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ON THE C^1 NON-INTEGRABILITY OF THE BELOUSOV–ZHABOTINSKII SYSTEM

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ABSTRACT. We use the stability or instability of the singular points together with results of Poincaré on the multipliers of a periodic orbit for studying the C^1 non-integrability of the Belousov–Zhabotinskii system.

1. Introduction and statements of the main results

The Belousov–Zhabotinskii system (see [2]):

$$\dot{x} = s(x + y - qx^2 - xy), \quad \dot{y} = s^{-1}(-y + fz - xy), \quad \dot{z} = w(x - z), \tag{1}$$

is one of the most interesting and best understood dynamical oscillators, where x, y, z are real variables; and s, f, q, w are real parameters and $s \neq 0$. This system has been intensively investigated as a dynamical system (see for instance [4, 5, 10, 12, 13, 14]). Here the dot denotes derivative with respect to the time t.

In MathSciNet there are now 265 published papers with some relation with the Belousov–Zhabotinskii system but very few is known about the integrability or non–integrability of this system.

A global analytic first integral or simply in what follows *an analytic first integral* is a non–constant analytic function $H\colon \mathbb{R}^3 \to \mathbb{R}$ whose domain of definition is $\mathbb{R}^3$, and it is constant on the solutions of system (1), i.e.

$$\mathcal{X}H = s(x + y - qx^2 - xy)\frac{\partial H}{\partial x} + s^{-1}(-y + fz - xy)\frac{\partial H}{\partial y} + w(x - z)\frac{\partial H}{\partial z} = 0. \tag{2}$$

In [8] the authors proved the following result concerning the analytic integrability of system (1).

Theorem 1. *The following statements hold for the Belousov–Zhabotinskii system.*

(1) *The unique global analytic first integrals for system* (1) *with $w = 0$ are of the form $g(z)$ where g is an arbitrary global analytic function.*
(2) *System* (1) *with $w \neq 0$ has no global analytic first integrals which are also analytic in the parameter w in a neighborhood of $w = 0$.*

We note that Theorem 1 characterizes all the global analytic first integrals for system (1) with $w = 0$, $q, f, s \in \mathbb{R}$ with $s \neq 0$. However it only provides partial information of the first integrals when $w \neq 0$.

Now we shall study the C^1-nonintegrability of system (1). Let U be an open subset of $\mathbb{R}^3$ and $H\colon U \to \mathbb{R}$ be a C^1 non-constant function satisfying (2). Then

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H is called a C^1-*first integral* of system (1) on U. If $U = \mathbb{R}^3$ then H is called a C^1-*global first integral*.

The next result studies the C^1 non-integrability of the Belousov–Zhabotinskii system using the Hopf bifurcation at the origin together with Poincaré's results on the multipliers of a periodic orbit (see section 2 for details). We introduce the notation

$$
\begin{aligned}
F &= A_{101} + B_{011},\\
G &= C_{020} + C_{200},\\
D &= c\big(-4aC_{002} + dF\big),\\
E &= D^2 + 8ac^2F\big(-2aC_{002} + dF\big).
\end{aligned} \tag{3}
$$

Instead of working with the initial parameters f, q, s, w of the Belousov–Zhabotinskii system we shall work with the parameters a, b, q, s, ε, where a and b are defined through the following expressions

$$
\begin{aligned}
f &= -\frac{(b^2 + (s - a\,\varepsilon)^2)(1 + b^2s^2 + 2as\varepsilon + a^2s^2\varepsilon^2)}{s(s(1 + b^2) + 2(s^2 - 1)a\,\varepsilon - 3a^2s\varepsilon^2)},\\
w &= \frac{(1 + b^2)s - 3a^2s\varepsilon^2 + 2a(-1 + s^2)\varepsilon}{1 - s^2 + 2as\varepsilon}.
\end{aligned} \tag{4}
$$

The reason of changing these parameters is that with the new parameters we will be able to compute explicitly the family of periodic orbits and their multipliers (see Theorem 5 and its proof for details).

