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ON THE 16TH HILBERT PROBLEM FOR ALGEBRAIC LIMIT CYCLES ON NONSINGULAR ALGEBRAIC CURVES

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Abstract. We give an upper bound for the maximum number N of algebraic limit cycles that a planar polynomial vector field can exhibit if the vector field has exactly k non-singular invariant algebraic curves. Additionally we provide sufficient conditions in order that all the algebraic limit cycles are hyperbolic. For $k = 1, 2, n-1$ we also give a lower bounds for N.

1. Introduction and statement of the main results

A *(planar) polynomial differential system* is a system of the form

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y), \tag{1}$$

where P and Q are polynomials in the variables x and y. In this work the dependent variables x and y, the independent variable t, and the coefficients of the polynomials P and Q are all real because we are interested in the real algebraic limit cycles of system (1). The *degree* n of the polynomial system (1) is the maximum of the degrees of the polynomials P and Q.

Associated to the polynomial differential system (1) there is the *polynomial vector field*

$$\mathcal{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}, \tag{2}$$

or simply $\mathcal{X} = (P,Q)$.

Let $\mathbb{R}[x,y]$ be the ring of all real polynomials in the variables x and y. Let $f = f(x,y) \in \mathbb{R}[x,y]$. The algebraic curve $f(x,y) = 0$ of $\mathbb{R}^2$ is an *invariant algebraic curve* of the polynomial vector field $\mathcal{X}$ if for some polynomial $K \in \mathbb{R}[x,y]$ we have

$$\mathcal{X}f = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{3}$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$. We note that since the polynomial system has degree n, then any cofactor has at most degree $n-1$.

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Since on the points of an invariant algebraic curve $f = 0$ the gradient $(\partial f/\partial x,\ \partial f/\partial y)$ of the curve is orthogonal to the vector field $\mathcal{X}$ (see (3)), the vector field $\mathcal{X}$ is tangent to the curve $f = 0$. Hence the curve $f = 0$ is formed by orbits of the vector field $\mathcal{X}$. This justifies the name of invariant algebraic curve given to the algebraic curve $f = 0$ satisfying (3) for some polynomial K, because it is *invariant* under the flow defined by $\mathcal{X}$.

An invariant algebraic curve $f = 0$ is called *irreducible* if the polynomial f is irreducible in $\mathbb{R}[x, y]$.

We recall that a *limit cycle* of a polynomial vector field $\mathcal{X}$ is an isolated periodic orbit in the set of all periodic orbits of $\mathcal{X}$. An *algebraic limit cycle* of degree m of $\mathcal{X}$ is an oval of a real irreducible (on $\mathbb{R}[x, y]$) invariant algebraic curve $f = 0$ of degree m which is a limit cycle of $\mathcal{X}$.

Consider the set Σ_n of all real polynomial vector fields (2) of degree n. Hilbert in [7] asked: *Is there an uniform upper bound on the number of limit cycles for all polynomial vector field of* Σ_n? This is a version of the second half part of Hilbert's 16th problem.

Consider the set Σ'_n of all real polynomial vector fields (2) of degree n having irreducible invariant algebraic curves. A simpler version of the second part of the 16th Hilbert's problem is: *Is there an uniform upper bound on the number of algebraic limit cycles of any polynomial vector field of* Σ'_n? In [9] we give the answer to this last question when all the invariant algebraic curves are generic in the following sense

(i) There are no points at which $f_j = 0$ and its first derivatives all vanish (i.e. $f_j = 0$ is a non–singular algebraic curve).
(ii) The highest order homogeneous terms of f_j have no repeated factors.
(iii) If two curves intersect at a point in the affine plane, they are transversal at this point.
(iv) There are no more than two curves $f_j = 0$ meeting at any point in the affine plane.
(v) There are no two curves having a common factor in the highest order homogeneous terms.

In [9] the following theorem is proved.

Theorem 1. *For a polynomial vector field $\mathcal{X}$ of degree n having all its irreducible invariant algebraic curves generic, the maximum number of algebraic limit cycles is at most $1 + (n - 1)(n - 2)/2$ if n is even, and $(n - 1)(n - 2)/2$ if n is odd. Moreover these upper bounds are reached.*

Also in [9] is stated the following conjecture.

Conjecture 2. *Is $1+(n-1)(n-2)/2$ the maximum number of algebraic limit cycles that a polynomial vector field of degree n can have?*

As usual we denote by f_x the partial derivative of the function f with respect to the variable x.

In this paper we study the 16th Hilbert problem for algebraic limit cycles contained in nonsingular invariant algebraic curves. Our main results are the following.

Theorem 3. *Let $g_\nu = g_\nu(x, y) = 0$ for $\nu = 1, 2, \ldots, k$ are the unique non-singular invariant algebraic curves of the polynomial vector field $\mathcal{X}$ of degree n and let $A(k, n)$ is the maximum number of algebraic limit cycles of $\mathcal{X}$, then if*

(a) *the curves are non-singular and irreducible then*

$$A(k, n) \leq k\big(\frac{(n-1)n}{2} + 1\big).$$

(b) *the degree of the curves are such that $\sum_{\nu=1}^{k} \deg g_\nu \leq n + 1$, then*

$$A(k, n) \leq \begin{cases} 1 + \frac{1}{2}(n-1)n & \text{if } n \text{ is even,} \\ \frac{1}{2}(n-1)n & \text{if } n \text{ is odd.} \end{cases}$$

(c) *the curves are generic in the above sense, then*

$$A(k, n) \leq A(1, n) = \begin{cases} 1 + \frac{1}{2}(n-1)(n-2) & \text{if } n \text{ is even,} \\ \frac{1}{2}(n-1)(n-2) & \text{if } n \text{ is odd.} \end{cases}$$

(d) *the vector $\mathcal{X}$ does not admits a rational first integral then*

$$A(k, n) \leq \big(\frac{(n+1)n}{2} + 1\big)\big(\frac{(n-1)n}{2} + 1\big) = \frac{n^4 + n^2 + 4}{4}.$$

The proof of Theorem 3 is given in section 2.

