorems see [3].

Theorem 2. *Consider the differential system*

$$(4) \quad \dot{x}(t) = \varepsilon F_1(x, t) + \varepsilon^2 R(x, t, \varepsilon),$$

where $F_1 : D \times \mathbb{R} \rightarrow \mathbb{R}^n$, $R : D \times \mathbb{R} \times (-\varepsilon_f, \varepsilon_f) \rightarrow \mathbb{R}^n$ are continuous functions, T -periodic in the first variable and D is an open subset of \mathbb{R}^n . Assume that the following hypotheses (i) and (ii) hold.

- (i) F_1 and R are locally Lipschitz with respect to x . We define $F_{10} : D \rightarrow \mathbb{R}^n$ as

$$F_{10}(z) = \frac{1}{T} \int_0^T F_1(z, s) ds.$$

- (ii) For $a \in D$ with $F_{10}(a) = 0$, there exists a neighborhood V of a such that $F_{10}(z) \neq 0$ for all $z \in \bar{V} \setminus \{a\}$ and $d_B(F_{10}, V, a_\varepsilon) \neq 0$.

Then for $|\varepsilon| > 0$ sufficiently small there exists a T -periodic solution $\varphi(\cdot, \varepsilon)$ of system (4) such that $\varphi(0, \varepsilon) \rightarrow a$ as $\varepsilon \rightarrow 0$.

The expression $d_B(F_{10}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} : V \rightarrow \mathbb{R}^n$ at the fixed point a is not zero. A sufficient condition in order that this inequality is true is that the Jacobian of the function F_{10} at a is not zero.

Theorem 3. Consider the differential system

$$(5) \quad \dot{x}(t) = \varepsilon F_1(x, t) + \varepsilon^2 F_2(x, t) + \varepsilon^3 R(x, t, \varepsilon),$$

where $F_1, F_2 : D \times \mathbb{R} \rightarrow \mathbb{R}^n$, $R : D \times \mathbb{R} \times (-\varepsilon_f, \varepsilon_f) \rightarrow \mathbb{R}^n$ are continuous functions, T -periodic in the first variable and D is an open subset of \mathbb{R}^n . Assume that the following hypotheses (i) and (ii) hold.

- (i) $F_1(\cdot, t) \in C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, R and $D_x F_1$ are locally Lipschitz with respect to x , and R is differentiable with respect to ε . We define $F_{10}, F_{20} : D \rightarrow \mathbb{R}^n$ as

$$F_{10}(z) = \frac{1}{T} \int_0^T F_1(z, s) ds = 0.$$

$$F_{20}(z) = \frac{1}{T} \int_0^T \left[D_z F_1(z, s) \cdot \int_0^s F_1(z, t) dt + F_2(z, s) \right] ds.$$

- (ii) For $V \subset D$ an open and bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, there exists $a_\varepsilon \in V$ such that $F_{20}(a_\varepsilon) = 0$ and $d_B(F_{20}, V, a_\varepsilon) \neq 0$.

Then for $|\varepsilon| > 0$ sufficiently small, there exists a T -periodic solution $\varphi(\cdot, \varepsilon)$ of system (5) such that $\varphi(0, \varepsilon) \rightarrow a_\varepsilon$ as $\varepsilon \rightarrow 0$.

3. PROOF OF STATEMENT (a) OF THEOREM 1

The proof of statement (a) of Theorem 1 is based on the first order averaging theory present in the previous section.

We consider $k = 1$. We write the polynomials $f_{n_1}^1(x)$, $g_{n_2}^1(x)$, $h_{n_3}^1(x)$ appearing in (3) as $f_{n_1}^1(x) = \sum_{i=0}^{n_1} a_i x^i$, $g_{n_2}^1(x) = \sum_{i=0}^{n_2} b_i x^i$ and $h_{n_3}^1(x) = \sum_{i=0}^{n_3} c_i x^i$.

By means of the change of variables $x = r \cos \theta$, $y = r \sin \theta$, system (3) in the region $r > 0$ can be written as

$$(6) \quad \begin{aligned} \dot{r} &= -\varepsilon \sin \theta P(r, \theta), \\ \dot{\theta} &= -1 - \frac{\varepsilon}{r} \cos \theta P(r, \theta), \end{aligned}$$

where

$$(7) \quad P(r, \theta) = d_0^1 r^3 \sin^3 \theta + \sum_{i=0}^{n_1} a_i r^i \cos^i \theta + \sum_{i=0}^{n_2} b_i r^{i+1} \cos^i \theta \sin \theta + \sum_{i=0}^{n_3} c_i r^{i+2} \cos^i \theta \sin^2 \theta.$$

Now taking θ as the new independent variable system (6) becomes

$$(8) \quad \frac{dr}{d\theta} = \varepsilon \sin \theta P(r, \theta) + \mathcal{O}(\varepsilon^2) = \varepsilon F_1(r, \theta) + \mathcal{O}(\varepsilon^2),$$

which is in the standard form for applying the averaging theory. Then from Theorem 2 we get that

$$F_{10}(r) = \frac{1}{2\pi} \int_0^{2\pi} \sin \theta P(r, \theta) d\theta.$$

In order to calculate the exact expression of F_{10} we use the following formulas

$$(9) \quad \begin{aligned} \int_0^{2\pi} \cos^k \theta \sin \theta d\theta &= 0, & k = 0, 1, \dots \\ \int_0^{2\pi} \cos^{2k+1} \theta \sin^2 \theta d\theta &= 0, & k = 0, 1, \dots \\ \int_0^{2\pi} \cos^{2k} \theta \sin^2 \theta d\theta &= \alpha_{2k} \neq 0, & k = 0, 1, \dots \\ \int_0^{2\pi} \cos^k \theta \sin^3 \theta d\theta &= 0, \\ \int_0^{2\pi} \sin^4 \theta d\theta &= \frac{3\pi}{4}. \end{aligned}$$