We define

$$
K = as\left(s^4 + \left(b^2 - 1\right)s^2 + 1\right)\left((q-1)s^4 + (b^2(q-1) - 1)s^2 + 1\right)\Delta_{12}, \tag{5}
$$

where

$$
\begin{aligned}
\Delta_{12} = \;& \Big(\left(((4q-1)b^2 + 1\right)s^8 + \left(b^4 + 3b^2 + 2\left(b^4 - 4b^2 + 2\right)q - 4\right)s^6\\
&+ \Big(b^6 + b^4 + b^2 - 2\left(b^2 - 2\right)^2 q + 5\Big)s^4 + \Big(b^4 - (4q+1)b^2 + 6q + 2\Big)s^2\\
&+ b^2 - 2q - 5\Big)s^2 + 2.
\end{aligned}
$$

In the next theorem we prove that system (1) under the assumptions of Theorem 2 has no global C^1 first integrals.

Theorem 2. *Consider the one–parameter family in ε of the Belousov–Zhabotinskii systems (1) with a, b, s and q satisfying the open condition $K \neq 0$. If $K > 0$ there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ in a neighborhood of the $(0,0,0) \in \mathbb{R}^3$ the Belousov–Zhabotinskii system (1) has no limit cycles if $\varepsilon < 0$, and has a unique limit cycle γ_ε if $\varepsilon > 0$. The limit cycle γ_ε tends to $(0,0,0)$ when $\varepsilon \to 0$. For $K < 0$ the limit cycle γ_ε only exists for $\varepsilon < 0$.*

If $(D \pm \sqrt{E})/(2c^2F) \neq 1$ with D, E, F defined in (3) and where A_{101}, B_{011}, C_{200}, C_{020} and C_{002} are given in (10), then the Belousov–Zhabotinskii system for $\varepsilon \in (0, \varepsilon_0)$ when $K > 0$, and for $\varepsilon \in (-\varepsilon_0, 0)$ when $K < 0$ has no C^1 first integrals $F(x, y, z)$ defined in a neighborhood of the zero-Hopf periodic orbit γ_ε satisfying that $\nabla F(x, y, z)$ and $(s(x + y - qx^2 - xy), s^{-1}(-y + fz - xy), w(x - z))$ are linearly independent on the points of γ_ε.

Theorem 2 is proved in section 2.

We introduce the following notation:

$$
\begin{aligned}
\alpha_1 &= \frac{s^2-ws-1}{s},\\
\alpha_2 &= w(1+f),\\
\alpha_3 &= (f+1)w+\frac{(s^2-ws-1)(s+(s^2-1)w)}{s^2},\\
S &= \sqrt{f^2+2(3q-1)f+(q+1)^2},\\
\alpha_4 &= -\frac{S(f+q+S-1)w}{2q},\\
\alpha_5 &= \frac{-q(s^2+2)+f(qs^2+2)+2S+qs(3s(q+S)-4w)-2}{4qs},\\
\alpha_6 &= \frac{1}{16q^2}\bigg(\frac{1}{s^2}\left(-q(s^2+2)+f(qs^2+2)+2S+qs(3s(q+S)-4w)-2\right)\big(-2sf^2+\\
&\quad (2w+s(-4S+q(sw-6)+2))f-2sS(q+S-1)-2(q-S+1)w+qs^2(3q+3S-1)w\big)\\
&\quad -8qS(f+q+S-1)w\bigg),\\
\alpha_7 &= -\frac{S(-f-q+S+1)w}{2q},\\
\alpha_8 &= \frac{f(qs^2+2)-2(S+1)+q((3q-3S-1)s^2-4ws-2)}{4qs},\\
\alpha_9 &= \frac{1}{16q^2}\bigg(8q(f+q-S-1)Sw-\frac{1}{s^2}\left(f(qs^2+2)-2(S+1)+q((3q-3S-1)s^2-4ws-2)\right)\\
&\quad \big(2sf^2-(2w+s(4S+q(sw-6)+2))f+2sS(-q+S+1)+2(q+S+1)w+\\
&\quad qs^2(-3q+3S+1)w\big)\bigg).
\end{aligned}
$$

In the next result it is used the existence of attractor or repeller singular points for studying the C^1 non-integrability of the Belousov–Zhabotinskii system.