Theorem 4. *Let $g = g(x, y) = 0$ be a unique non-singular irreducible algebraic curve invariant of the vector field $\mathcal{X}$ associated to polynomial differential system*

$$\dot{x} = \lambda_3 g + \lambda_1 g_y = P(x, y), \quad \dot{y} = \lambda_3 g - \lambda_1 g_x = Q(x, y). \tag{4}$$

where $\lambda_\nu = \lambda_\nu(x, y)$ for $\nu = 1, 2, 3$ are polynomials.

Assume that the following conditions hold.

(i) *Intersection of the ovals of $g = 0$ with the algebraic curve $\lambda_1 = 0$ is empty.*

(ii) *The polynomial $(\lambda_3(\lambda_1)_y + \lambda_2(\lambda_2)_x)g|_{r=0}$ is not zero in $\mathbb{R}^2/\{g = 0\}$,*

(iii) *if γ is a isolated periodic solution of (4) which does not intersect the curve $\lambda_1 = 0$, then*

$$I_1 = \oint_\gamma \frac{1}{\lambda_1}(\lambda_2 dx - \lambda_3 dy) = -\iint_\Gamma \left(\left(\frac{\lambda_2}{\lambda_1}\right)_y + \left(\frac{\lambda_3}{\lambda_1}\right)_x\right) dxdy \neq 0;$$

where Γ is the bounded region limited by γ, and

(iv) $\max\Big(\deg(\lambda_3 g + \lambda_1 g_y), \deg(\lambda_3 g - \lambda_1 g_x)\Big) = n.$

Then (4) *is a polynomial differential system of degree n for which the following statements hold.*

(a) *The curve $g = 0$ is an invariant algebraic curve.*
(b) *All the ovals of $g = 0$ are hyperbolic limit cycles. Furthermore system* (4) *has no other limit cycles.*
(c) *Assume that $\alpha \in \mathbb{R}\backslash\{0\}$ and $G = G(x, y)$ is an arbitrary polynomial of degree $n - 2$ such that the algebraic curve*

$$f = ax^{n+1} + G(x, y) = 0 \tag{5}$$

is nonsingular and irreducible. We denote by $B_1(n)$ (respectively $B_2(n)$) the maximum number of ovals of all curves $f = 0$ when n is odd (respectively even). If $A(1, n)$ is the maximum number of algebraic limit cycles of system (4)*, then*

$$\max\left(\frac{(n-1)(n-2)}{2},\, B_1(n)\right) \leq A(1, n),$$

when n is odd and

$$\max\left(\frac{(n-1)(n-2)}{2} + 1,\, B_2(n)\right) \leq A(1, n),$$

when n is even.

The polynomial differential systems (4) provide the more general polynomial differential systems having $g = 0$ as irreducible invariant algebraic curve, for more details see [4].

Corollary 5. *Under the assumptions of Theorem 4 we have that*

(a) $A(1, 2) = 1$,
(b) $2 \leq A(1, 3) \leq 4$, *and*
(c) $6 \leq A(1, 5) \leq 11$.

The proof of Theorem 4 and corollary 5 is given in section 3.

We note that Theorem 4 improves Theorem of Christopher [2].

Theorem 6. *Let $g_j = g_j(x, y) = 0$, for $j = 1, 2$ are the unique non-singular irreducible algebraic curves invariant of the vector field $\mathcal{X}$ associated to polynomial differential system*

$$\dot{x} = -r_1(g_1)_y - r_2(g_2)_y + g_1g_2\lambda_4, \quad \dot{y} = r_1(g_1)_x + r_2(g_2)_x + g_1g_2\lambda_3 \tag{6}$$

where $r_1 = \lambda_1 g_2$, $r_2 = \lambda_2 g_1$. and $\lambda = \lambda_j(x, y)$ for $j = 1, 2, 3, 4$ be polynomials.

Assuming that the following conditions hold.

(i) *The intersection of the ovals of $g_\nu = 0$ with the algebraic curve $r_\nu = 0$, for $\nu = 1, 2$ are empty.*
(ii) *The two polynomials*

$$\lambda_4(r_1)_x + \lambda_3(r_1)_y + \lambda_2\{\lambda_1,\, g_1\})g_1g_2|_{r_1=0},$$
$$(\lambda_4(r_2)_x + \lambda_3(r_2)_y + \lambda_1\{\lambda_2,\, g_2\})g_1g_2|_{r_2=0},$$

are not zero in $\mathbb{R}^2/\{g_1g_2 = 0\}$, where $\{f,\, g\} = f_xg_y - f_yg_x$.