Hence

$$(10) \quad F_{10}(r) = \frac{3}{8} d_0^1 r^3 + \frac{1}{2\pi} \sum_{\substack{i=0 \\ i \text{ even}}}^{n_2} b_i \alpha_i r^{i+1}.$$

Then the polynomial $F_{10}(r)$ has at most $\max \left\{ \left[\frac{n_2}{2} \right], 1 \right\}$ positive roots, and we can choose the coefficients b_i with i even in such a way that $F_{10}(r)$ has exactly $\left[\frac{n_2}{2} \right]$, or 1 simple positive roots. Hence, by Theorem 2, statement (a) of Theorem 1 is proved.

4. PROOF OF STATEMENT (b) OF THEOREM 1

For proving statement (b) of Theorem 1 we shall use the second order averaging theory.

If we write

$$\begin{aligned} f_{n_1}^1(x) &= \sum_{i=0}^{n_1} a_i x^i, & f_{n_1}^2(x) &= \sum_{i=0}^{n_1} p_i x^i, \\ g_{n_2}^1(x) &= \sum_{i=0}^{n_2} b_i x^i, & g_{n_2}^2(x) &= \sum_{i=0}^{n_2} q_i x^i, \\ h_{n_3}^1(x) &= \sum_{i=0}^{n_3} c_i x^i, & h_{n_3}^2(x) &= \sum_{i=0}^{n_3} s_i x^i, \end{aligned}$$

then system (3) with $k = 2$ in polar coordinates (r, θ) , $r > 0$ becomes

$$(11) \quad \begin{aligned} \dot{r} &= -\varepsilon \sin \theta P(r, \theta) - \varepsilon^2 \sin \theta Q(r, \theta), \\ \dot{\theta} &= -1 - \frac{\varepsilon}{r} \cos \theta P(r, \theta) - \frac{\varepsilon^2}{r} \cos \theta Q(r, \theta), \end{aligned}$$

where

$$(12) \quad \begin{aligned} Q(r, \theta) &= d_0^2 r^3 \sin^3 \theta + \sum_{i=0}^{n_1} p_i r^i \cos^i \theta + \\ &\quad \sum_{i=0}^{n_2} q_i r^{i+1} \cos^i \theta \sin \theta + \sum_{i=0}^{n_3} s_i r^{i+2} \cos^i \theta \sin^2 \theta. \end{aligned}$$

Taking θ as the new independent variable system (11) writes

$$\frac{dr}{d\theta} = \varepsilon F_1(r, \theta) + \varepsilon^2 F_2(r, \theta) + O(\varepsilon^3),$$

where

$$\begin{aligned} F_1(r, \theta) &= \sin \theta P(r, \theta), \\ F_2(r, \theta) &= \sin \theta \left(Q(r, \theta) - \frac{\cos \theta}{r} P^2(r, \theta) \right). \end{aligned}$$

In order to apply the averaging theory of second order, F_{10} must be identically zero. Therefore from (10) F_{10} is identically zero if and only if $b_i = 0$ for all $i = 0, 4, 6, 8, \dots$ and $b_2 = -3 d_0^1$.

Now we determine the corresponding function F_{20} . For this we compute

$$\begin{aligned} \frac{d}{dr} F_1(r, \theta) &= 3d_0^1 r^2 \sin^4 \theta + \sum_{i=1}^{n_1} i a_i r^{i-1} \cos^i \theta \sin \theta + \\ &\quad \sum_{i=0}^{n_2} (i+1) b_i r^i \cos^i \theta \sin^2 \theta + \sum_{i=0}^{n_3} (i+2) c_i r^{i+1} \cos^i \theta \sin^3 \theta, \end{aligned}$$

and

$$\begin{aligned} \int_0^\theta F_1(r, \phi) d\phi &= d_0^1 r^3 \int_0^\theta \sin^4 \phi d\phi + \sum_{i=0}^{n_1} a_i r^i \int_0^\theta \cos^i \phi \sin \phi d\phi + \\ &\quad \sum_{i=0}^{n_2} b_i r^{i+1} \int_0^\theta \cos^i \phi \sin^2 \phi d\phi + \sum_{i=0}^{n_3} c_i r^{i+2} \int_0^\theta \cos^i \phi \sin^3 \phi d\phi. \end{aligned}$$

We have that

$$\int_0^\theta \sin^4 \phi \, d\phi = \frac{1}{32} (12\theta - 8\sin(2\theta) + \sin(4\theta)),$$

$$\int_0^\theta \cos^i \phi \sin \phi \, d\phi = \frac{1}{i+1} (1 - \cos^{i+1} \theta),$$

$$\int_0^\theta \cos^2 \phi \sin^2 \phi \, d\phi = \frac{1}{32} (4\theta - \sin(4\theta)),$$

$$\int_0^\theta \cos^i \phi \sin^2 \phi \, d\phi = \sum_{j=1}^{\frac{i+3}{2}} \alpha_j^i \sin((2j-1)\theta) \quad \text{if } i \text{ odd,}$$

$$\int_0^\theta \cos^i \phi \sin^3 \phi \, d\phi = \beta_1^i + \sum_{j=2}^{\frac{i+6}{2}} \beta_j^i \cos((2j-3)\theta) \quad \text{if } i \text{ even,}$$

$$\int_0^\theta \cos^i \phi \sin^3 \phi \, d\phi = \gamma_1^i + \sum_{j=2}^{\frac{i+5}{2}} \gamma_j^i \cos((2j-2)\theta) \quad \text{if } i \text{ odd,}$$

where α_j^i , β_j^i and γ_j^i are constant.