Theorem 3. *The Belousov–Zhabotinskii system has no C^1 global first integrals if one of the following conditions hold.*

(a) $\alpha_1 > 0$, $\alpha_2 > 0$ *and* $\alpha_3 > 0$,
(b) $\alpha_1 < 0$, $\alpha_2 < 0$ *and* $\alpha_3 < 0$,
(c) $S^2 > 0$, $\alpha_4 > 0$, $\alpha_5 > 0$ *and* $\alpha_6 > 0$,
(d) $S^2 > 0$, $\alpha_4 < 0$, $\alpha_5 < 0$ *and* $\alpha_6 < 0$,
(e) $S^2 > 0$, $\alpha_7 > 0$, $\alpha_8 > 0$ *and* $\alpha_9 > 0$,
(f) $S^2 > 0$, $\alpha_7 < 0$, $\alpha_8 < 0$ *and* $\alpha_9 < 0$.

Theorem 3 is proved in Section 3.

2. Proof of Theorem 2

Now we will use the multipliers of a periodic orbit for studying the C^1 non-integrability of the Belousov–Zhabotinskii system. To state the results we shall introduce some notation.

We continue to consider system (11) and we write its general solution as $\phi(t,\mathbf{x}_0)$ with $\phi(0,\mathbf{x}_0)=\mathbf{x}_0 \in U$ and t belonging to its maximal interval of definition.

We say that the solution $\phi(t,\mathbf{x}_0)$ is T*-periodic* with $T>0$ if and only if $\phi(T,\mathbf{x}_0)=\mathbf{x}_0$ and $\phi(t,\mathbf{x}_0)\neq \mathbf{x}_0$ for $t\in(0,T)$. The *periodic orbit* associated to the periodic solution $\phi(t, \mathbf{x}_0)$ is $\gamma = \{\phi(t, \mathbf{x}_0), t \in [0, T]\}$. The *variational equation* associated to the T-periodic solution $\phi(t, \mathbf{x}_0)$ is

$$\dot{M} = \left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Big|_{\mathbf{x}=\phi(t,\mathbf{x}_0)} \right) M, \tag{6}$$

where M is an $n \times n$ matrix. Of course $\partial f(\mathbf{x})/\partial \mathbf{x}$ denotes the Jacobian matrix of f with respect to $\mathbf{x}$. The *monodromy matrix* associated to the T-periodic solution $\phi(t, \mathbf{x}_0)$ is the solution $M(T, \mathbf{x}_0)$ of (6) satisfying that $M(0, \mathbf{x}_0)$ is the identity matrix. The eigenvalues of the monodromy matrix associated to the periodic solution $\phi(t, \mathbf{x}_0)$ are called the *multipliers* of the periodic orbit.

The *gradient* of a C^1 function H is defined as

$$\nabla H(\mathbf{x}) = \left(\frac{\partial H}{\partial x_1}, \ldots, \frac{\partial H}{\partial x_n} \right).$$

In [9] the authors proved the following theorem which goes back to Poincaré (see [11]).

Theorem 4. *Consider the C^2 differential system* (11). *If there is a periodic orbit γ having only one multiplier equal to* 1, *then system* (11) *has no C^1 first integrals H defined in a neighborhood of γ such that the vectors $\nabla H(\mathbf{x})$ and $f(\mathbf{x})$ are linearly independent on the points $\mathbf{x} \in \gamma$.*

We will use Theorem 4. The main difficulty for applying this theorem is to find for our system (1) periodic orbits having multipliers different from one. For the Belousov–Zhabotinskii system we shall find periodic orbits and compute their multipliers using the following theorem.