(iii) *If γ_1 (respectively γ_2) is a isolated periodic solution of* (6) *which does not intersect the curve $r_1 = 0$ (respectively $r_2 = 0$), then*

$$I_1 = \oint_{\gamma_1} \frac{1}{\lambda_1}(-\lambda_3 dx + \lambda_4 dy) - \oint_{\gamma_1} \frac{\lambda_2}{\lambda_1} d\log|g_2| \neq 0, \tag{7}$$

$$I_2 = \oint_{\gamma_2} \frac{1}{\lambda_2}(-\lambda_3 dx + \lambda_4 dy) - \oint_{\gamma_2} \frac{\lambda_1}{\lambda_2} d\log|g_1| \neq 0. \tag{8}$$

(iv) $\max\Big(\deg(g_1g_2\lambda_4 - r_1(g_1)_y - r_2(g_2)_y), \deg(+g_1g_2\lambda_3 + r_1(g_1)_x + r_2(g_2)_x)\Big) = n$.

Then (6) *is a polynomial differential system of degree n for which the following statements hold.*

(a) *The curves $g_j = 0$ for $j = 1, 2$ are invariant.*
(b) *All the ovals of $g_j = 0$ for $j = 1, 2$ are hyperbolic limit cycles. Furthermore system* (6) *has no other limit cycles.*
(c) *Assume that $\alpha \in \mathbb{R}\backslash\{0\}$ and $G = G(x, y)$ is an arbitrary polynomial of degree $n - 1$ such that the algebraic curve*

$$f = ax^n + G(x, y) = 0 \tag{9}$$

is nonsingular and irreducible. We denote by $b_1(n)$ (respectively $b_2(n)$) the maximum number of ovals of all curves $f = 0$ when n is odd (respectively even). If $A(2, n)$ is the maximum number of algebraic limit cycles of system (6)*, then*

$$b_1(n) \leq A(2, n),$$

when n is odd, and

$$b_2(n) \leq A(2, n),$$

when n is even. Here

$$\begin{aligned} \frac{(n-2)(n-3)}{2} &\leq b_1(n) \leq \frac{(n-1)(n-2)}{2} \\ \frac{(n-2)(n-3)}{2} + 1 &\leq b_2(n) \leq \frac{(n-1)(n-2)}{2} + 1 \end{aligned} \tag{10}$$

Corollary 7. *Under the assumptions of Theorem 6 we have that*

$$2 \leq A(2, 3) \leq 6.$$

The proofs of Theorem 6 and its corollary are presented in section 5.

Theorem 8. *Let $g_\nu = g_\nu(x, y) = 0$, for $\nu = 1, 2 \ldots, k$ are the unique nonsingular irreducible algebraic curves invariant of the vector field $\mathcal{X}$ associated to polynomial differential system*

$$\dot{x} = \lambda_{k+2} g - \sum_{j=1}^{k} r_\nu (g_\nu)_y, \quad \dot{y} = \lambda_{k+1} g + \sum_{\nu=1}^{k} r_\nu (g_\nu)_x, \tag{11}$$

where $g=\prod_{\nu=1}^{k} g_\nu$, $\lambda_j=\lambda_j(x,y)$, *for* $j=1,2,\ldots k+2$ *are polynomial, and* $r_\nu=\lambda_\nu\prod_{j\neq\nu} g_j$, *for* $\nu=1,2,\ldots`,k$.

Assume that

(i) *The intersection of the ovals of* $g_\nu=0$ *and* $r_\nu=0$ *for* $\nu=1,2,\ldots,k$ *are empty.*

(ii) *The polynomials*

$$\Big(\lambda_{k+2}(r_\nu)_x+\lambda_{k+1}(r_\nu)_y+\sum_{j\neq\nu}^{k}\lambda_j\{\lambda_\nu,g_j\}\prod_{m\neq j,\,\nu}^{k} g_m\Big)\prod_{j=1}^{k} g_j|_{r_\nu=0}$$

for $\nu=1,2,\ldots k$, *are not zero in* $\mathbb{R}^2/\{\prod_{j=1}^{k} g_j=0\}$.

(iii) *if* γ_ν *is a isolated periodic solutions which does not intersect the curve* $r_\nu=0$, *then*

$$I_\nu=\oint_{\gamma_\nu}\frac{1}{\lambda_\nu}(-\lambda_{k+1}dx+\lambda_{k+2}dy)-\sum_{j\neq\nu}^{k}\oint_{\gamma_\nu}\frac{\lambda_j}{\lambda_\nu}d\log|g_j|\neq 0,\qquad \nu=1,2,\ldots k;$$

and

(iv)

$$\max\Big(\deg\big(\lambda_{k+2}g\big)-\sum_{j=1}^{k} r_j(g_j)_y\big),\ \deg\big(\lambda_{k+1}g+\sum_{j=1}^{k} r_j(g_j)_x)\big)\Big)=n,$$

Then (11) *is a polynomial differential system of degree n for which the following statements hold.*

(a) *The curve* $g_\nu=0$ *for* $\nu=1,2,\ldots,k$ *are invariant algebraic curves.*

(b) *All the ovals of* $g_\nu=0$, *for* $\nu=1,2,\ldots,k$ *are hyperbolic limit cycles. Furthermore system* (11) *has no other limit cycles.*

Corollary 9. *Under the assumptions of Theorem 8 we have that*

$$n-1\leq A(n-1,n).$$

The proof of Theorem 8 is analogous to the proof of Theorem 6 and the proof of corollary 9 is given in section 6.