The integral $\int_0^{2\pi} \frac{d}{dr} F_1(r, \theta) \left(\int_0^\theta F_1(r, \phi) d\phi \right) d\theta$ will be given into several lemmas.

Lemma 4. *The integral*

$$\int_0^{2\pi} \left(\sum_{i=1}^{n_1} ia_i r^{i-1} \cos^i \theta \sin \theta \right) \left(\int_0^\theta F_1(r, \phi) d\phi \right) d\theta$$

is in the variable r the polynomial

$$(13) \quad - \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_1} \frac{1}{32} ia_i b_2 r^{i+2} M'_i + \sum_{\substack{i=2 \\ i \text{ even}}}^{n_1} \sum_{\substack{j=1 \\ j \text{ odd}}}^{n_2} ia_i b_j r^{i+j} N_{ij} - \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_1} \frac{1}{32} ia_i d_0^1 r^{i+2} O_i,$$

where M'_i , N_{ij} and O_i are real constants.

Proof. We have that

$$(a_1) \quad \int_0^{2\pi} \left(\sum_{i=1}^{n_1} ia_i r^{i-1} \cos^i \theta \sin \theta \right) \left(\sum_{j=0}^{n_1} a_j r^j \frac{1}{j+1} (1 - \cos^{j+1} \theta) \right) d\theta = 0 \text{ due}$$

to (9).

$$(b_1) \quad \int_0^{2\pi} \left(\sum_{i=1}^{n_1} ia_i r^{i-1} \cos^i \theta \sin \theta \right) \left(b_2 r^3 \frac{1}{32} (4\theta - \sin(4\theta)) \right) d\theta =$$

$$\sum_{i=1}^{n_1} \frac{4}{32} i a_i b_2 r^{i+2} \int_0^{2\pi} \cos^i \theta \sin \theta \theta \, d\theta - \sum_{i=1}^{n_1} \frac{1}{32} i a_i b_2 r^{i+2} \int_0^{2\pi} \cos^i \theta \sin \theta \sin(4\theta) \, d\theta =$$

$$\sum_{i=1}^{n_1} \frac{4}{32} i a_i b_2 r^{i+2} M_i - \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_1} \frac{1}{32} i a_i b_2 r^{i+2} M'_i,$$

where $M_i = \int_0^{2\pi} \cos^i \theta \sin \theta \theta \, d\theta \neq 0$ for $i = 1, \dots, n_1$ and $M'_i = \int_0^{2\pi} \cos^i \theta \sin \theta \sin(4\theta) \, d\theta \neq 0$ for $i \geq 1$ odd.

$$(c_1) \int_0^{2\pi} \left(\sum_{i=1}^{n_1} i a_i r^{i-1} \cos^i \theta \sin \theta \right) \left(\sum_{\substack{j=1 \\ j \text{ odd}}}^{n_2} \sum_{k=1}^{\frac{j+3}{2}} b_j r^{j+1} \alpha_k^j \sin((2k-1)\theta) \right) d\theta =$$

$$\sum_{\substack{i=2 \\ i \text{ even}}}^{n_1} \sum_{\substack{j=1 \\ j \text{ odd}}}^{n_2} i a_i b_j r^{i+j} N_{ij},$$

where $N_{ij} = \int_0^{2\pi} \cos^i \theta \sin \theta \sum_{k=1}^{\frac{j+3}{2}} \alpha_k^j \sin((2k-1)\theta) \, d\theta \neq 0$ for $i \geq 2$ even and $j \geq 1$ odd.

(d₁)

$$\int_0^{2\pi} \left(\sum_{i=1}^{n_1} i a_i r^{i-1} \cos^i \theta \sin \theta \right) \left(\sum_{\substack{j=0 \\ j \text{ even}}}^{n_3} c_j r^{j+2} \left(\beta_1^j + \sum_{k=2}^{\frac{j+6}{2}} \beta_k^j \cos((2k-3)\theta) \right) \right) d\theta =$$

0.