Theorem 5. *Consider a C^3 differentiable system in $\mathbb{R}^3$ having the origin as a singular point with eigenvalues $\varepsilon a \pm ci$ and εd. Then such a system can be written as*

$$\begin{aligned} \dot{x} &= p(x,y,z) = \varepsilon a x - c y + \sum_{i+j+k=2} A_{ijk} x^i y^j z^k + O_3(x,y,z), \\ \dot{y} &= q(x,y,z) = c x + \varepsilon a y + \sum_{i+j+k=2} B_{ijk} x^i y^j z^k + O_3(x,y,z), \\ \dot{z} &= r(x,y,z) = \varepsilon d z + \sum_{i+j+k=2} C_{ijk} x^i y^j z^k + O_3(x,y,z), \end{aligned} \tag{7}$$

where $O_3(x,y,z)$ denotes the terms of order at least three in x, y, z. Let D, E, F, G the constants defined in (3). *Assume that $(E - D^2)/(FG) > 0$ and that $(D \pm \sqrt{E})/(2c^2F) \neq 1$. Then system* (7) *has a limit cycle γ_ε tending to the origin as ε tends to zero. Moreover there exists $\varepsilon_0 > 0$ such that for either $\varepsilon \in (-\varepsilon_0, 0)$ or $\varepsilon \in (0, \varepsilon_0)$ system* (7) *has no C^1 first integrals H defined in a neighborhood of γ_ε such that the vectors $\nabla H(x,y,z)$ and $\big(p(x,y,z), q(x,y,z), r(x,y,z)\big)$ are linearly independent on the points of γ_ε.*

Proof. From Theorem 1 of [3] it follows the existence of the limit cycle γ_ε when $(E^2 - D^2)/(FG) > 0$. In the proof of that theorem the multipliers of γ_ε are computed and they are 1 and $(D \pm \sqrt{E})/(2c^2F)$. So from Theorem 4 the proof of this theorem follows. □

We should apply Theorem 5 to the Belousov–Zhabotinskii system to study its Hopf bifurcation at the singular point located at the origin.

In order to apply Theorem 5 to system (1) we need to write it into the normal form of system (7). The characteristic polynomial of the linear part at the origin of system (1) is (6). On the other hand the characteristic polynomial of the linear part of system (7) is

$$\lambda^3 - (2a+c)\varepsilon\lambda^2 + (b^2 + a^2\varepsilon^2 + 2ac\varepsilon^2)\lambda - (b^2 + a^2\varepsilon^2)c\varepsilon. \tag{8}$$

Changing the parameters f and w of the Belousov–Zhabotinskii systems by the parameters a and b through the implicit equations given in (4) and defining

$$c = \frac{-1 + s^2 - sw - 2as\varepsilon}{s\varepsilon}, \tag{9}$$

we obtain that the two characteristic polynomials (6) and (8) coincide. Hence using the new parameters a, b, s, q the singular point at the origin of the Belousov–Zhabotinskii system (1) has the eigenvalues $a\,\varepsilon \pm b\,i$ and $c\,\varepsilon$.

Proof of Theorem 2. Consider the Belousov–Zhabotinskii system (1) with the parameters a, b, s and f defined using (4).

We consider the change of coordinates $(U, V, W) = B^{-1}(x, y, z)$ which pass the linear part at the origin of system (1) to its real Jordan normal form; i.e. to the linear part at the origin of system (7).

The expressions of the coefficients of the quadratic terms of the Belousov–Zhabotinskii system in the coordinates (U, V, W) are very large and we do not give them here. According to Theorem 5, the important coefficients for studying the Hopf bifurcation and its stability are

