This is a preprint of: "Limit cycles bifurcating from a perturbed quartic center", Bartomeu Coll, Jaume Llibre, Rafel Prohens, *Chaos Solitons Fractals*, vol. 44, 317–334, 2011.
DOI: [10.1016/j.chaos.2011.02.009]

LIMIT CYCLES BIFURCATING FROM A PERTURBED QUARTIC CENTER

BARTOMEU COLL[1], JAUME LLIBRE[2] AND RAFEL PROHENS[1]

Abstract. We consider the quartic center $\dot{x} = -yf(x,y)$, $\dot{y} = xf(x,y)$, with $f(x,y) = (x+a)(y+b)(x+c)$ and $abc \neq 0$. Here we study the maximum number σ of limit cycles which can bifurcate from the periodic orbits of this quartic center when we perturb it inside the class of polynomial vector fields of degree n, using the averaging theory of first order. We prove that $4[(n-1)/2]+4 \leq \sigma \leq 5[(n-1)/2+14$, where $[\eta]$ denotes the integer part function of η.

1. Introduction and statement of the main results

In the qualitative theory of real planar polynomial differential systems the problem of studying how many limit cycles bifurcate perturbing the periodic orbits of a center has been extensively considered in literature. Basically, four methods have been used to perform such studies and they are based on: the Poincaré return map (see for instance [7, 8, 18]), the Poincaré-Pontrjagin-Melnikov integrals or Abelian integrals that are equivalent in the plane (see [2, 11, 3, 4, 6, 10, 25, 26]), the inverse integrating factor (see [12, 13, 14, 24]), and the averaging method which in the plane is also equivalent to the Abelian integrals (see for instance [5, 17, 19]).

Roughly speaking the averaging method gives a quantitative relation between the solutions of a non-autonomous periodic differential system and the solutions of its averaged differential system, which is autonomous. In particular the number of hyperbolic equilibrium points of the averaged differential system up to first order gives a lower bound of the maximum number of limit cycles of the non-autonomous periodic differential system, for more details see Theorem 2.6.1 of Sanders and Verhulst [22] and Theorem 11.5 of Verhulst [23]. Whenever the first averaged function vanishes, the number of limit cycles depends on the second averaged function, and so on (see for more details [5]). In some cases by using the second order averaging method the number of limit cycles increases, even more than the double see for instance [20].

Using the averaging theory of first order in [6], and the Melnikov method in [3], the authors give upper and lower bounds for the maximum number of limit cycles bifurcating from the period annulus of a cubic or quintic center, respectively. The lower bounds in both cases are $3[(n-1)/2]+2$ and $3[(n+1)/2]+1$, respectively.

In this paper we bound the maximum number of limit cycles that bifurcate from the period annulus surrounding the origin of the quartic polynomial differential system

$$\begin{aligned} \dot{x} &= \quad -y(x+a)(y+b)(x+c), \\ \dot{y} &= \quad x(x+a)(y+b)(x+c), \end{aligned} \tag{1}$$

where a, b, $c \in \mathbb{R}\backslash\{0\}$, when we perturb it inside the class of all polynomial differential systems of degree $n \in \mathbb{N}$ and using the averaging theory of first order. That

2000 *Mathematics Subject Classification.* Primary 34C07, Secondary: 34C23, 37G15.

is, we want to study the maximum number of limit cycles of the differential systems

$$\begin{aligned} \dot{x} = & \quad -y(x+a)(y+b)(x+c) + \varepsilon P_n(x,y), \\ \dot{y} = & \quad x(x+a)(y+b)(x+c) + \varepsilon Q_n(x,y), \end{aligned} \tag{2}$$

which bifurcate from the period annulus surrounding the origin of the unperturbed system (1), where $P_n(x,y),\ Q_n(x,y) \in \mathbb{R}_n[x,y]$ (being $\mathbb{R}_n[x,y]$ the ring of real polynomials in the variables x and y of degree $n \in \mathbb{N}$) where ε is a small parameter.

For doing this study we transform the differential system (2) in the equivalent differential equation

$$\frac{dr}{d\theta} = \varepsilon\, \frac{P_n(r\cos\theta, r\sin\theta)\,\cos\theta + Q_n(r\cos\theta, r\sin\theta)\,\sin\theta}{(r\cos\theta + a)(r\sin\theta + b)(r\cos\theta + c)} + \varepsilon^2 G(\theta, r, \varepsilon), \tag{3}$$

by using the change to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ and taking θ as the new independent variable of the differential system. System (3) is defined on the disc $r < d$, with $d = \min\{|a|, |b|, |c|\}$ and being G a 2π-periodic function of θ.

From the averaging theory of first order, if r_0 is a hyperbolic equilibrium point of the averaged differential system

$$\frac{dr}{d\theta} = \varepsilon \mathcal{F}^0(r), \tag{4}$$

associated to system (3) where $\mathcal{F}^0 : (0,d) \to \mathbb{R}$ is defined as

$$\mathcal{F}^0(r) = \frac{1}{2\pi}\int_0^{2\pi} \frac{P_n(r\cos\theta, r\sin\theta)\,\cos\theta + Q_n(r\cos\theta, r\sin\theta)\,\sin\theta}{(r\cos\theta + a)(r\sin\theta + b)(r\cos\theta + c)}\, d\theta,$$

$0 < r < d$, then there exists a periodic orbit of the differential equation (3) which tends to $r = r_0$ as $\varepsilon \to 0$.

In order to obtain lower and upper bounds of the maximum number of limit cycles of differential system (2) bifurcating from the periodic orbits of the center (1) using the averaging theory of first order, we need to control the number of simple zeroes of the function $\mathcal{F}^0(r)$. We note that $\mathcal{F}^0(r)$ has the same zeroes as the first-order Melnikov or Abelian integral associated to differential system (2), in the interval $(0,d)$, see for more details [5]. Also we note that in this interval the zeroes of the function $\mathcal{F}^0(r)$ coincide with the zeroes of the function $r\mathcal{F}^0(r)$. Hence, in order to simplify further computations we consider the function $f^0(r) = r\mathcal{F}^0(r)$.

Our main results are the following theorem and proposition.

Theorem 1. *Consider system* (2) *with P_n and Q_n polynomials of degree n and with $|a| \neq |b|$, $|a| \neq |c|$ and $|b| \neq |c|$. An upper bound for the maximum number of limit cycles bifurcating from the period annulus of system* (2) *when $\varepsilon = 0$, and using averaging theory of first order is $5[(n-1)/2]+14$. Moreover there are systems* (2) *with at least $4[(n-1)/2]+4$ limit cycles bifurcating from the period annulus.*

For the other values of the parameters a, b and c, upper and lower bounds are also obtained. In the next proposition we summarize these results.

Proposition 2. *Consider system* (2) *with P_n and Q_n polynomials of degree n. It is not restrictive to assume that $a < 0$, $b < 0$ and that $|a| = \min(|a|, |c|)$. In Table 1 we present the upper and lower bounds for the maximum number of limit cycles bifurcating from the period annulus of system* (2) *when $\varepsilon = 0$, and either $|a| = |b|$, or $|a| = |c|$, or $|b| = |c|$, using the averaging theory of first order. There are systems* (2) *having at least the number of limit cycles given by the lower bounds.*

The main idea for obtaining the lower bounds is to express the function $f^0(r)$ as a linear combination of a minimal number of generating functions. These calculations start by writing the function $f^0(r)$ as a linear combination of some functions, that we call generating functions, where these coefficients depend on the arbitrary

Cases	Lower bound	Upper bound
$a = b = -c$	$2N + 1$	$2N + 2$
$a = b = c$	$2N + 3$	$2N + 6$
$c = -b$, $b < a$	$3N + 2$	$4N + 6$
$c = -a$, $a < b$ $c = -a$, $b < a$ $c = b$, $b < a$	$3N + 3$	$4N + 6$
$a = b$, $c < -a$ $a = b$, $c < a$	$3N + 3$	$4N + 10$
$a = c$, $c < b$ $a = c$, $b < c$	$3N + 5$	$4N + 14$

TABLE 1. Lower and upper bounds for the maximum number of limit cycles bifurcating from system (2), using the averaging theory of first order, under the assumptions of Proposition 2. Here $N = [\frac{n-1}{2}]$.

coefficients of the polynomial perturbation. In fact, for determining the function $f^0(r)$ in terms of a minimal number of generating functions, two main strategies are used. One for preserving the arbitrariness of the coefficients of such functions and the other for obtaining some relations between the generating functions which allow to clarify which coefficients are independent, and also to reduce their number. This is a laborious part of our study. Finally the lower bounds follow from the calculation of the precise number of linearly independent generating functions which appear in $f^0(r)$.

By comparing our results with the previous similar ones we point out that by increasing the degeneracy of the center at the origin a higher lower bound for the maximum number of limit cycles is obtained. For instance, by taking degree four we obtain $[(n-1)/2]$ limit cycles more than the number of limit cycles obtained in [6], where the degree of the unperturbed center was three.

In the proofs of the upper bounds for the maximum number of limit cycles and in the more generic case we use the Variation of the Argument Principle applied to a suitable complex extension of the function $f^0(r)$ plus some techniques of the complex analysis. See [15] for a similar application of this principle.

The organization of the paper is as follows. In section 2 we introduce some auxiliary functions and some relations between them that are used in section 3. There we reduce the initial number of functions which generate the function $f^0(r)$. In section 4 we use the Variation of the Argument Principle to obtain the upper bound for the maximum number of limit cycles when $|a| \neq |b|$, $|a| \neq |c|$ and $|b| \neq |c|$, and also we obtain upper bounds for the rest of the cases. In section 5 we prove Theorem 1 and Proposition 2. Finally in section 6 we provide some numerical computations for the exact number of limit cycles, and we compare the lower and upper bounds obtained with them. We do that for $n = 4, \ldots, 9$.

2. Auxiliary functions

In this section we introduce a set of functions and relations that are useful in the reduction process for obtaining the minimal number of generating functions for the function $f^0(r)$.

From equation (1) and for integers numbers $i,\ j \geq 0$, we define the functions:

$$A_{i,j} = A_{i,j}(r) = \int_0^{2\pi} \frac{\cos^i\theta \sin^j\theta}{(r\cos\theta + a)}\,d\theta,$$

$$\begin{aligned}
B_{i,j} = B_{i,j}(r) &= \int_0^{2\pi} \frac{\cos^i\theta \sin^j\theta}{(r\sin\theta + b)}\,d\theta,\\
C_{i,j} = C_{i,j}(r) &= \int_0^{2\pi} \frac{\cos^i\theta \sin^j\theta}{(r\cos\theta + c)}\,d\theta,\\
AB_{i,j} = AB_{i,j}(r) &= \int_0^{2\pi} \frac{\cos^i\theta \sin^j\theta}{(r\cos\theta + a)(r\sin\theta + b)}\,d\theta,\\
AC_{i,j} = AC_{i,j}(r) &= \int_0^{2\pi} \frac{\cos^i\theta \sin^j\theta}{(r\cos\theta + a)(r\cos\theta + c)}\,d\theta,\\
BC_{i,j} = BC_{i,j}(r) &= \int_0^{2\pi} \frac{\cos^i\theta \sin^j\theta}{(r\sin\theta + b)(r\cos\theta + c)}\,d\theta,\\
I_{i,j} = I_{i,j}(r) &= \int_0^{2\pi} \frac{\cos^i\theta \sin^j\theta}{(r\cos\theta + a)(r\sin\theta + b)(r\cos\theta + c)}\,d\theta.
\end{aligned} \tag{5}$$

For each integer $p \geq 0$ we introduce the function

$$m(p) = \frac{1}{2\pi}\int_0^{2\pi} \cos^p\theta\,d\theta = \begin{cases} 1 & \text{if } p = 0,\\ \dfrac{(p-1)!!}{p!!} & \text{if } p \text{ is even and } p \geq 2,\\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

where $p!! = \prod_{m=0}^{[\frac{p}{2}]-1}(p-2m)$. We take $0!! = 1$ and $1!! = 1$. From the previous definitions we obtain the next results.