(e₁)

$$\int_0^{2\pi} \left(\sum_{i=1}^{n_1} i a_i r^{i-1} \cos^i \theta \sin \theta \right) \left(\sum_{\substack{j=1 \\ j \text{ odd}}}^{n_3} c_j r^{j+2} \left(\gamma_1^j + \sum_{k=2}^{\frac{j+6}{2}} \gamma_k^j \cos((2k-2)\theta) \right) \right) d\theta =$$

0.

$$(f_1) \int_0^{2\pi} \left(\sum_{i=1}^{n_1} i a_i r^{i-1} \cos^i \theta \sin \theta \right) \left(\frac{d_0^1}{32} r^3 (12\theta - 8 \sin(2\theta) + \sin(4\theta)) \right) d\theta =$$

$$\int_0^{2\pi} \sum_{i=1}^{n_1} \frac{12}{32} i a_i d_0^1 r^{i+2} \cos^i \theta \sin \theta \, d\theta +$$

$$\int_0^{2\pi} \sum_{i=1}^{n_1} \frac{1}{32} i a_i d_0^1 r^{i+2} \cos^i \theta \sin \theta (-8 \sin(2\theta) + \sin(4\theta)) \, d\theta =$$

$$\sum_{i=1}^{n_1} \frac{12}{32} i a_i d_0^1 r^{i+2} M_i + \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_1} \frac{1}{32} i a_i d_0^1 r^{i+2} O_i,$$

where $O_i = \int_0^{2\pi} \cos^i \theta \sin \theta (-8 \sin(2\theta) + \sin(4\theta)) \, d\theta \neq 0$ for $i \geq 1$ odd.

Since $d_0^1 = -\frac{b_2}{3}$ we have that the sum of the integrals (a_1) to (f_1) is polynomial (13). This ends the proof of the lemma. \square

Lemma 5. *The integral*

$$\int_0^{2\pi} \left(\sum_{i=0}^{n_2} (i+1) b_i r^i \cos^i \theta \sin^2 \theta \right) \left(\int_0^\theta F_1(r, \phi) \, d\phi \right) d\theta$$

is in the variable r the polynomial

$$(14) \quad \sum_{i=0}^{n_1} \frac{3\pi}{4(i+1)} a_i b_2 r^{i+2} - \sum_{\substack{0 \leq i \leq n_2 \\ 0 \leq j \leq n_1 \\ i+j \text{ odd}}} \frac{i+1}{j+1} a_j b_i r^{i+j} P_{ij} +$$

$$\sum_{\substack{j=0 \\ j \text{ even}}}^{n_3} 3c_j b_2 r^{j+4} Q_j + \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_2} \sum_{\substack{j=0 \\ j \text{ even}}}^{n_3} (i+1) b_i c_j r^{i+j+2} R_{ij} + \sum_{\substack{j=1 \\ j \text{ odd}}}^{n_3} 3b_2 c_j r^{j+4} S_j,$$

where P_{ij} , Q_j , R_{ij} and S_j are real constants.

Proof. We have that

$$(a_2) \quad \int_0^{2\pi} \left(\sum_{i=0}^{n_2} (i+1) b_i r^i \cos^i \theta \sin^2 \theta \right) \left(\sum_{j=0}^{n_1} a_j r^j \frac{1}{j+1} (1 - \cos^{j+1} \theta) \right) d\theta =$$

$$\sum_{j=0}^{n_1} \frac{3\pi}{4(j+1)} a_j b_2 r^{j+2} - \sum_{\substack{0 \leq i \leq n_2 \\ 0 \leq j \leq n_1 \\ i+j \text{ odd}}} \frac{i+1}{j+1} a_j b_i r^{i+j} P_{ij},$$

where $P_{ij} = \int_0^{2\pi} \cos^{i+j+1} \theta \sin^2 \theta \, d\theta \neq 0$ if $i+j$ odd.

$$(b_2) \quad \int_0^{2\pi} \left(\sum_{i=0}^{n_2} (i+1) b_i r^i \cos^i \theta \sin^2 \theta \right) \left(b_2 r^3 \frac{1}{32} (4\theta - \sin(4\theta)) \right) d\theta = \frac{3\pi^2}{32} b_2^2 r^5.$$

$$(c_2) \int_0^{2\pi} \left(\sum_{i=0}^{n_2} (i+1) b_i r^i \cos^i \theta \sin^2 \theta \right) \left(\sum_{\substack{j=1 \\ j \text{ odd}}}^{n_2} \sum_{k=1}^{\frac{j+3}{2}} b_j r^{j+1} \alpha_k^j \sin((2k-1)\theta) \right) d\theta = 0.$$

$$(d_2) \int_0^{2\pi} \left(\sum_{i=0}^{n_2} (i+1) b_i r^i \cos^i \theta \sin^2 \theta \right) \left(\sum_{\substack{j=0 \\ j \text{ even}}}^{n_3} c_j r^{j+2} \left(\beta_1^j + \sum_{k=2}^{\frac{j+6}{2}} \beta_k^j \cos((2k-3)\theta) \right) \right) d\theta =$$

$$\sum_{\substack{j=0 \\ j \text{ even}}}^{n_3} 3b_2 c_j r^{j+4} Q_j + \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_2} \sum_{\substack{j=0 \\ j \text{ even}}}^{n_3} (i+1) b_i c_j r^{i+j+2} R_{ij},$$

where $Q_j = \frac{\beta_1^j \pi}{4} \neq 0$ and $R_{ij} = \int_0^{2\pi} \cos^i \theta \sin^2 \theta \left(\beta_1^j + \sum_{k=2}^{\frac{j+6}{2}} \beta_k^j \cos((2k-3)\theta) \right) d\theta < 0$ for $i \geq 1$ odd and $j \geq 2$ even.