$$\begin{aligned}
A_{101} &= \frac{\Delta_1}{\Delta_2} + \frac{a\Delta_3}{\Delta_2^2}\varepsilon + O(\varepsilon^2),\\
B_{011} &= \frac{(s^2-1)^3\Delta_4}{s^2\Delta_2} + \frac{a(s^2-1)^2\Delta_5}{\Delta_2^2}\varepsilon + O(\varepsilon^2),\\
C_{200} &= \frac{s^2(b^2+1)(b^2+s^2)\Delta_6}{\Delta_2} + \frac{as^2(b^2+1)\Delta_7}{\Delta_2^2}\varepsilon + O(\varepsilon^2),\\
C_{020} &= -\frac{b^2s^2(s^2-1)(b^2+s^2)\Delta_8}{\Delta_2} + \frac{b^2s^2\Delta_9}{\Delta_2^2}\varepsilon + O(\varepsilon^2),\\
C_{002} &= \frac{(s^2-1)^3\Delta_{10}}{s^2\Delta_2} + \frac{2a(s^2-1)\Delta_{11}}{s^2\Delta_2^2}\varepsilon + O(\varepsilon^2),
\end{aligned} \tag{10}$$

where the Δ_i for $i = 1, \ldots, 11$ are

$$\begin{aligned}
\Delta_1 &= \left(s^2-1\right)\left(\left(2\left(b^2+1\right)q-1\right)s^8+\left(b^2-1\right)\left(b^2+2\left(b^2+1\right)q\right)s^6\right. \\
&\quad \left.+\left(b^2+1\right)\left(b^4-2qb^2+3\right)s^4+b^2\left(b^2+3\right)s^2-2b^2-1\right), \\
\Delta_2 &= \left(b^2+1\right)s\left(s^8+\left(3b^2-2\right)s^6+\left(b^4-4b^2+3\right)s^4+\left(3b^2-2\right)s^2+1\right), \\
\Delta_3 &= -\left(b^2+2\left(b^2+1\right)q-1\right)s^{20}-\left((8q+6)b^4-(2q+9)b^2-10q+3\right)s^{18} \\
&\quad -\left(10(2q+1)b^6+(4q-41)b^4+(16q+7)b^2+32q-8\right)s^{16}-\left((18q+7)b^8\right. \\
&\quad \left.-2(5q+31)b^6+4(5q+9)b^4-22b^2-48q-3\right)s^{14}-\left(\left(\left((8q+2)b^8-2(q+16)b^6\right.\right.\right. \\
&\quad \left.\left.+40(q+2)b^4+(5-22q)b^2-32q+36\right)b^2+40q+33\right)s^{12}-2\left(b^2+1\right)\left(b^{10}\right. \\
&\quad \left.+(3-4q)b^8-(q-30)b^6-(25q+32)b^4+7(5q+4)b^2-8q-21\right)s^{10} \\
&\quad -\left(\left(\left(2\left(b^4-(3q+2)b^2+2q+15\right)b^2-44q+73\right)b^2-56q-54\right)b^2-2q\right. \\
&\quad \left.+21\right)s^8+\left(7b^8+2(q-1)b^6-4(4q-9)b^4-10(2q+9)b^2-2q-11\right)s^6 \\
&\quad +\left(16b^6+(2q-3)b^4+(2q+63)b^2+16\right)s^4+\left(8b^4-15b^2-5\right)s^2+3b^2+1, \\
\Delta_4 &= \left(2s^4-2q\left(s^4-s^2+1\right)+b^2\left((2q-1)\left(s^2-2\right)s^2+3\right)\right)s^2+2, \\
\Delta_5 &= \left(b^2+1\right)\left(s^2-1\right)\left(s^8+\left(3b^2-2\right)s^6+\left(b^4-4b^2+3\right)s^4+\left(3b^2-2\right)s^2+1\right) \\