Lemma 3. *Functions* (5) *satisfy:*

(i) $AC_{0,0} = \dfrac{1}{c-a}(A_{0,0} - C_{0,0})$, *if* $a \neq c$,

(ii) $I_{0,0} = \dfrac{1}{c-a}(AB_{0,0} - BC_{0,0})$, *if* $a \neq c$,

(iii) $I_{i,j+1} = \dfrac{1}{r}(AC_{i,j} - b\,I_{i,j})$,

(iv) $I_{i+1,j} = \dfrac{1}{r}(AB_{i,j} - c\,I_{i,j})$,

(v) $AB_{i,0} = \dfrac{1}{r}(B_{i-1,0} - a\,AB_{i-1,0})$,

(vi) $AC_{i+1,j} = \dfrac{1}{r}(A_{i,j} - c\,AC_{i,j})$,

(vii) $AC_{i,2j} = \displaystyle\sum_{k=0}^{j} \binom{j}{k} (-1)^k AC_{i+2k,0}$

(viii) $I_{i,2j} = \displaystyle\sum_{k=0}^{j} \binom{j}{k} (-1)^k I_{i+2k,0}$,

(ix) $A_{i,0} = \dfrac{1}{r}(2\pi m(i-1) - a\,A_{i-1,0})$,

(x) $B_{2i,0} = \dfrac{1}{r^2}(2\pi b\,m(2i-2) - (b^2 - r^2)B_{2i-2,0})$,

(xi) $I_{i,0} = \dfrac{1}{r^2}(c^2\,I_{i-2,0} - (a+c)\,AB_{i-2,0} + B_{i-2,0})$, $i \geq 2$,

for all integers $i,\ j \geq 0$.

Proof. These relations follow from the definitions of the corresponding functions plus some additional considerations. For instance, we get the relations (vii) and (viii) by using formula $\sin^{2i}\theta = \sum_{s=0}^{i} \left(\begin{array}{c} i \\ s \end{array} \right) (-1)^s \cos^{2s}\theta$. Relations (ix) and (x) follow from the definition of $m(p)$. To obtain relation (xi) we proceed as follows. From relation (iv) we have

$$I_{i,0} = \frac{1}{r}(AB_{i-1,0} - c\, I_{i-1,0}). \tag{7}$$

Then, by using relation (v) we obtain

$$I_{i,0} = \frac{1}{r}(\frac{1}{r}B_{i-2,0} - \frac{a}{r}AB_{i-2,0} - c\, I_{i-1,0}).$$

Using (7) in this last expression the proof follows. □

In next lemma we present how to write some previous functions after an iterative application of formulas given in Lemma 3.

Lemma 4. *Let i be a positive integer number. Functions $A_{2i,0}$, $AR_{2i,0}$, $B_{2i,0}$, $AB_{2i,0}$ and $I_{2i,0}$ satisfy:*

(i) $r^{2i}A_{2i,0} = a^{2i}A_{0,0} - 2\pi \sum_{j=1}^{i} a^{2j-1}m(2i-2j)r^{2i-2j}$;

(ii) $r^2 AC_{2,0} = 2\pi - (a+c)A_{0,0} + c^2\, AC_{0,0}$, *and if $i \geq 2$ then*

$$r^{2i}AC_{2i,0} = \frac{c^{2i} - a^{2i}}{a-c}A_{0,0} + c^{2i}\, AC_{0,0} + 2\pi \sum_{j=1}^{i-1} m(2j)c^{2(i-j-1)}r^{2j}$$
$$+ 2\pi(a+c)\sum_{j=1}^{i-1} c^{2(i-j-1)} \sum_{k=1}^{j} a^{2k-1}m(2j-2k)r^{2j-2k};$$

(iii) $r^{2i}B_{2i,0} = 2\pi b \sum_{j=1}^{i}(-1)^{j-1}(b^2 - r^2)^{j-1}r^{2(i-j)}m(2i-2j)$ $+(-1)^i(b^2-r^2)^i B_{0,0}$;

(iv) $r^2\, AB_{2,0} = a^2\, AB_{0,0} - a\, B_{0,0}$, *and if $i \geq 2$ then*

$$r^{2i}AB_{2i,0} = a^{2i-1}(a\, AB_{0,0} - B_{0,0}) + a(b^2 - r^2)\frac{(r^2-b^2)^{i-1} - a^{2i-2}}{a^2 + b^2 - r^2}B_{0,0}$$
$$- 2\pi b \sum_{j=1}^{i-1}\sum_{k=1}^{i-j}(-1)^{k-1}a^{2j-1}(b^2-r^2)^{k-1}m(2(i-j-k))r^{2(i-j-k)};$$

(v) $r^2 I_{2,0} = -(a+c)\, AB_{0,0} + B_{0,0} + c^2\, I_{0,0}$, *and if $i \geq 2$ then*

$$r^{2i}I_{2i,0} = -\frac{a^{2i} - c^{2i}}{a-c}AB_{0,0} + P^{i-1}_{a,b,c}(r^2)B_{0,0} + Q^{i-2}_{a,b,c}(r^2) + c^{2i}\, I_{0,0},$$

where $P^{i-1}_{a,b,c}(r^2)$ (resp. $Q^{i-2}_{a,b,c}(r^2)$) is a polynomial of degree $i-1$ (resp. $i-2$) in r^2, with no arbitrary coefficients.

Proof. The proof of statements (i) and (iii) follows easily from direct computations. To get (ii) we iteratively apply (vi) and (ix) of Lemma 3 for obtaining only the even first subindex for the $A_{i,0}$ functions. Finally we use equality (i) and statement (ii) follows.

For obtaining (iv) we iteratively apply Lemma 3(v) and, as $B_{2i+1,0} = 0$, only the functions $B_{2i,0}$ remain. Finally we use (iii) and we get statement (iv).

For having (v) we iteratively apply Lemma 3(xi), obtaining

$$r^{2i}I_{2i,0} = -(a+c)\sum_{j=1}^{i} c^{2j-2}(AB_{2i-2j,0} + B_{2i-2j,0})r^{2i-2j} + c^{2i}\, I_{0,0}.$$

Finally by using (iii) and (iv) the proof of statement (v) follows. □

In next lemma for a particular choice of the parameters a, b and c, we obtain the values of functions $A_{0,0}$, $B_{0,0}$, $C_{0,0}$, $AB_{0,0}$ and $BC_{0,0}$ and some new relations between them.

Lemma 5. *Let a, b and c be non-zero real numbers. We assume that $a \le -1$, $b \le -1$, $|c| \ge 1$ and $r \in (0, d)$, where $d = \min\{|a|, |b|, |c|\}$.*

(i) The following equalities hold:

$$A_{0,0} = -\frac{2\pi}{\sqrt{a^2-r^2}}, \quad B_{0,0} = -\frac{2\pi}{\sqrt{b^2-r^2}}, \quad C_{0,0} = \pm\frac{2\pi}{\sqrt{c^2-r^2}},$$
$$AB_{0,0} = -\frac{2\pi(a\sqrt{a^2-r^2}+b\sqrt{b^2-r^2})}{(a^2+b^2-r^2)\sqrt{a^2-r^2}\sqrt{b^2-r^2}},$$
$$BC_{0,0} = \frac{2\pi(\pm b\sqrt{b^2-r^2}+c\sqrt{c^2-r^2})}{(b^2+c^2-r^2)\sqrt{b^2-r^2}\sqrt{c^2-r^2}},$$

where in $\pm$ we take the plus (resp. minus) sign when $c \ge d$ (resp. $c \le -d$).

(ii) For any real polynomial $P(r^2)$ of degree m in the variable r^2, we have

$$P(r^2)AB_{0,0} = p_0\, AB_{0,0} + P_1(r^2)\, A_{0,0} + P_2(r^2)\, B_{0,0},$$
$$P(r^2)BC_{0,0} = q_0\, BC_{0,0} + Q_1(r^2)\, B_{0,0} + Q_2(r^2)\, C_{0,0},$$

where p_0 and q_0 are arbitrary constants depending linearly on the independent term of $P(r^2)$, and $P_i(r^2)$ and $Q_i(r^2)$ for $i = 1, 2$ are polynomials of degree $m-1$ in the variable r^2.

(iii) If $a = c$ then for any real polynomial $P(r^2)$ of degree m in the variable r^2 we have

$$AC_{0,0} = \frac{-2\pi a}{(a^2-r^2)^{3/2}},$$
$$I_{0,0} = \frac{-2\pi((a^2-r^2)^{3/2}(a^2-b^2+r^2)+ab(b^2-r^2)^{1/2}(3a^2-b^2+3r^2))}{(a^2-r^2)^{3/2}(b^2-r^2)^{1/2}(a^2+b^2-r^2)^2},$$
$$P(r^2)\, AC_{0,0} = p_0\, AC_{0,0} + P_1(r^2)\, A_{0,0},$$
$$P(r^2)\, I_{0,0} = P_1(r^2)\, B_{0,0} + P_2(R^2)\, A_{0,0} + p_0\, AC_{0,0} + p_1\, I_{0,0} + p_2\, r^2 I_{0,0},$$

where p_i for $i = 0, 1, 2$, are arbitrary constants depending linearly on the independent term of $P(r^2)$, and $P_i(r^2)$ for $i = 1, 2$ is a polynomial of degree $m - i$ in the variable r^2.

Proof. Formulas of (i) follow easily from the residue integration method.

From (i) we obtain

$$r^2 AB_{0,0} = (a^2+b^2)\, AB_{0,0} - a\, B_{0,0} - b\, A_{0,0}, \tag{8}$$
$$r^2 BC_{0,0} = (b^2+c^2)\, BC_{0,0} - c\, B_{0,0} - b\, C_{0,0}. \tag{9}$$

Equalities (ii) follow from a recursive use of formulas (8) and (9).

The first two formulas of (iii) follow again from the residue integration method.

As in the proof of the first equality of (ii), we can obtain the third equality of (iii) from a recursive use of

$$r^2\, AC_{0,0} = -a\, A_{0,0} + a^2\, AC_{0,0}. \tag{10}$$

This expression is obtained from the first two formulas of (iii) and (i).