$$(e_2) \int_0^{2\pi} \left(\sum_{i=0}^{n_2} (i+1) b_i r^i \cos^i \theta \sin^2 \theta \right) \left(\sum_{\substack{j=1 \\ j \text{ odd}}}^{n_3} c_j r^{j+2} \left(\gamma_1^j + \sum_{k=2}^{\frac{j+5}{2}} \gamma_k^j \cos((2k-2)\theta) \right) \right) d\theta =$$

$$\sum_{\substack{j=1 \\ j \text{ odd}}}^{n_3} 3b_2 c_j r^{j+4} S_j,$$

where $S_j = \int_0^{2\pi} \cos^2 \theta \sin^2 \theta \left(\gamma_1^j + \sum_{k=2}^{\frac{j+5}{2}} \gamma_k^j \cos((2k-2)\theta) \right) d\theta \neq 0$ for $j \geq 1$ odd.

$$(f_2) \int_0^{2\pi} \left(\sum_{i=0}^{n_2} (i+1) b_i r^i \cos^i \theta \sin^2 \theta \right) \left(\frac{d_0^1 r^3}{32} (12\theta - 8 \sin(2\theta) + \sin(4\theta)) \right) d\theta =$$

$$\frac{9\pi^2}{32} b_2 d_0^1 r^5$$

Now using the relation $d_0^1 = -\frac{b_2}{3}$ we have that the sum of the integrals (a_2) to (f_2) is polynomial (14). This ends the proof of the lemma. \square

Lemma 6. *The integral*

$$\int_0^{2\pi} \left(\sum_{i=0}^{n_3} (i+2) c_i r^{i+1} \cos^i \theta \sin^3 \theta \right) \left(\int_0^\theta F_1(r, \phi) d\phi \right) d\theta$$

is in the variable r the polynomial

$$(15) \quad - \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_3} \frac{1}{32} (i+2) b_2 c_i r^{i+4} T_i + \sum_{\substack{i=0 \\ i \text{ even}}}^{n_3} \sum_{\substack{j=1 \\ j \text{ odd}}}^{n_2} c_i b_j (i+2) r^{i+j+2} U_{ij} + \\ \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_3} \frac{1}{32} c_i d_0^1 (i+2) r^{i+4} V_i.$$

where T_i , U_{ij} and V_i are real constants.

Proof. We have that

$$(a_3) \quad \int_0^{2\pi} \left(\sum_{i=0}^{n_3} (i+2) c_i r^{i+1} \cos^i \theta \sin^3 \theta \right) \left(\sum_{j=0}^{n_1} a_j r^j \frac{1}{j+1} (1 - \cos^{j+1} \theta) \right) d\theta = 0. \\ (b_3) \quad \int_0^{2\pi} \left(\sum_{i=0}^{n_3} (i+2) c_i r^{i+1} \cos^i \theta \sin^3 \theta \right) \left(b_2 r^3 \frac{1}{32} (4\theta - \sin(4\theta)) \right) d\theta = \\ - \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_3} \frac{1}{32} (i+2) c_i b_2 r^{i+4} T_i + \sum_{i=0}^{n_3} \frac{4}{32} (i+2) c_i b_2 r^{i+4} T'_i,$$

where $T_i = \int_0^{2\pi} \cos^i \theta \sin^3 \theta \sin(4\theta) d\theta \neq 0$ for $i \geq 1$ odd, and $T'_j = \int_0^{2\pi} \cos^i \theta \sin^3 \theta \theta d\theta \neq 0$ if $i \geq 0$.

$$(c_3) \quad \int_0^{2\pi} \left(\sum_{i=0}^{n_3} (i+2) c_i r^{i+1} \cos^i \theta \sin^3 \theta \right) \left(\sum_{\substack{j=1 \\ j \text{ odd}}}^{n_2} \sum_{k=1}^{\frac{j+3}{2}} b_j r^{j+1} \alpha_k^j \sin((2k-1)\theta) \right) d\theta = \\ \sum_{\substack{i=0 \\ i \text{ even}}}^{n_3} \sum_{\substack{j=1 \\ j \text{ odd}}}^{n_2} (i+2) c_i b_j r^{i+j+2} U_{ij},$$

where $U_{ij} = \int_0^{2\pi} \cos^i \theta \sin^3 \theta \left(\sum_{k=1}^{\frac{j+3}{2}} \alpha_k^j \sin((2k-1)\theta) \right) d\theta \neq 0$ for $i \geq 0$ even and $j \geq 1$ odd.

$$(d_3) \quad \int_0^{2\pi} \left(\sum_{i=0}^{n_3} (i+2) c_i r^{i+1} \cos^i \theta \sin^3 \theta \right) \left(\sum_{\substack{j=0 \\ j \text{ even}}}^{n_3} c_j r^{j+2} \left(\beta_1^j + \sum_{k=2}^{\frac{j+6}{2}} \beta_k^j \cos((2k-3)\theta) \right) \right) d\theta = \\ 0.$$

$$(e_3) \int_0^{2\pi} \left(\sum_{i=0}^{n_3} (i+2)c_i r^{i+1} \cos^i \theta \sin^3 \theta \right) \left(\sum_{\substack{j=0 \\ j \text{ odd}}}^{n_3} c_j r^{j+2} \left(\gamma_1^j + \sum_{k=2}^{\frac{j+5}{2}} \gamma_k^j \cos((2k-2)\theta) \right) \right) d\theta = 0.$$

$$(f_3) \int_0^{2\pi} \left(\sum_{i=0}^{n_3} (i+2)c_i r^{i+1} \cos^i \theta \sin^3 \theta \right) \left(\frac{d_0^1 r^3}{32} (12\theta - 8 \sin(2\theta) + \sin(4\theta)) \right) d\theta = \sum_{i=0}^{n_3} \frac{12}{32} c_i d_0^1 (i+2) r^{i+4} T_i' + \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_3} \frac{1}{32} c_i d_0^1 (i+2) r^{i+4} V_i,$$

where $V_i = \int_0^{2\pi} \cos^i \theta \sin^3 \theta (-8 \sin(2\theta) + \sin(4\theta)) d\theta \neq 0$ for $i \geq 1$ odd.