We note that system (11) is the more general polynomial differential system having the invariant algebraic curves $g_\nu=0$, for $\nu=1,2,\ldots k$. For more details see [11].

2. Proof of Theorem 3

The proof of statement (i) is obtained as follows. First if we denote by m_ν, and K_ν the degree and the maximum number of ovals of the curve $g_\nu=0$, then in view of the Harnack theorem (see for more details [13, 14]) $K_\nu\leq 1+\dfrac{(m_\nu-1)(m_\nu-2)}{2}$. On the other hand if $g_\nu=0$ is non-singular and irreducible, from [1] the degree of g_ν does not exceed $n+1$, hence

$$K_\nu \leq 1 + \frac{(m_\nu - 1)(m_\nu - 2)}{2} \leq 1 + \frac{(n-1)n}{2}.$$

Let $A(k,n)$ be the maximum number of algebraic limit cycles of the given polynomial planar vector field of degree n, with k irreducible non-singular invariant algebraic curve, then

$$A(k,n) \leq \sum_{\nu=1}^{k} K_\nu \leq \sum_{\nu=1}^{k} \big(1 + \frac{(m_\nu - 1)(m_\nu - 2)}{2}\big) \leq k(1 + \frac{(n-1)n}{2}).$$

So the statement (a) is proved.

From proposition 8 of [9] we obtain that if $\sum_{\nu=1}^{k} \deg g_\nu \leq n+1$ then

$$\kappa(m_1, m_2, \ldots, m_k) = \sum_{\nu=1}^{k} \big(\frac{(m_\nu - 1)(m_\nu - 2)}{2}\big) + \sum_{\nu=1}^{k} a_\nu \leq \sum_{\nu=1}^{k} a_\nu + \frac{(n+1-k)(n-k)}{2}\big)$$

where $a_\nu = 1$ if m_ν is even, and $a_\nu = 0$ if m_ν is odd. After some calculations (see for more details [9]) we obtain that

$$\sum_{\nu=1}^{k} a_\nu + \frac{(n+1-k)(n-k)}{2}\big) \leq \bar{\kappa}$$

where $\bar{\kappa}$ is equal to $n(n-1)/2$ when n is odd, and $1 + n(n-1)/2$ when n is even.

Hence, by considering that

$$A(k,n) \leq \kappa(m_1, m_2, \ldots, m_k) \leq \bar{\kappa},$$

we obtain the proof of statement (b).

The proof of the statement (c) is given in [9].

In view of the Jounolous's theorem (see [8], or in a shorter proof [3]) we obtain that if the number k of the given invariant curves is large than $\frac{n(n-1)}{2}+1$, then there exists a rational first integral. Since by assumption there is not a rational first integral then $k \leq \frac{n(n-1)}{2} + 1$.

By considering that the curves $g_\nu = 0$ are non-singular and irreducible then their degree is at most $n+1$ (see [3]). On the other hand from Harcnak's theorem we deduce the given upper bound for $A(k,n)$.

This completes the proof of Theorem 3.

3. Proof of Theorem 4 and its corollary

From (3) we have $\mathcal{X} g = (\lambda_3 g_y + \lambda_2 g_x) g$, consequently $g = 0$ is an invariant algebraic curve with cofactor $K = \lambda_3 g_y + \lambda_2 g_x$. Therefore statement (a) is proved. Clearly a singular point on $g = 0$ satisfies either $\lambda_1 = 0$, or $g_x = g_y = 0$. Due to our assumptions any of these two cases cannot occur. Thus each oval of $g = 0$ must be a periodic solution of system (4). Now we shall show that these periodic solutions are in fact hyperbolic limit cycles.

Consider an oval γ of $g=0$. From our choice of λ_1 we know that γ does not intersect the curve $\lambda_1=0$. In order to see that γ is a hyperbolic algebraic limit cycle we must show that

$$I=\oint_\gamma \text{div}(t)dt=\oint_\gamma\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)(x(t),y(t))dt\neq 0, \tag{12}$$

where $(x(t),y(t))$ is the parameterizations of the periodic orbit γ, see for more details Theorem 1.2.3 of [5].

After straightforward calculations and by considering that

$$g_y=\frac{\dot{x}-\lambda_2 g}{\lambda_1},\quad -g_x=\frac{\dot{y}-\lambda_3 g}{\lambda_1},$$

we obtain that

$$\begin{aligned}\text{div}(t)= &\ (\lambda_1)_x g_y+\lambda_1 g_{xy}+(\lambda_2)_x g+\lambda_1 g_x-(\lambda_1)_y g_x-\lambda_1 g_{xy}+(\lambda_3)_y g+\lambda_2 g_y\\ = &\ ((\lambda_1)_x+\lambda_3)g_y-((\lambda_1)_y-\lambda_2)g_x+g((\lambda_2)_x+(\lambda_3)_y)\\ = &\ ((\lambda_1)_x+\lambda_3)\frac{\dot{x}-\lambda_2 g}{\lambda_1}+((\lambda_1)_y-l)\frac{\dot{y}-\lambda_3 g}{(\lambda_1)}+g((\lambda_2)_x+(\lambda_3)_y)\\ = &\ \frac{(\lambda_1)_x\dot{x}+(\lambda_1)_y\dot{y}}{\lambda_1}+\frac{\lambda_3}{\lambda_1}(\dot{x}-\lambda_2 g)-\frac{\lambda_2}{\lambda_1}(\dot{y}-\lambda_3 g)\\ &-\frac{g}{\lambda_1}(l(\lambda_1)_x+\lambda_3(\lambda_1)_y)+g((\lambda_2)_x+(\lambda_3)_y).\end{aligned}$$