We prove the last formula of (iii). From the expression of $I_{0,0}$ and using the expressions of $AC_{0,0}$ and (i), we get

$$(a^2+b^2-r^2)^2\, I_{0,0} = (a^2-b^2+r^2)\, B_{0,0} + b(3a^2-b^2+3r^2)\, AC_{0,0}.$$

From this equality, by using expression (10), we obtain

$$\begin{aligned} r^4 I_{0,0} = &- (a^2+b^2)^2 I_{0,0} + 2(a^2+b^2) r^2 I_{0,0} + (a^2-b^2+r^2) B_{0,0} \\ &- 3ab A_{0,0} + b(6a^2-b^2) AC_{0,0}. \end{aligned}$$

By using recursively this expression combined with formula (10), we obtain the last formula of (iii). Concerning the arbitrariness of p_i for $i = 0, 1, 2$, we note that it is a consequence of the arbitrariness of the coefficient of degree i in $P(r^2)$. □

Remark 6. *Under the hypotheses of Lemma 5 we have that*

$$\begin{aligned} AB_{0,0} &= \frac{a}{a^2+b^2-r^2} B_{0,0} + \frac{b}{a^2+b^2-r^2} A_{0,0}, \\ BC_{0,0} &= \frac{-c}{b^2+c^2-r^2} B_{0,0} + \frac{b}{b^2+c^2-r^2} C_{0,0}. \end{aligned}$$

3. The generating functions of the function $f^0(r)$

The objective of this section is to obtain the minimum number of generating functions for the function $f^0(r)$.

In equation (2) we write

$$P_n(x,y) = \sum_{k=0}^{n} \sum_{i+j=k} p_{i,j} x^i y^j, \quad Q_n(x,y) = \sum_{k=0}^{n} \sum_{i+j=k} q_{i,j} x^i y^j,$$

where the coefficients $p_{i,j}$ and $q_{i,j}$ are arbitrary. We can express the function $f^0(r)$ as

$$f^0(r) = \frac{1}{2\pi} \sum_{k=0}^{n} \int_0^{2\pi} \frac{f_k(\theta) r^{k+1}}{(r\cos\theta + a)(r\sin\theta + b)(r\cos\theta + c)}\, d\theta, \tag{11}$$

where

$$f_k(\theta) = \cos\theta \sum_{i+j=k} p_{i,j} \cos^i\theta \sin^j\theta + \sin\theta \sum_{i+j=k} q_{i,j} \cos^i\theta \sin^j\theta.$$

Using the functions $I_{i,j}$ we obtain

$$f^0(r) = \sum_{\substack{i+j=k \\ k=0}}^{n} (p_{i,j} I_{i+1,j} + q_{i,j} I_{i,j+1}) r^{k+1}, \tag{12}$$

where for simplicity we renamed the coefficients $p_{i,j}/(2\pi)$ and $q_{i,j}/(2\pi)$ as $p_{i,j}$ and $q_{i,j}$, respectively.

A direct calculation shows that not all the coefficients $p_{i,j}$ and $q_{i,j}$ are necessary to express $f^0(r)$ as an arbitrary linear combination of functions $I_{i,j}$. In fact, we can write $f^0(r)$ as

$$\sum_{k=0}^{n} (p_{k,0} I_{k+1,0} + \sum_{\substack{i+j=k \\ j\neq 0}} (p_{i,j} + q_{i+1,j-1}) I_{i+1,j} + q_{0,k} I_{0,k+1}) r^{k+1}. \tag{13}$$

In this last expression, without loss of generality, we rename the coefficient $p_{i,j} + q_{i+1,j-1}$ as $p_{i,j}$ and the coefficient $q_{0,k}$ as $p_{-1,k+1}$. Hence $f^0(r)$ writes as

$$f^0(r) = \sum_{k=0}^{n} \sum_{j=0}^{k+1} p_{k-j,j} I_{k-j+1,j} r^{k+1}. \tag{14}$$

Note that all coefficients $p_{i,j}$ remain arbitrary. Hence, according to the goal of this section, $f^0(r)$ is a linear combination with arbitrary coefficients of the functions

$$I_{k-j+1,j} r^{k+1}, \quad 0 \le k \le n, \quad 0 \le j \le k+1.$$

Now we shall reduce the number of functions needed to express the function $f^0(r)$ given in (14). To achieve it we do several reductions. First, by using Lemma 3(iii) we transform $f^0(r)$ in

$$\begin{aligned} f^0(r) &= (p_{0,1}AC_{1,0} - (bp_{0,1} - p_{0,0})I_{1,0} + p_{-1,1}I_{0,1})r \\ &+ \sum_{s=1}^{N}\sum_{i=0}^{s}(p_{2s-2i-1,2i+1}AC_{2s-2i,2i}r^{2s} + p_{2s-2i,2i+1}AC_{2s-2i+1,2i}r^{2s+1}) \\ &+ \sum_{s=1}^{N+1}\sum_{i=0}^{s}p_{2s-2i-1,2i}I_{2s-2i,2i}r^{2s} + \sum_{s=1}^{N}\sum_{i=0}^{s}p_{2s-2i,2i}I_{2s-2i+1,2i}r^{2s+1}, \end{aligned} \tag{15}$$

where $N = [(n-1)/2]$.

Second, we shall do all the second subscripts of the functions involved in expression (15) equal to zero. We do it by using formulas (vii) and (viii) of Lemma 3. Hence (15) becomes

$$\begin{aligned} f^0(r) &= p_{-1,1}(AC_{0,0} - bI_{0,0}) + (p_{0,1}AC_{1,0} - (bp_{0,1} - p_{0,0})I_{1,0})r \\ &+ AC_{0,0}\sum_{s=1}^{N}c_{s,0}^{1,1}r^{2s} + \sum_{t=1}^{N}AC_{2t,0}(\sum_{s=t}^{N}c_{s,t}^{1,1}r^{2s-2t})r^{2t} \\ &+ AC_{1,0}\sum_{s=1}^{N}c_{s,0}^{0,1}r^{2s+1} + \sum_{t=1}^{N}AC_{2t+1,0}(\sum_{s=t}^{N}c_{s,t}^{0,1}r^{2s-2t})r^{2t+1} \\ &+ I_{0,0}\sum_{s=1}^{N+1}c_{s,0}^{1,0}r^{2s} + \sum_{t=1}^{N+1}I_{2t,0}(\sum_{s=t}^{N+1}c_{s,t}^{1,0}r^{2s-2t})r^{2t} \\ &+ I_{1,0}\sum_{s=1}^{N}c_{s,0}^{0,0}r^{2s+1} + \sum_{t=1}^{N}I_{2t+1,0}(\sum_{s=t}^{N}c_{s,t}^{0,0}r^{2s-2t})r^{2t+1}, \end{aligned} \tag{16}$$

where

$$c_{s,t}^{u,v} = \sum_{j=s-t}^{s}(-1)^{j-s+t}p_{2s-2j-u,2j+v}\begin{pmatrix} j \\ j-s+t \end{pmatrix}. \tag{17}$$

Collecting the coefficients of the same powers in the variable r (16) writes as

$$\begin{aligned} f^0(r) &= p_{-1,1}(AC_{0,0} - bI_{0,0}) + (p_{0,1}AC_{1,0} - (bp_{0,1} - p_{0,0})I_{1,0})r \\ &+ AC_{0,0}P_0^N(r^2) + AC_{1,0}P_1^N(r^2)r + I_{0,0}Q_0^{N+1}(r^2) + I_{1,0}Q_1^N(r^2)r \\ &+ \sum_{t=1}^{N}\big[AC_{2t,0}P_{2t}^{N-t}(r^2)r^{2t} + AC_{2t+1,0}P_{2t+1}^{N-t}(r^2)r^{2t+1} \\ &+ I_{2t,0}Q_{2t}^{N-t+1}(r^2)r^{2t} + I_{2t+1,0}Q_{2t+1}^{N-t}(r^2)r^{2t+1}\big] \\ &+ I_{2N+2,0}Q_{2N+2}^{0}(r^2)r^{2N+2}, \end{aligned} \tag{18}$$

where $P_{2t}^{N-t}(r^2)$, $P_{2t+1}^{N-t}(r^2)$ and $Q_{2t+1}^{N-t}(r^2)$ are real polynomials in the variable r^2, of degree $N-t$ for each $t = 0, \ldots, N$ and $Q_{2t}^{N-t+1}(r^2)$ also is a real polynomial in the variable r^2, but of degree $N-t+1$. We point out that $P_0^N(r^2)$, $P_1^N(r^2)$, $Q_0^{N+1}(r^2)$ and $Q_1^N(r^2)$ have no independent term. More precisely

$$P_0^N(r^2) = \sum_{s=1}^{N}c_{s,0}^{1,1}r^{2s}, \qquad P_1^N(r^2) = \sum_{s=1}^{N}c_{s,0}^{0,1}r^{2s},$$

$$Q_0^{N+1}(r^2) = \sum_{s=1}^{N+1}c_{s,0}^{1,0}r^{2s}, \qquad Q_1^N(r^2) = \sum_{s=1}^{N}c_{s,0}^{0,0}r^{2s},$$

$$P_{2t}^{N-t}(r^2) = \sum_{s=t}^{N} c_{s,t}^{1,1} r^{2s-2t}, \qquad P_{2t+1}^{N-t}(r^2) = \sum_{s=t}^{N} c_{s,t}^{0,1} r^{2s-2t}, \tag{19}$$
$$Q_{2t}^{N-t+1}(r^2) = \sum_{s=t}^{N+1} c_{s,t}^{1,0} r^{2s-2t}, \qquad Q_{2t+1}^{N-t}(r^2) = \sum_{s=t}^{N} c_{s,t}^{0,0} r^{2s-2t},$$

for $t = 0, \ldots, N$, except for the polynomial $Q_{2t}^{N-t+1}(r^2)$ which also is defined when $t = N+1$. We shall prove that all the coefficients $c_{s,t}^{u,v}$ defined in (17), are arbitrary. Hence, as a consequence, we will get that all polynomials in expression (18) are, also, arbitrary. We note that the coefficients $c_{s,t}^{u,v}$ depend only on the coefficients of the polynomial perturbation, i.e. on $P_n(x,y)$ and $Q_n(x,y)$.

Lemma 7. *From expression* (17)*, we have that* $c_{s_1,t_1}^{u_1,v_1} = c_{s_2,t_2}^{u_2,v_2}$ *if and only if* $(s_1,t_1) = (s_2,t_2)$.

Proof. From expression (17), if either $u_1 \neq u_2$ or $v_1 \neq v_2$, as $u_i,\ v_i \in \{0,1\}$, $i \in \{0,1\}$, then we have that $c_{s_1,t_1}^{u_1,v_1} \neq c_{s_2,t_2}^{u_2,v_2}$. Hence, it is enough to prove our result only when $(u,v) = (u_1,v_1) = (u_2,v_2)$ and $(s_1,t_1) \neq (s_2,t_2)$. Without loss of generality we may assume that $s_1 + t_1 \geq s_2 + t_2$. We prove that in the sum defining the coefficient $c_{s_1,t_1}^{u,v}$, given by (17), there is a term whose corresponding $p_{2s_1-2j-u,2j+v}$ is such that it does not appear in the sum defining the coefficient $c_{s_2,t_2}^{u,v}$.