From $d_0^1 = -\frac{b_2}{3}$ we have that the sum of the integrals (a_3) to (f_3) is polynomial (15). This completes the proof of the lemma. \square

Lemma 7. *The integral*

$$\int_0^{2\pi} (3d_0^1 r^2 \sin^4 \theta) \left(\int_0^\theta F_1(r, \phi) d\phi \right) d\theta$$

is in the variable r the polynomial

$$(16) \quad \sum_{i=0}^{n_1} \frac{9\pi}{4(i+1)} a_i d_0^1 r^{i+2} - \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_1} \frac{3}{i+1} a_i d_0^1 r^{i+2} W_i + \sum_{\substack{j=0 \\ j \text{ even}}}^{n_3} \frac{9\pi}{4} \beta_1^j d_0^1 c_j r^{j+4} + \sum_{\substack{j=1 \\ j \text{ odd}}}^{n_3} 3d_0^1 c_j r^{j+4} Y_j.$$

where W_i and Y_j are real constants.

Proof. We have that

$$(a_4) \int_0^{2\pi} (3d_0^1 r^2 \sin^4 \theta) \left(\sum_{j=0}^{n_1} a_j r^j \frac{1}{j+1} (1 - \cos^{j+1} \theta) \right) d\theta = \sum_{i=0}^{n_1} \frac{9\pi}{4(i+1)} d_0^1 a_i r^{i+2} - \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_1} \frac{3}{i+1} d_0^1 a_i r^{i+2} W_i,$$

where $W_i = \int_0^{2\pi} \sin^4 \theta \cos^{i+1} \theta d\theta \neq 0$ for $i \geq 1$ odd.

$$(b_4) \int_0^{2\pi} (3d_0^1 r^2 \sin^4 \theta) \left(b_2 r^3 \frac{1}{32} (4\theta - \sin(4\theta)) \right) d\theta = \frac{9\pi^2}{32} d_0^1 b_2 r^5.$$

$$\begin{aligned}
(c_4) \int_0^{2\pi} (3d_0^1 r^2 \sin^4 \theta) & \left(\sum_{\substack{j=1 \\ j \text{ odd}}}^{n_2} \sum_{k=1}^{\frac{j+3}{2}} b_j r^{j+1} \alpha_k^j \sin((2k-1)\theta) \right) d\theta = 0. \\
(d_4) \int_0^{2\pi} (3d_0^1 r^2 \sin^4 \theta) & \left(\sum_{\substack{j=0 \\ j \text{ even}}}^{n_3} c_j r^{j+2} \left(\beta_1^j + \sum_{k=2}^{\frac{j+6}{2}} \beta_k^j \cos((2k-3)\theta) \right) \right) d\theta = \\
& \sum_{\substack{j=0 \\ j \text{ even}}}^{n_3} \frac{9\pi}{4} \beta_1^j d_0^1 c_j r^{j+4}. \\
(e_4) \int_0^{2\pi} (3d_0^1 r^2 \sin^4 \theta) & \left(\sum_{\substack{j=0 \\ j \text{ odd}}}^{n_3} c_j r^{j+2} \left(\gamma_1^j + \sum_{k=2}^{\frac{j+5}{2}} \gamma_k^j \cos((2k-2)\theta) \right) \right) d\theta = \\
& \sum_{\substack{j=0 \\ j \text{ odd}}}^{n_3} 3d_0^1 c_j r^{j+4} Y_j,
\end{aligned}$$

where $Y_j = \int_0^{2\pi} \sin^4 \theta \left(\gamma_1^j + \sum_{k=2}^{\frac{j+5}{2}} \gamma_k^j \cos((2k-2)\theta) \right) d\theta \neq 0$ for $j \geq 0$ odd.

$$(f_4) \int_0^{2\pi} (3d_0^1 r^2 \sin^4 \theta) \left(\frac{d_0^1 r^3}{32} (12\theta - 8 \sin(2\theta) + \sin(4\theta)) \right) d\theta = \frac{27\pi^2}{32} (d_0^1)^2 r^5.$$

From $d_0^1 = -\frac{b_2}{3}$ we have that the sum of the integrals (a_4) to (f_4) is polynomial (16). This completes the proof of the lemma. \square

Now we calculate the integral $\int_0^{2\pi} F_2(r, \theta) d\theta$.

Proposition 8. *The integral $\int_0^{2\pi} F_2(r, \theta) d\theta$ is in the variable r the polynomial*

$$\begin{aligned}
(17) \sum_{\substack{i=0 \\ i \text{ even}}}^{n_2} q_i r^{i+1} A_i + \frac{3\pi}{4} d_0^2 r^3 + 2 & \sum_{\substack{0 \leq i \leq n_1 \\ 0 \leq j \leq n_2 \\ i+j \text{ odd}}} a_i b_j r^{i+j} C_{ij} + 2 \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_1} a_i d_0^1 r^{i+2} D_i + \\
& 2 \sum_{\substack{0 \leq i \leq n_2 \\ 0 \leq j \leq n_3 \\ i+j \text{ odd}}} b_i c_j r^{i+j+2} E_{ij} + 2 \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_3} d_0^1 c_i r^{i+4} F_i,
\end{aligned}$$

where A_i, C_{ij}, D_i, E_{ij} and F_i are real constants.