Therefore

$$\text{div}(t)dt=d(\log|\lambda_1|)+\frac{\lambda_3 dx-\lambda_2 dy}{\lambda_1}+\frac{g}{\lambda_3}(-\lambda_2(\lambda_1)_x-\lambda_3(\lambda_1)_y+\lambda_1 l_x+\lambda_1(\lambda_3)_y)dt.$$

Hence, by assumption we have that

$$\oint_\gamma \text{div}(t)dt=\oint_\gamma\frac{\lambda_3 dx-\lambda_2 dy}{\lambda_1}\neq 0.$$

Thus every oval of the algebraic curve $g=0$ is a hyperbolic algebraic limit cycle.

We suppose that there is a limit cycle $\tilde{\gamma}$ which is not contained in $g=0$. This limit cycle can be or not algebraic. On the algebraic curve $\lambda_1=0$ we have

$$\begin{aligned}\dot{\lambda_1}= &\ (\lambda_1)_x\dot{x}+(\lambda_1)_y\dot{y}\\ = &\ (\lambda_1)_x(\lambda_1 g_y+\lambda_2 g)+(\lambda_1)_y(-\lambda_1 g_x+\lambda_3 g)\\ = &\ \lambda_1((\lambda_1)_x g_y-(\lambda_1)_y g_x)+g(\lambda_2(\lambda_1)_x+\lambda_3(\lambda_1)_y).\end{aligned}$$

Hence by assumption (i) we have that

$$\dot{\lambda_1}|_{\lambda_1=0}=(\lambda_3(\lambda_1)_y+\lambda_2(\lambda_1)_x)g|_{\lambda_1=0}\neq 0, \tag{13}$$

in $\mathbb{R}^2/\{g=0\}$. Thus $\tilde{\gamma}$ cannot intersect the curve $\lambda_1=0$. Therefore $\tilde{\gamma}$ lies in a connected component U of $\mathbb{R}^2/\{\lambda_1 g=0\}$, so that g and λ_1 have constant sign on U. In view of the relation

$$\dot g = g_x \dot x + g_y \dot y = -\frac{\dot y - \lambda_3 g}{\lambda_1}\,\dot x + \frac{\dot x - \lambda_2 g}{\lambda_1}\,\dot y = \frac{g}{\lambda_1}(\lambda_3 \dot x - \lambda_2 \dot y).$$

So $d(\log|g|) = (\lambda_3 dx - \lambda_2 dy)/\lambda_1$, and by assumption (ii) we get the contradiction

$$0 = \oint_{\tilde\gamma} d\log|g| = \oint_{\tilde\gamma} \frac{1}{\lambda_1}(\lambda_3 dx - \lambda_2 dy) \neq 0.$$

In short statement (b) is proved.

The lower bound is deduced as follows. First we note that the differential system already considered in [2]

$$\dot x = (Ax + By + C)g_y + \alpha g, \quad \dot y = -(Ax + By + C)g_x + \beta g, \tag{14}$$

where $A, B, C, \alpha, \beta \in \mathbb{R}$, $\beta B + \alpha A \neq 0$, and $g = 0$ is an arbitrary algebraic curve of degree n such that $g|_{\lambda_1=0} \neq 0$ in $\mathbb{R}^2 \setminus \{g = 0\}$, is a particular case of system (4). In fact for this system we have that

$$\lambda_1 = Ax + By + C, \quad \lambda_2 = \alpha, \quad \lambda_3 = \beta.$$

Therefore from (13) we obtain

$$\begin{aligned} \dot\lambda_1|_{\lambda_1=0} &= (\beta B + \alpha A \neq 0) g|_{\lambda_1=0} \quad \text{in } \mathbb{R}^2 \setminus \{g = 0\}, \\ \oint_\gamma \frac{1}{\lambda_1}(\lambda_3 dx - \lambda_2 dy) &= (\beta B + \alpha A) \iint_\Gamma \frac{dxdy}{(Ax + By + C)^2} \neq 0. \end{aligned}$$

Hence conditions (i) and (ii) hold for system (14).

By choosing $g = 0$ an M-curve of degree n having the maximum numbers of ovals in the affine plane, i.e. $1 + (n-1)(n-2)/2$ ovals when n is even, and $(n-1)(n-2)/2$ ovals when n is odd. Hence we obtain that $A(1, n) \geq 1 + (n-1)(n-2)/2$ when n is even, and $A(1, n) \geq (n-1)(n-2)/2$ when n is odd.