In the expression of $c_{s_1,t_1}^{u,v}$ we consider the first term of the sum, i.e. the term corresponding to $j = s_1 - t_1$. In this term $p_{2s_1-2j-u,2j+v} = p_{2t_1-u,2(s_1-t_1)+v}$. If in the expression of the coefficient $c_{s_2,t_2}^{u,v}$ there is, for some $j_0 \in \{s_2 - t_2, ..., s_2\}$, a term in the sum containing $p_{2t_1-u,2(s_1-t_1)+v}$, then we should have $p_{2s_2-2j_0-u,2j_0+v} = p_{2t_1-u,2(s_1-t_1)+v}$. Hence $s_2 = s_1$ and $j_0 = s_1 - t_1$. Therefore, since $s_2 - t_2 \leq j_0 \leq s_2$ we get $t_1 \leq t_2$.

From the hypothesis, i.e. since $(s_1,t_1) \neq (s_2,t_2)$ then, necessarily, $t_1 < t_2$. This contradicts our hypothesis, $s_1 + t_1 \geq s_2 + t_2$. $\square$

One step more in the reduction of the number of functions involved in expression (18) is to get only even subscripts in the functions $AC_{p,0}$ and $I_{p,0}$. We do it by using formulas (iv) and (vi) of Lemma 3. Hence, expression (18) writes as

$$\begin{aligned}
f^0(r) = {} & (p_{0,1} + P_1^N(r^2))A_{0,0} + (-cp_{0,1} + p_{-1,1} + P_0^N(r^2) - cP_1^N(r^2))AC_{0,0} \\
& + (-bp_{0,1} + p_{0,0} + Q_1^N(r^2))AB_{0,0} + (bcp_{0,1} - bp_{-1,1} - cp_{0,0} + Q_0^{N+1}(r^2) \\
& - cQ_1^N(r^2))I_{0,0} + I_{2N+2,0}Q_{2N+2}^0(r^2)r^{2N+2} \\
& + \sum_{t=1}^{N} P_{2t+1}^{N-t}(r^2)A_{2t,0}r^{2t} + \sum_{t=1}^{N}(P_{2t}^{N-t}(r^2) - cP_{2t+1}^{N-t}(r^2))AC_{2t,0}r^{2t} \\
& + \sum_{t=1}^{N} Q_{2t+1}^{N-t}(r^2)AB_{2t,0}r^{2t} + \sum_{t=1}^{N}(Q_{2t}^{N-t+1}(r^2) - cQ_{2t+1}^{N-t}(r^2))I_{2t,0}r^{2t},
\end{aligned}$$

or equivalently

$$\begin{aligned}
f^0(r) = {} & P_1^N(r^2)A_{0,0} + P_0^N(r^2)AC_{0,0} + Q_1^N(r^2)AB_{0,0} + Q_0^{N+1}(r^2)I_{0,0} \\
& + \sum_{t=1}^{N} P_{2t+1}^{N-t}(r^2)A_{2t,0}r^{2t} + \sum_{t=1}^{N} P_{2t}^{N-t}(r^2)AC_{2t,0}r^{2t} \\
& + \sum_{t=1}^{N} Q_{2t+1}^{N-t}(r^2)AB_{2t,0}r^{2t} + \sum_{t=1}^{N+1} Q_{2t}^{N-t+1}(r^2)I_{2t,0}r^{2t}
\end{aligned} \tag{20}$$

where $P_0^N(r^2)$, $P_1^N(r^2)$, $Q_0^{N+1}(r^2)$, $Q_1^N(r^2)$, $P_{2t}^{N-t}(r^2)$ and $Q_{2t}^{N-t+1}(r^2)$ are new polynomials in the variable r^2 with arbitrary coefficients except the polynomial $Q_0^{N+1}(r^2)$ which has no arbitrary independent term.

The next step in the reduction process is to reduce the number of generating functions of $f^0(r)$. In (20) $f^0(r)$ is given by a lineal combinations of the functions $r^{2i}A_{2t,0}$, $r^{2i}AC_{2t,0}$, $r^{2i}AB_{2t,0}$, $r^{2i}I_{2t,0}$ for suitable integers $i \geq 0$ plus some polynomial. In the next proposition we express $f^0(r)$ in terms of a less number of generating functions, namely we express $f^0(r)$ as an arbitrary lineal combination of the functions $r^{2i}A_{0,0}$, $r^{2i}B_{0,0}$, $r^{2i}C_{0,0}$, $r^{2i}AB_{0,0}$ and $r^{2i}BC_{0,0}$ for suitable integrers $i \geq 0$ plus some polynomial. This is performed, basically by applying Lemmas 3 and 4 and by taking care of the coefficients.

Proposition 8. *The function $f^0(r)$ can be written in the following way.*

(i) If $a \neq c$, then

$$f^0(r) = \overline{P}^N(r^2)\,A_{0,0} + S^N(r^2)\,B_{0,0} + Q^N(r^2)\,C_{0,0} + \overline{S}^{N+1}(r^2)\,AB_{0,0} + T^{N+1}(r^2)\,BC_{0,0} + U^{N-1}(r^2),$$

(ii) If $a = c$, then

$$f^0(r) = P^N(r^2)\,A_{0,0} + Q^N(r^2)\,AC_{0,0} + R^N(r^2)\,AB_{0,0} + S^N(r^2)\,B_{0,0} + T^{N+1}(r^2)\,I_{0,0} + U^{N-1}(r^2),$$

where the functions: P^N, $\overline{P}^N$, Q^N, R^N, S^N, $\overline{S}^{N+1}$, T^{N+1} and U^{N-1} are real polynomials in the variable r^2, whose degree is the corresponding superindex and with arbitrary coefficients, except the leading term of $\overline{S}^{N+1}$ which is not.

Proof. The idea of the proof is to apply Lemma 4 to expression (20) and to make a suitable election of the coefficients of each one of the polynomials multiplying the functions $A_{0,0}$, $B_{0,0}$, $AB_{0,0}$, $AC_{0,0}$ and $I_{0,0}$, to ensure arbitrariness of their coefficients. By applying Lemma 4 to expression (20) we get

$$\begin{aligned}
&\sum_{t=1}^{N} P_{2t+1}^{N-t}(r^2)A_{2t,0}r^{2t} = S_{1,1}^{N-1}A_{0,0} + S_{1,2}^{N-1},\\
&\sum_{t=1}^{N} P_{2t}^{N-t}(r^2)AC_{2t,0}r^{2t} = S_{2,1}^{N-1}A_{0,0} + S_{2,2}^{N-1}AC_{0,0} + S_{2,3}^{N-1},\\
&\sum_{t=1}^{N} Q_{2t+1}^{N-t}(r^2)AB_{2t,0}r^{2t} = S_{3,1}^{N-1}AB_{0,0} + S_{3,2}^{N-1}B_{0,0} + S_{3,3}^{N-2},\\
&\sum_{t=1}^{N+1} Q_{2t}^{N-t+1}(r^2)I_{2t,0}r^{2t} = S_{4,1}^{N}AB_{0,0} + S_{4,2}^{N}B_{0,0} + S_{4,3}^{N}I_{0,0} + S_{4,4}^{N-1},
\end{aligned} \tag{21}$$

where $S_{i,j}^{N-k}$ is a real polynomial in the variable r^2 of degree $N-k$.

From Lemma 4(i) we have

$$S_{1,2}^{N-1} = -2\pi\sum_{t=1}^{N} P_{2t+1}^{N-t}(r^2)\sum_{j=1}^{t} a^{2j-1}m(2t-2j)r^{2t-2j}.$$

Hence the arbitrariness of the coefficients of the polynomial $S_{1,2}^{N-1}$ is a consequence of the arbitrariness of the coefficients of the polynomial $P_3^{N-1}(r^2)$, which includes an arbitrary independent term. About the arbitrariness of the coefficients of the polynomial $S_{4,2}^N$ from Lemma 4(v), we have

$$S_{4,2}^{N-1} = Q_2^N(r^2) + \sum_{t=2}^{N+1} Q_{2t}^{N-t+1}(r^2) P_{a,b,c}^{t-1}(r^2).$$

It follows from the arbitrariness of the coefficients of the polynomial $Q_2^N(r^2)$, which also includes an arbitrary independent term.

From (20) and (21) we obtain

$$\begin{aligned} f^0(r) = \; & P^N(r^2)\, A_{0,0} + Q^N(r^2)\, AC_{0,0} + R^N(r^2)\, AB_{0,0} + S^N(r^2)\, B_{0,0} \\ & + T^{N+1}(r^2)\, I_{0,0} + U^{N-1}(r^2), \end{aligned} \tag{22}$$

for some real polynomials P^N, Q^N, R^N, S^N, T^{N+1} and U^{N-1} in the variable r^2, whose degree is the corresponding superindex. The arbitrariness of the coefficients of these polynomials is based on the arbitrariness of the coefficients of the polynomials P_1^N, P_0^N, Q_1^N, $S_{4,2}^N$, Q_0^{N+1} and $S_{1,2}^{N-1}$, respectively. It is important to remark that, as the polynomial Q_0^{N+1} has no independent term, the existence of arbitrary independent term for polynomial T^{N+1} is guaranteed from the existence of arbitrary independent term in $S_{4,3}^N$. Check this fact from (21). This proves statement (ii) of this proposition.

If we assume $a \neq c$ then, from (22) and by using (i) and (ii) of Lemma 3, we obtain statement (i), where $\overline{P}^N$ and $\overline{S}^{N+1}$ are new real polynomials in the variable r^2 satisfying the requirements of the statement of this proposition. □

The last step in this reduction process is next result. For a particular choice of the parameters a, b and c in equation (2), we use the residue integration method applied to the functions involved in Proposition 8(i) and (ii). In this way we get new relations between these functions that provide a new expression of the function $f^0(r)$. This is the goal of next proposition.

Proposition 9. *We consider the function $f^0(r)$ and we assume that $a \leq -1$, $b \leq -1$ and $|c| \geq 1$. Then $f^0(r)$ can be written in the following way.*

(i) If $a \neq c$, then

$$\begin{aligned} f^0(r) \;\; = \;\; & P^N(r^2)\, A_{0,0} + Q^N(r^2)\, B_{0,0} + R^N(r^2)\, C_{0,0} + s_0\, AB_{0,0} \\ & + t_0\, BC_{0,0} + U^{N-1}(r^2), \end{aligned}$$

(ii) If $a = c$, then

$$\begin{aligned} f^0(r) \;\; = \;\; & P^N(r^2)\, A_{0,0} + Q^N(r^2)\, B_{0,0} + r_0\, AB_{0,0} + s_0\, AC_{0,0} \\ & + t_0\, I_{0,0} + u_0\, I_{0,0} r^2 + U^{N-1}(r^2), \end{aligned}$$

where the functions: P^N, Q^N, R^N and U^{N-1} are real polynomials in the variable r^2, whose degree is the corresponding superindex and with arbitrary coefficients. Here r_0, s_0, t_0 and u_0 are arbitrary costants.