Proof. We have that

$$\int_0^{2\pi} F_2(r, \theta) d\theta = \int_0^{2\pi} \sin \theta Q(r, \theta) d\theta - \int_0^{2\pi} \frac{1}{r} \sin \theta \cos \theta P^2(r, \theta) d\theta.$$

First we calculate $\int_0^{2\pi} \sin \theta Q(r, \theta) d\theta$. Noting that $Q(r, \theta)$ is given by (12), we have

$$\begin{aligned} (18) \quad \int_0^{2\pi} \sin \theta Q(r, \theta) d\theta &= \int_0^{2\pi} \sum_{i=0}^{n_1} p_i r^i \cos^i \theta \sin \theta d\theta + \int_0^{2\pi} \sum_{i=0}^{n_2} q_i r^{i+1} \cos^i \theta \sin^2 \theta d\theta + \\ &\quad \int_0^{2\pi} \sum_{i=0}^{n_3} s_i r^{i+1} \cos^i \theta \sin^3 \theta d\theta + \int_0^{2\pi} d_0^2 r^3 \sin^4 \theta d\theta \\ &= \sum_{\substack{i=0 \\ i \text{ even}}}^{n_2} q_i r^{i+1} A_i + \frac{3\pi}{4} d_0^2 r^3, \end{aligned}$$

where $A_i = \int_0^{2\pi} \cos^i \theta \sin^2 \theta d\theta \neq 0$ for $i \geq 0$ even.

Now noting that $P(r, \theta)$ is given by (7), we compute

$$\begin{aligned} &\int_0^{2\pi} \frac{1}{r} \sin \theta \cos \theta P^2(r, \theta) d\theta = \\ &\int_0^{2\pi} \left(\sum_{i=0}^{n_1} a_i r^i \cos^i \theta \right)^2 \frac{\sin \theta \cos \theta}{r} d\theta + \\ &\int_0^{2\pi} 2 \left(\sum_{i=0}^{n_1} a_i r^i \cos^i \theta \right) \left(\sum_{j=0}^{n_2} b_j r^{j+1} \cos^j \theta \sin \theta \right) \frac{\sin \theta \cos \theta}{r} d\theta + \\ &\int_0^{2\pi} \left(\sum_{i=0}^{n_2} b_i r^{i+1} \cos^i \theta \sin \theta \right)^2 \frac{\sin \theta \cos \theta}{r} d\theta + \\ &\int_0^{2\pi} 2 \left(\sum_{i=0}^{n_1} a_i r^i \cos^i \theta \right) \left(\sum_{j=0}^{n_3} c_j r^{j+2} \cos^j \theta \sin^2 \theta \right) \frac{\sin \theta \cos \theta}{r} d\theta + \\ &\int_0^{2\pi} 2 \left(\sum_{i=0}^{n_1} a_i r^i \cos^i \theta \right) (d_0^1 r^3 \sin^3 \theta) \frac{\sin \theta \cos \theta}{r} d\theta + \\ &\int_0^{2\pi} 2 \left(\sum_{i=0}^{n_2} b_i r^{i+1} \cos^i \theta \sin \theta \right) \left(\sum_{j=0}^{n_3} c_j r^{j+2} \cos^j \theta \sin^2 \theta \right) \frac{\sin \theta \cos \theta}{r} d\theta + \\ &\int_0^{2\pi} 2 \left(\sum_{i=0}^{n_2} b_i r^{i+1} \cos^i \theta \sin \theta \right) (d_0^1 r^3 \sin^3 \theta) \frac{\sin \theta \cos \theta}{r} d\theta + \\ &\int_0^{2\pi} \left(\sum_{i=0}^{n_3} c_i r^{i+2} \cos^i \theta \sin^2 \theta \right)^2 \frac{\sin \theta \cos \theta}{r} d\theta + \\ &\int_0^{2\pi} 2 \left(\sum_{i=0}^{n_3} c_i r^{i+2} \cos^i \theta \sin^2 \theta \right) (d_0^1 r^3 \sin^3 \theta) \frac{\sin \theta \cos \theta}{r} d\theta + \end{aligned}$$

$$\begin{aligned}
& \int_0^{2\pi} (d_0^1 r^3 \sin^3 \theta)^2 \frac{\sin \theta \cos \theta}{r} d\theta = \\
(19) \quad & 2 \sum_{\substack{0 \leq i \leq n_1 \\ 0 \leq j \leq n_2 \\ i+j \text{ odd}}} a_i b_j r^{i+j} C_{ij} + 2 \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_1} a_i d_0^1 r^{i+2} D_i + \\
& 2 \sum_{\substack{0 \leq i \leq n_2 \\ 0 \leq j \leq n_3 \\ i+j \text{ odd}}} b_i c_j r^{i+j+2} E_{ij} + 2 \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_3} d_0^1 c_i r^{i+4} F_i,
\end{aligned}$$

where $C_{ij} = \int_0^{2\pi} \cos^{i+j+1} \sin^2 \theta d\theta$, $D_i = \int_0^{2\pi} \cos^{i+1} \sin^4 \theta d\theta$, $E_{ij} = \int_0^{2\pi} \cos^{i+j+1} \sin^4 \theta d\theta$ and $F_i = \int_0^{2\pi} \cos^{i+1} \sin^6 \theta d\theta$.