Second the differential system of degree n studied in [11]

$$\dot x = (a + byx) f_y, \quad \dot y = -(a + byx) f_x + (n+1) by f, \tag{15}$$

where $a, b \in \mathbb{R}\{0\}$, has the invariant algebraic curve (9). It is easy to show that this system is a particular case of system (5). In fact in this case we have that

$$\lambda_1 = a + byx, \quad \lambda_3 = (n+1)by, \quad \lambda_2 = 0.$$

Therefore, from (13) we obtain

$$\dot\lambda_1|_{\lambda_1=0} = (n+1)bybxf|_{\lambda_1=0} = (n+1)b(\lambda_1 - a)f|_{\lambda_1=0} = -(n+1)abf|_{\lambda_1=0} \neq 0,$$

and

$$\oint_\gamma \frac{1}{\lambda_1}(\lambda_3 dx - \lambda_2 dy) = -(n+1)ab \iint_\Omega \frac{dxdy}{(a + bxy)^2} \neq 0.$$

Hence system (15) satisfies conditions (i) and (ii). So we obtain that $B_1(n) \leq A(1, n)$ if n is odd, and $B_2(n) \leq A(1, n)$ if n is even. In short statement (c) is proved.

4. On the number $B_1(n)$

In this section we analyze the existence of the upper and lower bounds for the maximum numbers of ovals of the curve (5), that we have denoted by $B_1(n)$.

Now for n odd we are interesting in showing

$$A(1,n) \geq \frac{(n-1)(n-2)}{2} + 1 \quad \text{for all } n \in \mathbb{N},$$

and that this lower bound is reached. So we want to determine the bounds of $B_1(n)$, and study its realization in the affine plane.

We shall study the curve (5) for $n = 2m-1$. It is easy to see that the maximum genus of the curve (5) is $g = 2(m-1)^2 - 1$, see for more details [6]. Hence the maximum number of ovals (which we denote by $B_1(m)$) is not greater than $g+1 = 2(m-1)^2$ in $\mathbb{RP}^2$ [14]. On the other hand Oleg Viro in a personal communication proved that the numbers of ovals of the curve (5) is at least $1 + (2m-2)(2m-3)/2 = 2(m-1)^2 - (m-2)$. As a consequence we have the following bounds in the projective plane

$$2(m-1)^2 - (m-2) \leq B_1(m) \leq 2(m-1)^2,$$

or equivalently

$$\frac{(n-1)(n-2)}{2} + 1 \leq B_1(n) \leq \left(\frac{(n-1)(n-2)}{2} + 1\right) + \frac{n-1}{2},$$

for $n = 2m-1$.

We are interested in realizing the lower bound of the previous inequality in the affine plane, because then we obtain the number of algebraic limit cycles stated in Conjecture 2 for n odd. In particular for $n = 3$, i.e., $m = 2$ we obtain that $B_1(3) = 2$.This number of ovals is realized in the affine plane as we can observe in the following particular algebraic curve

$$y^2 + (x^2 - 1/2)^2 - 1/5 = 0,$$

see for more details [10].

For the moment we are not able to realize in the affine plane this lower bound for $n > 3$.

Proof of Corollary 5. (a) The proof of the equality $A(2) = 1$ trivially follow from Theorem 3and Theorem 4.

(b) The proof of the inequality $2 \leq A(3) \leq 4$ is easily obtained from Theorem 3 and the fact that $B_1(3) = 2$.

(c) The inequality $6 \leq A(5) \leq 11$ follows from Theorem 3 and from the fact that $B_1(5) \geq 6$ because the algebraic curve

$$f = (x^2 - 1/9)(x^2 - 1)^2 + (y^2 - 1/2)^2 - 1/10 = 0,$$

has genus seven and admits six ovals(see for more details [11]). □

5. Proof of Theorem 6

The proof is obtained from the one of Theorem 4 by considering that system (6) admits the following equivalent two representations

$$\dot{x} = r_j(g_j)_y + l_j g_j, \quad \dot{y} = -r_j(g_j)_x + s_j g_j, \quad j = 1, 2 \tag{16}$$

where

$$\begin{aligned} r_1 &= -\lambda_1 g_2, \quad r_2 = -\lambda_2 g_1, \\ l_1 &= g_2\lambda_4 + \lambda_2(g_2)_y, \quad s_1 = g_2\lambda_3 + \lambda_2(g_2)_x, \\ l_2 &= g_1\lambda_4 + \lambda_1(g_1)_y \quad s_2 = g_1\lambda_3 + \lambda_1(g_1)_x. \end{aligned}$$

Let $\mathcal{X}$ be the vector field associated to system (6). Then we have

$$\begin{aligned} \mathcal{X}g_1 &= (g_2(\lambda_4(g_1)_x + \lambda_3(g_1)_y) + \lambda_2\{g_2, g_1\})g_1, \\ \mathcal{X}g_2 &= (g_1(\lambda_4(g_2)_x + \lambda_3(g_2)_y) + \lambda_2\{g_1, g_2\})g_2. \end{aligned}$$

Consequently $g_j = 0$ for $j = 1, 2$ are invariant algebraic curves. Then statement (a) is proved.

Clearly a singular point on $g_1 = 0$ satisfies either $r_1 = 0$, or $(g_1)_x = (g_1)_y = 0$. Due to our assumptions any of these two cases cannot occur. Thus each oval of $g_1 = 0$ must be a periodic solution of system (6). Analogously we prove that each oval of $g_2 = 0$ must be a periodic solution of system (6). Now we shall show that these periodic solutions are in fact hyperbolic limit cycles.

Consider an oval γ_j of $g_j = 0$ for $j = 1, 2$. From our choice of r_j we know that γ_j does not intersect the curve $r_j = 0$. In order to see that γ_j is a hyperbolic algebraic limit cycle we must show that condition (12) holds for system (6).