Proof. We consider $a \neq c$. In this case function $f^0(r)$ is given in Proposition 8(i). If we express $AB_{0,0}$ and $BC_{0,0}$ as they are given in Lemma 5(ii), then statement (i) is obtained with new polynomials $P^N(r^2)$, $Q^N(r^2)$ and $R^N(r^2)$, and where s_0 and t_0 are constants.

In case $a = c$ the function $f^0(r)$ is given by Proposition 8(ii). By using $AB_{0,0}$, $AC_{0,0}$ and $I_{0,0}$ given in Lemma 5, we obtain

$$\begin{aligned} f^0(r) \;\; = \;\; & P^N(r^2)\, A_{0,0} + Q^N(r^2)\, B_{0,0} + r_0\, AB_{0,0} + s_0\, AC_{0,0} \\ & + t_0\, I_{0,0} + u_0\, I_{0,0} r^2 + U^{N-1}(r^2), \end{aligned}$$

with new polynomials $P^N(r^2)$ and $Q^N(r^2)$, and where r_0, s_0, t_0 and u_0 are constants. □

In next result we give the exact number of generating functions of $f^0(r)$, i.e. the basis in which the function $f^0(r)$ can be expressed as a linear combination. Its proof follows straightforward from Proposition 9.

Proposition 10. *We consider differential system* (2). *Assume that $a \le -1$, $b \le -1$, $|c| \ge 1$, $b \ne a$ and $b \ne c$. A basis for the generating functions of $f^0(r)$ is given by the following system of linearly independent functions, where the functions $A_{0,0}$, $B_{0,0}$, $C_{0,0}$, $AB_{0,0}$, $AC_{0,0}$, $BC_{0,0}$ and $I_{0,0}$ are given in Lemma 5.*

(i) If $|a| \ne |c|$ then a basis is given by the following $4N+5$ functions:

$$\{A_{0,0}, A_{0,0}r^2, A_{0,0}r^4, \ldots A_{0,0}r^{2N}, B_{0,0}, B_{0,0}r^2, B_{0,0}r^4, \ldots B_{0,0}r^{2N},$$
$$C_{0,0}, C_{0,0}r^2, C_{0,0}r^4, \ldots C_{0,0}r^{2N}, AB_{0,0}, BC_{0,0}, 1, r^2, r^4, \ldots r^{2(N-1)}\}.$$

(ii) If $a = c$ then a basis is given by the following $3N+6$ functions:

$$\{A_{0,0}, A_{0,0}r^2, A_{0,0}r^4, \ldots A_{0,0}r^{2N}, B_{0,0}, B_{0,0}r^2, B_{0,0}r^4, \ldots B_{0,0}r^{2N},$$
$$AB_{0,0}, AC_{0,0}, I_{0,0}, I_{0,0}r^2, 1, r^2, r^4, \ldots r^{2(N-1)}\}.$$

(iii) If $a = -c$ then a basis is given by the following $3N+4$ functions:

$$\{A_{0,0}, A_{0,0}r^2, A_{0,0}r^4, \ldots A_{0,0}r^{2N}, B_{0,0}, B_{0,0}r^2, B_{0,0}r^4, \ldots B_{0,0}r^{2N},$$
$$AB_{0,0}, BC_{0,0}, 1, r^2, r^4, \ldots r^{2(N-1)}\}.$$

Proof. We prove only statement (i). Similar arguments can be followed for proving (ii) and (iii). In the case $a^2 \ne c^2$ we take an arbitrary linear combination of the functions given in (i), as

$$p_N(r^2)A_{0,0} + q_N(r^2)B_{0,0} + r_N(r^2)C_{0,0} + s_0AB_{0,0} + t_0BC_{0,0} + U_{N-1}(r^2) = 0, \quad (23)$$

for all $r \in (0, d)$ and where d is defined as in (3). Hence $p_N(r^2)$, $q_N(r^2)$ and $r_N(r^2)$ are arbitrary real polynomials of degree N, s_0 and t_0 are arbitrary real constants and $U_{N-1}(r^2)$ is an arbitrary real polynomial of degree $N-1$. We shall prove that $p_N(r^2) = q_N(r^2) = r_N(r^2) = s_0 = t_0 = U_{N-1}(r^2) = 0$.

Without loss of generality we may assume that the period annulus of equation (1) is the open disc of radius either a or b. We assume that the radius of this disc is a. Analogous considerations can be done otherwise.

From Remark 6, (23) writes

$$\begin{aligned}(p_N(r^2) &+ \frac{bs_0}{a^2+b^2-r^2})A_{0,0} + (r_N(r^2) + \frac{bt_0}{b^2+c^2-r^2})C_{0,0} \\ &+(q_N(r^2) + \frac{as_0}{a^2+b^2-r^2} - \frac{ct_0}{b^2+c^2-r^2})B_{0,0} + U_{N-1}(r^2) = 0.\end{aligned} \quad (24)$$

By passing to the limit (24) when $r \nearrow |a|$, we get that $p_N(r^2) = s_0 = 0$. Now if we use the change of variables $t^2 = b^2 - r^2$ in (24), we get $(c^2+t^2)r_N(b^2-t^2) + t_0b = 0$. This equality forces that $r_N(r^2) = 0$ and $t_0 = 0$. Finally, since $B_{0,0}$ is not a rational function, from (24) we obtain $q_N(r^2) = 0$ and $U_{N-1}(r^2) = 0$. □

Remark 11. *The number of elements of a basis for the generating functions of $f^0(r)$ when $a \le -1$, $b \le -1$ and $|c| \ge 1$, and when the squares of two of these parameters coincide, also can be obtained from Proposition 9 by using analogous arguments as in Proposition 10. More precisely, the number of independent functions of a basis is*

(i) $3N+4$ *if either $a = b \ne c$ or $b = c \ne a$;*
(ii) $3N+3$ *if $a = b = -c$;*
(iii) $2N+4$ *if $a = b = c$;*
(iv) $2N+2$ *if $c = -d$ and $b < a$.*

4. Upper bounds

To give an upper bound for the maximum number of limit cycles of equation (2) via the averaging theory of first order, we need to bound the number of real zeroes of the function $f^0(r)$.

A first approximation to this bound is given in the next Proposition 13 which is useful to prove Proposition 2. In a further step, see Proposition 16, we refine this upper bound when the parameters a, b, and c are pairwise different. In the next lemma we give an equation satisfied for the zeroes of the function $f^0(r)$ when the parameters a, b, and c are pairwise different.

Lemma 12. *We consider differential system* (2). *Assume that* $a \le -1$, $b \le -1$, $|c| \ge 1$, $|a| \ne |c|$, $|b| \ne |a|$ *and* $|b| \ne |c|$. *Then the zeroes of the function* $f^0(r)$ *satisfy equation*

$$\begin{aligned} &P^{N+2}(r^2)\sqrt{b^2-r^2}\sqrt{c^2-r^2} + Q^{N+2}(r^2)\sqrt{a^2-r^2}\sqrt{c^2-r^2} + \\ &R^{N+2}(r^2)\sqrt{a^2-r^2}\sqrt{b^2-r^2} + U^{N+1}(r^2)\sqrt{a^2-r^2}\sqrt{b^2-r^2}\sqrt{c^2-r^2} = 0, \end{aligned} \tag{25}$$

where P^{N+2}, Q^{N+2}, R^{N+2} *and* U^{N+1} *are real polynomials in the variable* r^2, *whose degree is the corresponding superindex and with arbitrary coefficients.*

Proof. From Proposition 9(i) we have that

$$\begin{aligned} f^0(r) = {} & P^N(r^2)\,A_{0,0} + Q^N(r^2)\,B_{0,0} + R^N(r^2)\,C_{0,0} + s_0\,AB_{0,0} \\ & + t_0\,BC_{0,0} + U^{N-1}(r^2), \end{aligned}$$

where the functions: P^N, Q^N, R^N and U^{N-1} are real polynomials in the variable r^2, whose degree is the corresponding superindex.

From Remark 6 and the expressions of $A_{0,0}$, $B_{0,0}$ and $C_{0,0}$ given in Lemma 5(i), for each $r \in (0,|a|)$, we obtain that the zeroes of the function $f^0(r)$ satisfy the equation

$$\begin{aligned} (a^2+b^2-r^2)(b^2+c^2-r^2)f^0(r) = {} & P^{N+2}(r^2)\,\frac{1}{\sqrt{a^2-r^2}} \\ & + Q^{N+2}(r^2)\,\frac{1}{\sqrt{b^2-r^2}} + R^{N+2}(r^2)\,\frac{1}{\sqrt{c^2-r^2}} + U^{N+1}(r^2), \end{aligned}$$

where the functions: P^N, Q^N, R^N and U^{N-1} are real polynomials in the variable r^2, whose degree is the corresponding superindex. Hence an upper bound for the number of zeroes of $f^0(r)$ is an upper bound for the number of solutions of the equation (25), as we would prove. □

Proposition 13. *Let* a, b *and* c *be real numbers such that* $a \le -1$, $b \le -1$, $|c| \ge 1$, *and* $|a| = \min(|a|,|c|)$. *An upper bound for the maximum number of real zeroes of* $f^0(r)$ *is given in Table 2.*

Proof. We prove the upper bounds given in Table 2 when $|a| \ne |c|$, $|b| \ne |a|$ and $|b| \ne |c|$. To prove the rest of the cases one can proceed in an analogous way. We additionally assume that $a > b$. If $a < b$, then by exchanging the roles of a and b the proof also follows. From Lemma 12 the zeroes of function $f^0(r)$ satisfy the equation (25). Hence an upper bound for the number of zeroes of $f^0(r)$ is an upper bound for the number of solutions of equation (25). Therefore the zeroes of $f^0(r)$ are among the zeroes of some polynomial of degree $8N+24$. This can be seen eliminating the squareroots of (25), after raising to the square two times the expression (25). □

As it will be seen in Proposition 16 our objective now is to improve the former upper bound, when the parameters satisfy that $|a| \ne |c|$, $|b| \ne |a|$ and $|b| \ne |c|$. This result will allow to prove Theorem 1. The main tool in the proof of Proposition 16

Cases	*Upper bound*
$a = b = -c$	$2N + 2$
$a = b = c$	$2N + 6$
$c = -b,\ b < a$ $c = -a,\ a < b$ $c = -a,\ b < a$ $c = b,\ b < a$	$4N + 6$
$a = b,\ c < -a$ $a = b,\ c < a$	$4N + 10$
$a = c,\ c < b$ $a = c,\ b < c$	$4N + 14$
$\|a\| \neq \|b\| \neq \|c\|$	$8N + 24$

TABLE 2. Upper bounds for the maximum number of real roots of $f^0(r)$. Here $N = [\frac{n-1}{2}]$.

is the next theorem of complex analysis, known as the Variation of the Argument Principle. We start by recalling some useful definitions and results. For more details see for instance [1].