&\quad \left(\left(s^4-6s^2-2q\left(s^2-2\right)^2+b^2\left((3-6q)s^2+3\right)\right)s^2+7\right)-\frac{2}{s^2}\left(s^4+\left(b^2-1\right)s^2\right. \\
&\quad \left.+1\right)\left(\left(s^6-3s^4+(3b^4+7)s^2+2b^2\left(s^4+s^2+1\right)-3\right)s^2+1\right)\left(\left(2s^4\right.\right. \\
&\quad \left.\left.-2q\left(s^4-s^2+1\right)+b^2\left((2q-1)\left(s^2-2\right)s^2+3\right)\right)s^2+2\right), \\
\Delta_6 &= \left(qb^2+q-1\right)s^4-\left(b^2+1\right)s^2+1, \\
\Delta_7 &= -b^2s^{16}+\left(-5b^4-b^2+4\left(b^2+1\right)^2q-3\right)s^{14}+\left(-8b^6-2b^4-9b^2\right. \\
&\quad \left.+2\left(b^2+1\right)^2\left(5b^2-4\right)q+3\right)s^{12}+\left((8q-5)b^8+2(q-2)b^6-(14q+9)b^4\right. \\
&\quad \left.+(17-2q)b^2+6q+8\right)s^{10}-\left(b^2+1\right)\left(b^8+2(5q+1)b^6+(4q-6)b^4-4(q+5)b^2\right. \\
&\quad \left.+2q+13\right)s^8+\left(9b^8+20b^6+11b^4-7b^2-2\left(b^2+1\right)^2\left(3b^2+1\right)q+12\right)s^6 \\
&\quad +\left(2b^6+6b^4+21b^2-1\right)s^4-\left(b^4+9b^2+1\right)s^2+b^2, \\
\Delta_8 &= -(q-1)s^4+\left(b^2+q+1\right)s^2-1, \\
\Delta_9 &= -2a\left(b^2+1\right)\left(3(q-1)s^6+\left(2(q-2)b^2-4q-1\right)s^4-\left(b^4+2qb^2-q-4\right)s^2\right. \\
&\quad \left.+2b^2-1\right)\left(s^8+\left(3b^2-2\right)s^6+\left(b^4-4b^2+3\right)s^4+\left(3b^2-2\right)s^2+1\right)s^2 \\
&\quad -2a\left(1-s^2\right)\left(s^2-1\right)\left(b^2+s^2\right)\left(-(q-1)s^4+\left(b^2+q+1\right)s^2-1\right)\left(s^8-5s^6\right. \\
&\quad \left.+\left(-4b^4-6b^2+6\right)s^4-5s^2+1\right), \\
\Delta_{10} &= qs^8+\left(b^2(q+1)-q\right)s^6+\left(b^4-(q-1)b^2+1\right)s^4+s^2-1, \\
\Delta_{11} &= \left(1-s^2\right)^3\left(s^8-5s^6+\left(-4b^4-6b^2+6\right)s^4-5s^2+1\right)\left(qs^8\right. \\
&\quad \left.+\left(b^2(q+1)-q\right)s^6+\left(b^4-(q-1)b^2+1\right)s^4+s^2-1\right)-\left(b^2+1\right)s^2\left(s^2-1\right) \\
&\quad \left(s^8+\left(3b^2-2\right)s^6+\left(b^4-4b^2+3\right)s^4+\left(3b^2-2\right)s^2+1\right)\left(5qs^8+\left(4(q+1)b^2\right.\right. \\
&\quad \left.\left.-6q-1\right)s^6+\left(3b^4+(2-4q)b^2+q+4\right)s^4+5s^2-5\right).
\end{aligned}$$