After straightforward calculations and by considering that

$$(g_j)_y = \frac{\dot{x} - l_j g_j}{r_j}, \quad -(g_j)_x = \frac{\dot{y} - s_j g_j}{r_j}, \quad j = 1, 2,$$

we obtain in view of (**??**) that

$$\operatorname{div}(t)dt = d(\log|r_j|) + \frac{s_j dx - l_j dy}{r_j} + \frac{g_j}{r_j}(-l_j(r_j)_x - s_j(r_j)_y + r_j(l_j)_x + r_j(s_j)_y)dt,$$

for $j = 1, 2$. Hence, by assumption we have that

$$\oint_{\gamma_j} \operatorname{div}(t)dt = \oint_{\gamma_j} \frac{s_j dx - l_j dy}{r_j} \neq 0,$$

for $j = 1, 2$, which are equivalent to (7) and (8). Thus every oval of the algebraic curve $g_j = 0$ is a hyperbolic algebraic limit cycle.

We suppose that there is a limit cycle $\tilde{\gamma}_j$ which is not contained in $g_1 g_2 = 0$. This limit cycle can be or not algebraic. On the algebraic curve $r_j = 0$ for

$j = 1, 2$ we have

$$\begin{aligned}\dot{r_j} = & \ (r_j)_x\dot{x} + (r_j)_y\dot{y} \\ = & \ (r_j)_x(r_j(g_j)_y + l_jg_j) + (r_j)_y(-r_j(g_j)_x + s_jg_j) \\ = & \ r_j((r_j)_x(g_j)_y - (r_j)_y(g_j)_x) + g_j(l_j(r_j)_x + s_j(r_j)_y), \quad .\end{aligned}$$

Hence

$$\begin{aligned}\dot{r_1} = & \ -r_1\{g_1, r_1\} + g_1g_2(\lambda_4(r_1)_x + \lambda_3(r_1)_y + \lambda_2\{\lambda_1,\, g_2\}, \\ \dot{r_2} = & \ -r_2\{g_2, r_2\} + g_1g_2(\lambda_4(r_2)_x + \lambda_3(r_2)_y + \lambda_1\{\lambda_2,\, g_1\}.\end{aligned}$$

Therefore by assumption (i) we have that

$$\begin{aligned}\dot{r_1}|_{r_1=0} = g_1g_2(\lambda_4(r_1)_x + \lambda_3(r_1)_y + \lambda_2\{\lambda_1,\, g_2\})|_{r_1=0} \neq 0, \\ \dot{r_2}|_{r_2=0} = g_1g_2(\lambda_4(r_2)_x + \lambda_3(r_2)_y + \lambda_1\{\lambda_2,\, g_1\}|_{r_2=0} \neq 0.\end{aligned} \tag{17}$$

in $\mathbb{R}^2/\{g_1g_2 = 0\}$. Thus $\tilde{\gamma}_j$ cannot intersect the curve $r_j = 0$. So $\tilde{\gamma}_j$ lies in a connected component U_j of $\mathbb{R}^2/\{r_j = 0\}$, so that g_j and r_j for $j = 1, 2$ have constant sign on U_j. In view of the relation

$$\dot{g_j} = (g_j)_x\dot{x} + (g_j)_y\dot{y} = -\frac{\dot{y} - s_jg_j}{r_j}\,\dot{x} + \frac{\dot{x} - l_jg_j}{r_j}\dot{y} = \frac{g_j}{r_j}(s_j\dot{x} - l_j\dot{y}),$$

hence

$$\begin{aligned}d(\log|g_1|) = & \ (s_1dx - l_1dy)/r_1 = \frac{1}{\lambda_1}(\lambda_3dx + \lambda_4dy) - \frac{\lambda_2}{\lambda_1}d\log|g_2| \\ d(\log|g_2|) = & \ (s_2dx - l_2dy)/r_2 = \frac{1}{\lambda_2}(\lambda_3dx + \lambda_4dy) - \frac{\lambda_1}{\lambda_2}d\log|g_1|.\end{aligned}$$

By assumption (ii) we get the contradiction

$$0 = \oint_{\tilde{\gamma}_j} d\log|g_j| = \oint_{\tilde{\gamma}_j} \frac{1}{r_j}(s_jdx - l_jdy) \neq 0, \quad j = 1, 2.$$

So statement (b) is proved. The lower bound is deduced as follows. First we note that the differential system

$$\dot{x} = -\lambda(x - a)(g_2)_y, \quad \dot{y} = (n\lambda + K)g_2 + \lambda(x - a)(g_2)_x \tag{18}$$

studied in [12], is a particular case of the system (6), with

$$\lambda_1 = n\lambda + K, \quad \lambda_2 = \lambda, \quad \lambda_3 = 0, \quad \lambda_4 = 0,$$

where $\lambda = (Ax + By + C)$, A, B, C, K, α, a are real parameters and

$$g_1(x, y) = x - a, \quad g_2(x, y) = \alpha x^n + G(x, y),$$

with G an arbitrary polynomial of degree $n - 1$, such that the curve $g_2 = 0$ is irreducible and nonsingular. The parameter a is chosen in such a way that the straight line $x - a = 0$ does not cross the ovals of the curve $g_2 = 0$ and A, B, C, K are such that $BK \neq 0$.