A continuous function $\gamma : [0, 1] \to \mathbb{C} \setminus \{0\}$ is called a *path* in $\mathbb{C} \setminus \{0\}$. The *index* (or *winding number*) of a path γ in $\mathbb{C} \setminus \{0\}$ with respect to 0 is defined by

$$w(\gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z}.$$

In case that γ is a piecewise smooth path, then the index can be calculated as

$$w(\gamma, 0) = \frac{1}{2\pi i} \log \frac{\gamma(1)}{\gamma(0)} + \frac{1}{2\pi} \Delta\arg(\gamma), \tag{26}$$

where $\Delta\arg(\gamma)$ denotes the variation of the argument on the curve γ. In case that γ is a closed curve, then we have that

$$w(\gamma, 0) = \frac{1}{2\pi} \Delta\arg(\gamma).$$

Theorem 14. *Let G be a Jordan closed curve and we denote by D its interior. Let f be a holomorphic function in a neighborhood of $\bar{D}$ and such that it has no zeroes on G. We denote $N_0(f)$ the number of zeroes of f in D. Then*

$$N_0(f) = w(f(G), 0) = \frac{1}{2\pi i} \int_{f(G)} \frac{dz}{z} = \frac{1}{2\pi i} \int_G \frac{f'(z)}{f(z)}\, dz.$$

Proposition 15 ([6])**.** *Let γ and γ_1 be two paths in $\mathbb{C} \setminus \{0\}$ such that*

$$|\gamma(t) - \gamma_1(t)| \leq |\gamma_1(t)| \quad \text{for all } t \in [0, 1]. \tag{27}$$

Then connecting the points $P = \gamma(0)$ with $P_1 = \gamma_1(0)$, and $Q = \gamma(1)$ with $Q_1 = \gamma_1(1)$ by a segment having these endpoints, we obtain a closed curve that does not contain the origin inside. Moreover

$$w(\gamma, 0) = w(\gamma_1, 0) + w(PP_1, 0) - w(QQ_1, 0).$$

We obtain a new lower upper bound of the number of zeroes of $f^0(r)$ in a complex domain that includes the real interval where $f^0(r)$ defined. To do this we consider the complex extension of the function defined by the left-hand side of (25). By taking the following holomorphic branch of the complex square root function,

$$\sqrt{z} = \begin{cases} \sqrt{(|z| + \operatorname{Re} z)/2} + i\sqrt{(|z| - \operatorname{Re} z)/2}, & \text{if } \quad 0 \le \arg z < \pi, \\ \sqrt{(|z| + \operatorname{Re} z)/2} - i\sqrt{(|z| - \operatorname{Re} z)/2}, & \text{if } \quad -\pi \le \arg z < 0, \end{cases}$$

in the domain $\mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Im} z = 0 \text{ and } \operatorname{Re} z \le 0\}$, the complex extension of $f^0(r)$ given in (25) is

$$\begin{aligned} f(z) = \quad & P^{N+2}(z)\sqrt{b^2 - z}\sqrt{c^2 - z} + Q^{N+2}(z)\sqrt{a^2 - z}\sqrt{c^2 - z} + \\ & R^{N+2}(z)\sqrt{a^2 - z}\sqrt{b^2 - z} + U^{N+1}(z)\sqrt{a^2 - z}\sqrt{b^2 - z}\sqrt{c^2 - z}, \end{aligned} \tag{28}$$

which is holomorphic in the domain

$$\Omega = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Im} z = 0 \text{ and } \operatorname{Re} z \ge d\},$$

where $d = \min\{a^2, b^2, c^2\}$. Note that $|z| = r^2$.

Proposition 16. *Let $a < 0$, $b < 0$ and $c \neq 0$ be real numbers such that: $|a| \neq |b|$, $|a| \neq |c|$, $|b| \neq |c|$ and $|a| = \min(|a|, |c|)$. Let $f : \Omega \to \mathbb{C}$ be the function defined in (28) and $N_0(f)$ be the number of zeroes of f in Ω. Then $N_0(f) \le 5N + 13$.*

Proof. First we assume that $a > b$. If $a < b$, then by exchanging the roles of a and b the proof also follows. The zeroes of the function f are among the zeroes of some polynomial of degree $8N + 24$, as it was proved in Proposition 13. Hence, as $N_0(f)$ is finite, there exists a closed curve whose interior is included in Ω and contains all the zeroes of $f^0(r)$. The idea is to use Theorem 14 applied to this curve for obtaining a lower upper bound of $N_0(f)$.

From now on we will denote by ρ a positive real number large enough and ε a positive real number small enough. Let C_ρ be the circle centered at the origin and radius ρ and consider the points $A, A' \in C_\rho$ where $A = (x_A, \varepsilon)$, $A' = (x_A, -\varepsilon)$. Let $C_{\rho,\varepsilon}$ be the curve obtained by removing the arc AA' of the circle C_ρ, and let C_ε be the arc $\widehat{BB'}$ of the circle with center at $(a^2, 0)$ and radius ε, where $B = (a^2, \varepsilon)$, $B' = (a^2, -\varepsilon)$. The segment joining A and B (respectively A' and B') is denoted by L_+^ε (respectively L_-^ε) and also we introduce the piece of curve $I_{\rho,\varepsilon}$ given by $I_{\rho,\varepsilon} = L_+^\varepsilon \cup C_\varepsilon \cup L_-^\varepsilon$.

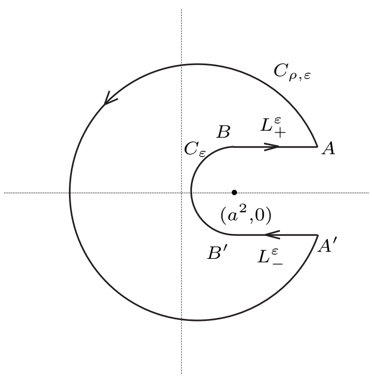

FIGURE 1. Graph of the closed curve G.

We define the closed curve

$$G = C_{\rho,\varepsilon} \cup I_{\rho,\varepsilon}$$

on the complex plane and denote by D its interior. Consider the counterclockwise orientation on G. See Figure 1.

Note that since $N_0(f)$ is finite, for ρ sufficiently large and ε sufficiently small, all the zeroes of f are in D. By applying Theorem 14 to the closed curve G we get

$$N_0(f) = \frac{1}{2\pi i}\int_{f(G)} \frac{dz}{z}$$

and by denoting

$$\begin{aligned} Z_1 &= \frac{1}{2\pi i}\int_{f(C_{\rho,\varepsilon})} \frac{dz}{z}, \\ Z_2 &= \frac{1}{2\pi i}\int_{f(I_{\rho,\varepsilon})} \frac{dz}{z}, \end{aligned}$$

we have that

$$N_0(f) = Z_1 + Z_2 = w(f(C_{\rho,\varepsilon}),0) + w(f(I_{\rho,\varepsilon}),0). \tag{29}$$

First we will estimate an upper bound for Z_1. From (28) we obtain

$$f(z) = \alpha_0 z^{N+3} + \alpha_1 z^{N+5/2} + \ell.o.t., \tag{30}$$

when $|z| \to +\infty$, with $\alpha_0, \alpha_1 \in \mathbb{C}$ and where $\ell.o.t.$ denotes lower order terms in z, as usual. Furthermore

$$|f(z) - \alpha_0 z^{N+3}| \le |\alpha_0 z^{N+3}| \quad \text{for} \quad z \in C_{\rho,\varepsilon}.$$

If we denote by $g(z) = \alpha_0 z^{N+3}$, then the hypotheses of Proposition 15 are fulfilled for the curves $\gamma = f(C_{\rho,\varepsilon})$ and $\gamma_1 = g(C_{\rho,\varepsilon})$. In that case $P = f(A)$, $Q = f(A')$, $P_1 = g(A)$ and $Q_1 = g(A')$, where $A = \sqrt{\rho^2 - \varepsilon^2} + \varepsilon i$ and $A' = \sqrt{\rho^2 - \varepsilon^2} - \varepsilon i$.

Note that $A' = \overline{A}$ and then $Q_1 = \overline{P_1}$ and $Q = \overline{P}$. Therefore $|P| = |Q|$, $|P_1| = |Q_1|$. Then, by applying Proposition 15, we get

$$w(f(C_{\rho,\varepsilon}),0) = w(g(C_{\rho,\varepsilon}),0) + w(PP_1,0) - w(QQ_1,0), \tag{31}$$

where

$$w(PP_1,0) = \frac{1}{2\pi i}\log\left|\frac{P_1}{P}\right| + \frac{1}{2\pi}\text{angle}(\widehat{POP_1}),$$

and

$$w(QQ_1,0) = \frac{1}{2\pi i}\log\left|\frac{Q_1}{Q}\right| + \frac{1}{2\pi}\text{angle}(\widehat{QOQ_1}).$$

So, since $Q_1 = \overline{P_1}$ and $Q = \overline{P}$, we have

$$\log\left|\frac{P_1}{P}\right| = \log\left|\frac{Q_1}{Q}\right|.$$

From straightforward computations we get that

$$\text{angle}(\widehat{POP_1}) = \arg\left(\frac{f(A)}{g(A)}\right) = \mathrm{O}(1/\rho),$$

and

$$\text{angle}(\widehat{QOQ_1}) = \arg\left(\frac{f(A')}{g(A')}\right) = \mathrm{O}(1/\rho),$$

where $O(1/\rho)$ denotes some function that goes to 0 when $\rho \to \infty$. Therefore

$$w(PP_1,0) - w(QQ_1,0) = \mathrm{O}(1/\rho). \tag{32}$$

For the estimation of the term $w(g(C_{\rho,\varepsilon}))$, a direct calculation shows that

$$w(g(C_{\rho,\varepsilon}),0) = \frac{1}{2\pi i}\int_{C_{\rho,\varepsilon}} \frac{g'(z)}{g(z)}dz = N + 3 + O(\varepsilon/\rho), \tag{33}$$

where $O(\varepsilon/\rho)$ denotes some function that goes to 0 when $\varepsilon \to 0$ and $\rho \to \infty$.