In order to apply Theorem 5 we compute the constant

$$\frac{E-D^2}{FG}=\frac{8ab^2\left(s^4+\left(b^2-1\right)s^2+1\right)\Delta_{12}}{\varepsilon s^5\left(b^2+s^2\right)\left(b^2s^2+1\right)\left((q-1)s^4+\left(b^2(q-1)-1\right)s^2+1\right)}+O(1),$$

where Δ_{12} is given in the statement of Theorem 2.

The hypothesis (5) implies $(E-D^2)/(FG) \neq 0$ for $\varepsilon > 0$ sufficiently small. According to Theorem 5 the Belousov–Zhabotinskii system (1) has a limit cycle γ_ε.

Computing the multipliers $(D \pm \sqrt{E})/(2c^2F)$ and considering when they are different from one, from Theorem 5, the proof of this theorem follows. □

3. Proof of Theorem 3

To prove Theorem 3 we need some auxiliary results. We consider the autonomous differential system

$$\dot{\mathbf{x}} = f(\mathbf{x}), \tag{11}$$

where $f\colon U \to \mathbb{R}^n$ is C^2, U is an open subset of $\mathbb{R}^n$ and the dot denotes the derivative with respect to the time t.

Theorem 6. *If system* (11) *has an isolated singular point p which is either attractor or repeller, then it has no C^1 first integrals defined in a neighborhood of p.*

Proof. We proceed by contradiction. Assume that system (11) has an isolated attractor singular point p and a C^1 first integral H. Let U be a neighborhood of p such that all the orbits starting in a point of U have p as ω-limit. Let $H(p) = h$. Clearly $H^{-1}(h)\bigcap U \neq \emptyset$. Then for $\varepsilon > 0$ sufficiently small $H^{-1}(h+\varepsilon)\bigcap U \neq \emptyset$. Since $p \notin H^{-1}(h+\varepsilon)$ and $H^{-1}(h+\varepsilon)$ is an invariant set by the flow of system (11), clearly p is not the ω-limit of the orbits starting at points of $H^{-1}(h+\varepsilon)\bigcap U \neq \emptyset$, a contradiction with the definition of U.

When system (11) has an isolated repeller singular point, reversing the time, it becomes an attractor, and the proof of the theorem follows. □

Theorem 7. *The zeros of $G(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ with $a_3 \neq 0$ have negative real part if and only if*

$$a_2 > 0, \quad a_0 > 0 \quad \text{and} \quad D_1 := a_1a_2 - a_3a_0 > 0. \tag{12}$$

Proof. This theorem is the well-known Roth-Hurwitz criterion. For a proof see for example [7, p.230]. □

Proof of Theorem 3. Computing the singular points of (1) they are of the form

$$\begin{aligned} s_0 &= (0,0,0),\\ s_1 &= \left(-\frac{f+q+S-1}{2q}, \frac{1}{4}(3f+q+S+1), -\frac{f+q+S-1}{2q} \right),\\ s_2 &= \left(-\frac{f+q-S-1}{2q}, \frac{1}{4}(3f+q-S+1), -\frac{f+q-S-1}{2q} \right), \end{aligned}$$

with s_2 and s_3 being real when $S^2 > 0$. The eigenvalues of the Jacobian matrix of system (1) at the singular point s_0 are the zeros of the characteristic polynomial

$$-u^3 + \frac{s^2 - ws - 1}{s}u^2 + \frac{s + (s^2-1)w}{s}u + (f+1)w = 0.$$

Therefore we have that

$$a_0 = w(1+f), \quad a_1 = \frac{s+(s^2-1)w}{s}, \quad a_2 = \frac{s^2 - ws - 1}{s}, \quad a_3 = -1.$$

In view of (12) the singular point s_0 is an attractor if and only if

$$(13)\qquad \begin{aligned} a_2 &= \frac{s^2-ws-1}{s} > 0,\\ a_0 &= w(1+f) > 0,\\ D_1 &= (f+1)w + \frac{(s^2-ws-1)(s+(s^2-1)w)}{s^2} > 0. \end{aligned}$$

Then statement (a) is proved.

Reversing time in equation (1). i.e., considering the system

$$(14)\qquad \dot{x} = -s(x+y-qx^2-xy), \quad \dot{y} = -s^{-1}(-y+fz-xy), \quad \dot{z} = -w(x-z),$$

we get that the eigenvalues of the Jacobian matrix of system (14) at the singular point s_0 are the zeros of the characteristic polynomial

$$-u^3 - \frac{s^2-ws-1}{s}u^2 + \frac{(s+(s^2-1)w)}{s}u - (f+1)w = 0.$$

Therefore in view of (12) the singular point s_0 is an attractor of system (14) if and only if

$$\frac{s^2-ws-1}{s} < 0, \; w(1+f) < 0, \; (f+1)w + \frac{(s^2-ws-1)(s+(s^2-1)w)}{s^2} < 0.$$

So statement (b) follows.