Under these hypothesis we can check that all the assumptions of Theorem 6 hold. Hence the ovals of the curve $g_2 = 0$ are hyperbolic limit cycles of system (18). If we denote by $b_1(n)$, (respectively $b_2(n)$) the maximum number of ovals of the curve $g_2 = 0$ when n is odd (respectively even), then we obtain the lower bound of $A(2, n)$, i.e., $A(2, n) \geq b_1(n)$ when n is odd, and $A(2, n) \geq b_2(n)$ when n is even. By considering that the curve $g_2 = 0$ is nonsingular, then the upper and lower bound inequalities of (10) are obtained from Harnack's Theorem.

Proof of Corollary 7. The lower bound $A(2, 3) \geq 2$ is obtain from the system

$$\dot{x} = (x + y)yg_1 - y(x + y - a)g_2, \quad \dot{y} = -(x - a)(x + y)g_1 + x(x + y - a)g_2,$$

where

$$g_1 = x^2 + y^2 - r^2, \quad g_2 = (x - a)^2 + y^2 - r^2, \quad r < \frac{a}{2}.$$

This system is a particular case of system (6) with $\lambda_2 = (x + y)$, $\lambda_1 = x + y - a$, $\lambda_3 = 0$, $\lambda_4 = 0$. Clearly conditions (i) and (ii) of Theorem 6 in this case hold. It is easy to show that the curves $g_1 = 0$ and $g_2 = 0$ are invariant circles with cofactor $K_1 = 2ay(x + y)$ and $K_2 = 2ay(x + y - a)$ respectively.

The condition (iii) of Theorem 6 in this case does hold. In fact, by considering that $r < a/2$ then the circle $x^2 + y^2 = r^2$ does not intersect the curve $(x + y - a)g_2 = 0$, and the circles $(x - a)^2 + y^2 = r^2$ does not intersect the curve $(x + y)g_1 = 0$ and in view of the relation

$$\oint_{\{(x-s)^2+(y-l)^2=R^2\}} \operatorname{div}(t)dt = 2a\pi(2R^2 + 5l^2 - s^2 + as + 6sl - 3al - r^2)$$

we deduced that

$$\oint_{\{x^2+y^2=r^2\}} \operatorname{div}(t)dt = \oint_{\{(x-a)^2+y^2=r^2\}} \operatorname{div}(t)dt = 2ar^2\pi$$

i.e., the curves $g_\nu = 0$ for $\nu = 1, 2$ are hyperbolic limit cycles, hence $A(2, 3) \geq 2$.

We note that for this system the infinity is a limit cycles.

This completes the proof of the corollary. □

6. Proof of Corollary 9

Proof of Corollary 9. To obtain the lower bound we introduce the following polynomial differential system of degree n already considered in [12]

$$\begin{aligned}
\dot{x} = {} & \Big((x+y)\textstyle\prod_{j=1}^{l}(x^2+y^2-r_j^2) - (x+y-a)\prod_{j=1}^{l}((x-a)^2+y^2-r_j^2)\Big)y \\
\dot{y} = - & \Big((x+y)\textstyle\prod_{j=1}^{l}(x^2+y^2-r_j^2) - (x+y-a)\prod_{j=1}^{l}((x-a)^2+y^2-r_j^2)\Big)x \\
& + a(x+y)\textstyle\prod_{j=1}^{l}(x^2+y^2-r_j^2) \\
& = \Big((x+y)\textstyle\prod_{j=1}^{l}(x^2+y^2-r_j^2) - (x+y-a)\prod_{j=1}^{l}((x-a)^2+y^2-r_j^2)\Big)(x-a) \\
& + a(x+y-a)\textstyle\prod_{j=1}^{l}((x-a)^2+y^2-r_j^2),
\end{aligned}$$

where $0 < r_1 < r_2 < \ldots < r_l < a/2$, which is a particular case of system (11) we easily obtain that the circles $g_j = x^2 + y^2 - r_j^2 = 0$ and $g_{j+l} =$

$(x-a)^2+y^2-r_j^2=0$ for $j=1,2,\dots l$ are invariant of the given polynomial differential system of degree $n=2l+1$, with cofactors

$$K_j = ay(x+y)\prod_{m\neq j}^{l}(x^2+y^2-r_m^2), \quad K_{j+l} = ay(x+y-a)\prod_{m\neq j}^{l}((x-a)^2+y^2-r_m^2)$$

for $j=1,2,\dots l$, respectively.

The conditions (i) and (ii) in this case hold.

by considering that the circles $x^2+y^2=r_j^2$ does not intersect the curves $(x+y-a)g_{j+l}=0$, and the circles $(x-a)^2+y^2=r^2$ does not intersect the curves $(x+y)g_j=0$ for $j=1,2,\dots l$ and in view of the relations

$$\oint_{\{x^2+y^2=r_j^2\}} \operatorname{div}(t)dt = \oint_{\{(x-a)^2+y^2=r_j^2\}} \operatorname{div}(t)dt = (-1)^j 2a\pi \prod_{m\neq j}^{l}(r_m^2-r_j^2)$$

for $j=1,2,\dots l$, we obtain that the given $n-1=2l$ circles are hyperbolic limit cycles. Hence we obtain the lower bound $A(n-1,n)\geq n-1$. In short corollary 9 is proved.

We note that for the constructed differential system the infinity is a limit cycles. □

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