Substituting (32) and (33) in (31), we obtain that

$$Z_1 \leq N+3+O(\varepsilon/\rho)+O(1/\rho). \tag{34}$$

The next step is to give an upper bound for Z_2. We note that

$$Z_2=\frac{1}{2\pi i}\int_{f(I_{\rho,\varepsilon})}\frac{dz}{z}=\int_{I_{\rho,\varepsilon}}\frac{f'(z)}{f(z)}dz=w(f(I_{\rho,\varepsilon}),0).$$

By taking account that $I_{\rho,\varepsilon}=L_+^\varepsilon\cup C_\varepsilon\cup L_-^\varepsilon$, we get

$$Z_2=w(f(L_+^\varepsilon),0)+w(f(L_-^\varepsilon),0)+\frac{1}{2\pi i}\int_{C_\varepsilon}\frac{f'(z)}{f(z)}dz. \tag{35}$$

To obtain an estimation on the number of zeroes of Z_2 we introduce the functions $\gamma_\pm:[a^2,\infty)\to\mathbb{C}$ given by

$$\gamma_\pm(x)=\lim_{\varepsilon\to 0}f(x\pm i\varepsilon),\quad \text{for}\quad x\geq a^2.$$

Therefore $\gamma_-(x)=\overline{\gamma_+(x)}$, for all $x\geq a^2$, and the explicit expression for $\gamma_+(x)$ is given by

$$\begin{array}{l}
-i\sqrt{x-a^2}\sqrt{c^2-x}\,Q^{N+2}(x)-i\sqrt{x-a^2}\sqrt{b^2-x}\,R^{N+2}(x)\\
-i\sqrt{x-a^2}\sqrt{b^2-x}\sqrt{c^2-x}\,U^{N+1}(x)+\sqrt{b^2-x}\sqrt{c^2-x}\,P^{N+2}(x)\\
\text{for } a^2\leq x\leq b^2,\\
-i\sqrt{x-b^2}\sqrt{c^2-x}\,P^{N+2}(x)-i\sqrt{x-a^2}\sqrt{c^2-x}\,Q^{N+2}(x)\\
-\sqrt{x-a^2}\sqrt{x-b^2}\,R^{N+2}(x)-\sqrt{x-a^2}\sqrt{x-b^2}\sqrt{c^2-x}\,U^{N+1}(x)\\
\text{for } b^2<x\leq c^2,\\
+i\sqrt{x-a^2}\sqrt{x-b^2}\sqrt{x-c^2}\,U^{N+1}(x)-\sqrt{x-b^2}\sqrt{x-c^2}\,P^{N+2}(x)\\
-\sqrt{x-a^2}\sqrt{x-c^2}\,Q^{N+2}(x)-\sqrt{x-a^2}\sqrt{x-b^2}\sqrt{x-c^2}\,R^{N+2}(x)\\
\text{for } c^2<x.
\end{array} \tag{36}$$

Case 1. $\gamma_+(x^*)\neq 0$ for all $x^*\in[a^2,\infty)$.

We take the parametrization $f(x+i\varepsilon)$ (resp. $f(x-i\varepsilon)$), $x\in[a^2,\rho]$, for the curve $f(L_+^\varepsilon)$ (resp. $-f(L_-^\varepsilon)$). Since γ_+ is continuous on $[a^2,\rho]$ and $\gamma_+(x)\neq 0$ for all $x\in[a^2,\rho]$, there exists η such that $\inf_x|\gamma_+(x)|=\eta>0$, $x\in[a^2,\rho]$.

By direct calculations we get that

$$|f(x+i\varepsilon)-\gamma_+(x)|\leq O(\varepsilon),$$

for all $x\in[a^2,\rho]$. This property means that the convergence $\{f(x\pm i\varepsilon)\}\to\gamma_\pm(x)$ as $\varepsilon\to 0$, is uniform on ε in the interval $[a^2,\rho]$. Then, for the previous η, there exists ε_0 such that, for all $0<\varepsilon\leq\varepsilon_0$,

$$|f(x+i\varepsilon)-\gamma_+(x)|<\eta\leq|\gamma_+(x)|,\quad x\in[a^2,\rho].$$

By applying Proposition 15 to the curves $f(\cdot+i\varepsilon)$ and γ_+, we get

$$w(f(L_+^\varepsilon),0)=w(\gamma_+,0)+w(PP_1,0)-w(QQ_1,0), \tag{37}$$

where $P=f(a^2+i\varepsilon)$, $Q=f(\rho+i\varepsilon)$, $P_1=\gamma_+(a^2)$ and $Q_1=\gamma_+(\rho)$.

Applying formula (26) we get

$$w(PP_1,0)=\frac{1}{2\pi i}\log\left|\frac{\gamma_+(a^2)}{f(a^2+i\varepsilon)}\right|-\frac{1}{2\pi}\arg(f(a^2+i\varepsilon))$$

and

$$w(QQ_1,0)=\frac{1}{2\pi i}\log\left|\frac{\gamma_+(\rho)}{f(\rho+i\varepsilon)}\right|-\frac{1}{2\pi}\arg(f(\rho+i\varepsilon)).$$

From the series expansion in ε of the former functions we obtain that

$$w(PP_1, 0) - w(QQ_1, 0) = O(\varepsilon).$$

Then from (37) we get

$$w(f(L_+^\varepsilon), 0) = w(\gamma_+, 0) + O(\varepsilon). \tag{38}$$

Applying similar arguments to the curves $f(\cdot - i\varepsilon)$ and γ_- we obtain

$$w(f(L_-^\varepsilon), 0) = -w(\gamma_-, 0) + O(\varepsilon). \tag{39}$$

By writing $\gamma_+(x) = r(x)\exp(i\theta(x))$ then $\gamma_-(x) = r(x)\exp(-i\theta(x))$ and using again formula (26) we have

$$w(\gamma_+, 0) = \frac{1}{2\pi i}\log\left|\frac{r(\rho)}{r(a^2)}\right| + \frac{1}{2\pi}[\theta(\rho) - \theta(a^2)],$$

and

$$w(\gamma_-, 0) = \frac{1}{2\pi i}\log\left|\frac{r(\rho)}{r(a^2)}\right| - \frac{1}{2\pi}[\theta(\rho) - \theta(a^2)],$$

where $[\theta(\rho) - \theta(a^2)]$ is the variation of the argument on the curve $\gamma_+([a^2, \rho])$. Then, by (38) and (39), we get that

$$w(f(L_+^\varepsilon), 0) + w(f(L_-^\varepsilon), 0) = \frac{1}{\pi}[\theta(\rho) - \theta(a^2)] + O(\varepsilon).$$

Then we have that (35) can be written as

$$Z_2 = \frac{1}{\pi}[\theta(\rho) - \theta(a^2)] + \frac{1}{2\pi i}\int_{C_\varepsilon}\frac{f'(z)}{f(z)}dz + O(\varepsilon).$$

But, since

$$\frac{1}{2\pi i}\int_{C_\varepsilon}\frac{f'(z)}{f(z)}dz = \frac{1}{2\pi i}\int_{f(C_\varepsilon)}\frac{dz}{z} = w(f(C_\varepsilon), 0),$$

by formula (26), we get

$$\begin{aligned} w(f(C_\varepsilon), 0) &= \frac{1}{2\pi i}\log\left|\frac{f(1+\varepsilon i)}{f(1-\varepsilon i)}\right| \\ &\quad + \frac{1}{2\pi}(\arg(f(1+\varepsilon i)) - \arg(f(1-\varepsilon i))) = O(\varepsilon), \end{aligned}$$

and, hence

$$Z_2 = \frac{1}{\pi}[\theta(\rho) - \theta(a^2)] + O(\varepsilon),$$

where the difference $\theta(\rho) - \theta(a^2)$ is the variation of the argument on the curve $\gamma_+([a^2, \rho])$. We observe that the starting point of γ_+ is on the real axis. About its ending point we can say that either it is very close to the real axis or to the imaginary one. Hence, if we define

$$\begin{aligned} R &= \#\{z \in D : \mathrm{Re}(\gamma_+)(z) = 0\} \quad \text{and} \\ I &= \#\{z \in D : \mathrm{Im}(\gamma_+)(z) = 0\}, \end{aligned}$$

then we have that

$$|\theta(\rho) - \theta(a^2)| \leq \min\{R, I+1\}\,\pi + O(1/\rho).$$

As usual # denotes the cardinal of a set. As a consequence

$$Z_2 \ \leq \ 4N + 9 + O(\varepsilon) + O(1/\rho). \tag{40}$$

Finally, using the upper bounds obtained in (34) and (40), from (29) we get the estimation for $N_0(f)$ given by

$$N_0(f) \ \leq \ 5N + 12. \tag{41}$$

Case 2. Suppose there exists some $x^* \in [a^2, \infty)$ such that $\gamma_+(x^*) = 0$.

For each $x^* \in (a^2, \infty)$ such that $\gamma_+(x^*) = 0$ we shall define a function h^*.

(i) If $x^* \in (a^2, b^2) \cup (b^2, c^2) \cup (c^2, \infty)$, then $h^*(z) = (z - x^*)^{k_*}$, where $k_* \geq 1$ is the multiplicity of x^* as zero of γ_+ in $(a^2, b^2) \cup (b^2, c^2) \cup (c^2, \infty)$. Note that γ_+ is analytic in this domain of definition.

(ii) If $x^* = a^2$, then we take $h^*(z) = (\sqrt{a^2 - z})^{k_a}$, where

$$k_a = 2\ \min\{k_{a_1}, k_{a_2} + 1/2\}.$$

Here k_{a_1} is the multiplicity of a^2 as a zero of P^{N+2}, while k_{a_2} is the multiplicity of a^2 as a zero of the function

$$\sqrt{c^2 - x}\, Q^{N+2}(x) + \sqrt{b^2 - x}\, R^{N+2}(x) + \sqrt{c^2 - x}\sqrt{b^2 - x}\, U^{N+1}(x).$$

Since a^2 is a zero of γ_+, we have that $k_{a_1} \geq 1$.

(iii) If $x^* = b^2$, then we take $h^*(z) = (\sqrt{b^2 - z})^{k_b}$, where

$$k_b = 2\ \min\{k_{b_1} + 1/2, k_{b_2}, k_{b_3} + 1/2\}.$$

In this case k_{b_1} is the multiplicity of b^2 as a zero of P^{N+2}, k_{b_2} is the multiplicity of b^2 as a zero of Q^{N+2}, while k_{b_3} is the multiplicity of b^2 as a zero of the function $R^{N+2}(x) + \sqrt{c^2 - x}\, U^{N+1}(x)$. Since b^2 is a zero of γ_+, we obtain that $k_{b_2} \geq 1$.

(iv) If $x^* = c^2$, then we take $h^*(z) = (\sqrt{c^2 - z})^{k_c}$, where

$$k_c = 2\min\{k_{c_1}, k_{c_2} + 1/2, k_{c_3} + 1/2\}.$$

Here k_{c_1} is the multiplicity of c^2 as a zero of R^{N+2}, k_{c_2} is the multiplicity of c^2 as a zero of U^{N+1}, while k_{c_3} is the multiplicity of c^2 as a zero of the function $\sqrt{x - a^2}\, Q^{N+2}(x) + \sqrt{x - b^2}\, P^{N+2}(x)$. Since c^2 is a zero of γ_+, we get that $k_{c_1} \geq 1$.