From (18) and (19) we obtain (17) and this completes the proof of the proposition. \square

From Lemmas 4-7, Proposition 8 and using the fact $d_0^1 = -\frac{b_2}{3}$ we have that F_{20} is the polynomial in the variable r given by

$$\begin{aligned}
& - \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_1} \frac{1}{32} i a_i b_2 r^{i+2} M'_i + \sum_{\substack{i=2 \\ i \text{ even}}}^{n_1} \sum_{\substack{j=1 \\ j \text{ odd}}}^{n_2} i a_i b_j r^{i+j} N_{ij} - \frac{1}{32} \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_1} i a_i d_0 r^{i+2} O_i \\
& - \sum_{\substack{0 \leq i \leq n_2 \\ 0 \leq j \leq n_1 \\ i+j \text{ odd}}} \frac{i+1}{j+1} a_j b_i r^{i+j} P_{ij} + \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_2} \sum_{\substack{j=0 \\ j \text{ even}}}^{n_3} (i+1) b_i c_j r^{i+j+2} R_{ij} + \\
& \sum_{\substack{j=1 \\ j \text{ odd}}}^{n_3} 3 b_2 c_j r^{j+4} S_j - \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_3} \frac{1}{32} (i+2) b_2 c_i r^{i+4} T_i + \sum_{\substack{i=0 \\ i \text{ even}}}^{n_3} \sum_{\substack{j=1 \\ j \text{ odd}}}^{n_2} c_i b_j (i+2) r^{i+j+2} U_{ij} + \\
& \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_3} \frac{1}{32} c_i d_0^1 (i+2) r^{i+4} V_i - \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_1} \frac{3}{i+1} a_i d_0^1 r^{i+2} W_i + \sum_{\substack{j=1 \\ j \text{ odd}}}^{n_3} 3 d_0^1 c_j r^{j+4} Y_j + \\
& \sum_{\substack{i=0 \\ i \text{ even}}}^{n_2} q_i r^{i+1} A_i + \frac{3\pi}{4} d_0^2 r^3 + 2 \sum_{\substack{0 \leq i \leq n_1 \\ 0 \leq j \leq n_2 \\ i+j \text{ odd}}} a_i b_j r^{i+j} C_{ij} + 2 \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_1} a_i d_0^1 r^{i+2} D_i + \\
& 2 \sum_{\substack{0 \leq i \leq n_2 \\ 0 \leq j \leq n_3 \\ i+j \text{ odd}}} b_i c_j r^{i+j+2} E_{ij} + 2 \sum_{\substack{i=1 \\ i \text{ odd}}}^{n_3} d_0^1 c_i r^{i+4} F_i.
\end{aligned}$$

Note that in order to find the positive roots of F_{20} after dividing by r , we must find the zeros of a polynomial in the variable r^2 of degree equal to the

$\max \left\{ \frac{(n_1'' + 2) - 1}{2}, \frac{(n_1' + n_2'') - 1}{2}, \frac{(n_1'' + n_2') - 1}{2}, \frac{(n_2'' + n_3' + 2) - 1}{2}, \right.$
 $\left. \frac{(n_2' + n_3'' + 2) - 1}{2}, \frac{(n_3'' + 4) - 1}{2}, \frac{(n_2' + 1) - 1}{2}, 1 \right\}$, where n_i' denote the greatest even number less than or equal to n_i and n_i'' denote the greatest odd number less than or equal to n_i , $i = 1, 2, 3$. We have the following relations

$$\frac{(n_1'' + 2) - 1}{2} = \left[\frac{n_1 + 1}{2} \right], \quad \frac{(n_1' + n_2'') - 1}{2} = \left[\frac{n_1}{2} \right] + \left[\frac{n_2 - 1}{2} \right],$$

$$\frac{(n_1'' + n_2') - 1}{2} = \left[\frac{n_2}{2} \right] + \left[\frac{n_1 - 1}{2} \right], \quad \frac{(n_2'' + n_3' + 2) - 1}{2} = \left[\frac{n_2 + 1}{2} \right] + \left[\frac{n_3}{2} \right],$$

$$\frac{(n_2' + n_3'' + 2) - 1}{2} = \left[\frac{n_3 + 1}{2} \right] + \left[\frac{n_2}{2} \right], \quad \frac{(n_3'' + 4) - 1}{2} = \left[\frac{n_3 + 3}{2} \right],$$

$$\frac{(n_2' + 1) - 1}{2} = \left[\frac{n_2}{2} \right],$$

where $[\cdot]$ denotes the integer part function. Thus we conclude the proof of statement (b) of Theorem 1.

We remark that following the proof of statement (b) of Theorem 1 it is not difficult to verify that the polynomials, whose roots provide the number of limit cycles which bifurcate from the periodic orbits of the linear center, have independent coefficients in function of the coefficients of the perturbed system. Therefore the upper bounds provided in statement (b) can be reached.

ACKNOWLEDGMENTS

The second author is partially supported by the grants MEC/FEDER MTM 2008-03437, CIRIT 2009SGR410, and ICREA Academia.

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