Computing the eigenvalues of the Jacobian matrix of system (1) at the singular point s_1 we get that they are the zeros of the characteristic polynomial

$$a_0 + a_1u + a_2u^2 + a_3u^3 = 0,$$

with

$$\begin{aligned} a_0 &= -\frac{S(f+q+S-1)w}{2q},\\ a_1 &= \frac{1}{4qs}\big(-2sf^2 + (2w + s(-4S + q(sw-6)+2))f - 2sS(q+S-1) -\\ &\quad 2(q-S+1)w + qs^2(3q+3S-1)w\big),\\ a_2 &= \frac{-q\left(s^2+2\right) + f\left(qs^2+2\right) + 2S + qs(3s(q+S)-4w) - 2}{4qs},\\ a_3 &= -1. \end{aligned}$$

Then using (12) the Jacobian matrix associated to system (1) at the singular point s_1 has all its eigenvalues with real negative part (i.e. s_1 is an attractor) if and only if $a_0 > 0$, $a_2 > 0$ and $D_1 > 0$. Computing them we get statement (c) of the theorem.

Moreover reversing the time and proceeding as for s_0, we get that the Jacobian matrix associated to system (14) at the singular point s_1 has all its eigenvalues with real negative part (i.e. they are repellers for system (1)) if and only if $a_0 < 0$, $a_2 < 0$ and $D_1 < 0$. Computing them we get statement (d) of the theorem.

Finally computing the eigenvalues of the Jacobian matrix of system (1) at the singular point s_2 we get that they are the zeros of the characteristic polynomial

$$a_0 + a_1u + a_2u^2 + a_3u^3 = 0,$$

with

$$
\begin{aligned}
a_0 &= -\frac{S(-f-q+S+1)w}{2q},\\
a_1 &= \frac{1}{4qs}\big(-2sf^2+(2w+s(4S+q(sw-6)+2))f+2sS(q-S-1)S+\\
&\qquad qs^2(3q-3S-1)w-2(q+S+1)w\big),\\
a_2 &= \frac{f\left(qs^2+2\right)-2(S+1)+q\left((3q-3S-1)s^2-4ws-2\right)}{4qs},\\
a_3 &= -1.
\end{aligned}
$$

Then using (12) the Jacobian matrix associated to system (1) at the the singular point s_2 has all its eigenvalues with real negative part if and only if $a_0 > 0$, $a_2 > 0$ and $D_1 > 0$. Computing them we get statement (e) of the theorem.

Moreover reversing the time and proceeding as for s_0 and s_1, we get that the Jacobian matrix associated to system (14) at the singular point s_2 has all its eigenvalues with real negative part if and only if $a_0 < 0$, $a_2 < 0$ and $D_1 < 0$. Computing them we get statement (f) of the theorem. This completes the proof of the theorem. □

4. Further comments and conclusions

In this paper we have applied a method which goes back to Poincaré for studying the C^1 integrability of the Belousov–Zhabotinskii system, see Theorem 2. In [8] the analytic integrability of this system has been studied, see Theorem 1.

This last 30 years several methods for studying the meromorphic integrability of differential systems have been developed, mainly these methods are due to Ziglin and Morales–Ramis. According with Arnold [1], Ziglin's and Morales–Ramis' theory are inspired in Kovalevskaya's ideas which go back to Poincaré. These two theories used the multipliers of the monodromy group of the variational equations associated to some orbits for studying the non-integrability of the differential equations. Poincaré's method allows to prove under convenient assumptions the non-existence of first integrals of class C^1. The main difficulty for applying the Poincaré non-integrability method to a given differential system is to find for such a system periodic orbits having multipliers different from 1. It seems that this result of Poincaré was forgotten by the mathematical community until modern Russian mathematicians (mainly Kozlov) have recently published on it, see [1, 6]. Here we have applied the Poincaré criterion to the Belousov–Zhabotinskii system (1) and we have shown the non-existence of C^1 first integrals under convenient assumptions.

We note that this method for studying the non-existence of C^1 first integrals can be applied to any differential system for which we know analytically some periodic orbits and their multipliers.

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