It is clear that γ_+ has finitely many zeroes in $[a^2, \infty)$. So we choose ρ sufficiently large in order that all the zeroes of γ_+ in $[a^2, \infty)$ are contained in $[a^2, \rho]$. We define the function h as the product of all the functions h^* defined before, for each $x^* \in [a^2, \infty)$ zero of γ_+, and the function f_1 as

$$f_1(z) = \frac{f(z)}{h(z)}.$$

Both h and f_1 are holomorphic in Ω and the number of zeroes of f in D is equal to the number of zeroes of f_1 in D, i.e.

$$N_0(f) = N_0(f_1) = \frac{1}{2\pi i} \int_{f_1(G)} \frac{dz}{z}.$$

If we denote by

$$Y_1 = \frac{1}{2\pi i} \int_{f_1(C_{\rho,\varepsilon})} \frac{dz}{z} \quad \text{and} \quad Y_2 = \frac{1}{2\pi i} \int_{f_1(I_{\rho,\varepsilon})} \frac{dz}{z},$$

then we have that

$$N_0(f_1) = Y_1 + Y_2 = w(f_1(C_{\rho,\varepsilon}), 0) + w(f_1(I_{\rho,\varepsilon})). \tag{42}$$

Since $f(z) = h(z) f_1(z)$ we have that Z_1 can be written as

$$Z_1 = Y_1 + \sum_{x^*} \frac{1}{2\pi i} \int_{C_{\rho,\varepsilon}} \frac{(h^*)'(z)}{h^*(z)} dz. \tag{43}$$

Moreover the expression $(h^*)'/h^*$ is either $k_*/(z-x^*)$ or $k_d/(2(d^2-z))$, where $d \in \{a,b,c\}$. We note that $(h^*)'/h^*$ is continuous on C_ρ. Then we can write

$$\frac{1}{2\pi i}\int_{C_{\rho,\varepsilon}} \frac{(h^*)'(z)}{h^*(z)}dz = \frac{1}{2\pi i}\int_{C_\rho} \frac{(h^*)'(z)}{h^*(z)}dz \; + \; O(\varepsilon) = \tilde{k} + \; O(\varepsilon),$$

where $\tilde{k} \in \{k_*, k_a/2, k_b/2, k_c/2\}$.

Denote by k the sum with respect to all zeroes of γ_+ for all positive numbers of the form: $k_*, k_a/2, k_b/2, k_c/2$.

From (34) and (43), we get that

$$Y_1 \;\leq\; N + 3 - k + O(\varepsilon) + O(1/\rho). \tag{44}$$

In order to give an estimation for Y_2, we define $h_+(x) = \lim_{\varepsilon\to 0} h(x+i\varepsilon)$, $h_-(x) = \lim_{\varepsilon\to 0} h(x-i\varepsilon)$, $\beta_+(x) = \lim_{\varepsilon\to 0} f_1(x+i\varepsilon)$ and $\beta_-(x) = \lim_{\varepsilon\to 0} f_1(x-i\varepsilon)$ for all $x \geq a^2$. It is not difficult to check that $h_-(x) = \overline{h_+(x)}$ and, as a consequence, $\beta_-(x) = \overline{\beta_+(x)}$.

Since $N_0(f) = N_0(f_1)$, we shall give an estimation for Y_2 considering an analogous function to the function γ_+ associated to Z_2 would be the function β_+, but now associated to Y_2. We note that, as in Case 1, the function β_+ has no zeroes on $[a^2, \infty)$.

We note that by construction of the function h, all zeroes of γ_+ in $[a^2,\infty)$ are zeroes of h_+ with the same multiplicity. Consequently, the function β_+ has no zeroes and is continuous in $[a^2,\infty)$.

Note that the upper bound for the maximum number of zeroes of β_+ is optimal when the function h is composed by only one factor whose zero is $z = d$, $d \in \{a^2, b^2, c^2\}$. In that case the degree of β_+ decreases only in one half correspondind to square root, while in the other cases the degree of β_+ decreases in a natural number n, $n \geq 1$. Hence it is enough to study only the cases

(c1) $x^* = a^2$, with $h(z) = \sqrt{a^2 - z}$,
(c2) $x^* = b^2$, with $h(z) = \sqrt{b^2 - z}$,
(c3) $x^* = c^2$, with $h(z) = \sqrt{c^2 - z}$.

For each case we can consider the associated function h_+ to obtain the corresponding $\beta_+ = \gamma_+/h_+$ function.

By repeating the same arguments as the ones used to get the upper bound for Z_2, we obtain

$$Y_2 = \frac{1}{\pi}[\theta(\rho) - \theta(a^2)] + O(\varepsilon),$$

where $[\theta(\rho) - \theta(a^2)]$ is the variation of the argument on the curve $\beta_+([a^2,\rho])$. We observe that the starting point of β_+ is on the real axis. About its ending point we can say that either it is very close to the real axis, or to the imaginary one. Hence if we define $R = \#\{z \in D : \mathrm{Re}(\beta_+)(z) = 0\}$ and $I = \#\{z \in D : \mathrm{Im}(\beta_+)(z) = 0\}$, then we have that

$$[\theta(\rho) - \theta(a^2)] \leq \min\{R, I+1\}\,\pi + O(1/\rho).$$

By doing a particular study for the considered cases, we get an upper bound for the number of zeroes of Y_2. That is,

$$Y_2 \leq \begin{cases} (4N+10) + O(\varepsilon) + O(1/\rho), \text{ for (c1)},\\ (4N+8) + O(\varepsilon) + O(1/\rho), \text{ for (c2)},\\ (4N+7) + O(\varepsilon) + O(1/\rho), \text{ for (c3)}. \end{cases}$$

Then, by taking the greatest of these bounds and from the estimation of Y_1 given in (44), we obtain that the upper bound for the maximum number of zeroes of the function f in (c2) is

$$N_0(f) = N_0(f_1) = Y_1 + Y_2 \ \leq \ 5N + 13. \tag{45}$$

Finally from expressions (41) and (45), the upper bound for $N_0(f)$ is given by $N_0(f) \leq 5N + 13$. □

5. Proof of the main results

We consider differential system (2), where a, b, $c \in \mathbb{R}\backslash\{0\}$ are such that $|a| \neq |b|$, $|a| \neq |c|$ and $|b| \neq |c|$ and $P_n(x,y)$, $Q_n(x,y) \in \mathbb{R}_n[x,y]$. First we do some restrictions on the set of parameters a, b and c that, without loss of generality, will allow to simplify the proof.

We can assume that $b < 0$. If it is not the case we obtain it doing the change of variables $(x,y,t) \mapsto (x,-y,t)$. We assume that the parameter a is such that $|a| = \min(|a|,|c|)$. Furthermore we can take $a < 0$, if not then we apply the change of variables $(x,y,t) \mapsto (-x,y,-t)$. In the proof of the theorem we will use next lemma.

Lemma 17 ([10])**.** *Consider $p+1$ linearly independent analytical functions $f_i : U \subset \mathbb{R} \to \mathbb{R}$, $i = 0,1,\ldots,p$, where $U \subset \mathbb{R}$ is an interval. If there exists $j \in \{0,1,\ldots,p\}$ such that $f_j|_U$ has constant sign, then it is possible to get an f given by $f(x) := \sum_{i=0}^{p} C_i f_i(x)$, such that it has at least p simple zeroes in U.*

Proof of Theorem 1. To obtain bounds of the maximum number of limit cycles of differential system (2), applying the averaging theory of first order, we need to control the number of simple zeroes of the function $f^0(r)$.

To get a lower bound, we need to prove that there exists a basis of the generating functions of $f^0(r)$ having at least $4[(n-1)/2]+4$ simple zeroes for $|\varepsilon|$ small enough. From Proposition 5(i), $f^0(r)$ is an arbitrary linear combination of $4[(n-1)/2]+5$ independent functions. As all these functions are analytic in $U = (0,d)$, where $d = \min\{|a|,|b|\}$, and some of them are strictly positive on U, the result follows as a consequence of Lemma 17. Hence there are systems (2) with at least $4[(n-1)/2]+4$ limit cycles bifurcating from the period annulus around the origin for $\varepsilon = 0$.

Now we shall get an upper bound. Clearly $f^0(0) = 0$ and hence, from Proposition 9(i) and Lemma 12, the following relation must hold,

$$P^{N+2}(0)|b|\,|c| + Q^{N+2}(0)|a|\,|c| + R^{N+2}(0)|a|\,|b| + U^{N+1}(0)|a|\,|b|\,|c| = 0.$$

Therefore from expression (28), also we have that $f(0) = 0$. By applying Proposition 16, we deduce that an upper bound for the number of zeroes of $f^0(r)$ in $(0,d)$ is $5N + 14$. □

Proof of Proposition 2. From the considerations on the set of parameters a, b and c at the beginning of this section, it is not restrictive to assume that $a < 0$, $b < 0$ and that $|a| = \min(|a|,|c|)$.

The lower bounds of Table 1 follow from the study of the number of generating functions of $f^0(r)$ in the same way as we proved Theorem 1, and using from Lemma 10 and Remark 11. The upper bounds again can be obtained as in Theorem 1 using Proposition 13. □

We point out that the lower bounds of Proposition 13 are smaller than the lower bound given in Theorem 1. See Remark 11.

6. Numerical Computations

In order to show how accurate are the bounds provided, we present some numerical computations, performed with the algebraic manipulator *Mathematica*.

We consider system (2), with $a = -1$ and being $b < -1$, $c > 1$. In this case the period annulus of system (2) for $\varepsilon = 0$ is given by the the disc $r < 1$. For each $n = 4, \ldots, 9$ we obtain the function $f^0(r)$, that we call $f_n^0(r)$, associated to (2):

$$f_n^0(r) = \sum_{i=1}^{7} p_n^i(r^2) f_i(r), \tag{46}$$

where for $i = 1, \ldots, 7$, $p_n^i(r^2)$ depends on b and c and it is a real polynomial in the variable r^2, whose coefficients are linear combinations of the coefficients of $P_n(x, y)$ and $Q_n(x, y)$. To simplify notation we rename these coefficients as c_i for $i = 1, \ldots, (n^2 + 5n)/2$ and we denote by $c = (c_1, \ldots, c_k)$, where $k = (n^2 + 5n)/2$. The functions $f_i(r)$ are defined as follows,

$$\begin{aligned}
f_1(r) &= \frac{1}{\sqrt{b^2 - r^2}(1 + c)r(b^2 + c^2 - r^2)},\\
f_2(r) &= \frac{\sqrt{b^2 - r^2}}{(1 + c)r(b^2 + c^2 - r^2)},\\
f_3(r) &= \frac{1}{\sqrt{c^2 - r^2}r(1 + c^2 - r^2)(b^2 + c^2 - r^2)},\\
f_4(r) &= \frac{\sqrt{c^2 - r^2}}{r(1 + c^2 - r^2)(b^2 + c^2 - r^2)},\\
f_5(r) &= \frac{1}{\sqrt{b^2 - r^2}(1 + c)r(1 + c^2 - r^2)},\\
f_6(r) &= \frac{\sqrt{b^2 - r^2}}{(1 + c)r(1 + c^2 - r^2)},\\
f_7(r) &= \frac{1}{r}.
\end{aligned}$$

For each value of n we define $N(n)$ as the total number of monomials of all the polynomials $p_n^i(r^2)$. In this way we have that

$$f_n^0(r) = \sum_{i=1}^{N(n)} B_i(c_1, c_2, \ldots, c_k)\varphi_i(r), \tag{47}$$

where $k = (n^2 + 5n)/2$, $B_i(c_1, c_2, \ldots, c_k)$ is the coefficient of the corresponding monomial and $\varphi_i(r)$ stands for the product of some power of r^2 times $f_j(r^2)$ for some $j = 1, \ldots, 7$. We remark that $B_i(c_1, c_2, \ldots, c_k)$ is a linear combination of the c_i.

From Proposition 18(ii) we get the minimum number for the maximum number of limit cycles of system (2) obtained using the averaging theory of first order. This number is denoted by $\mathrm{z}(n)$. We note that $\mathrm{z}(n)$ is equal to the number of linearly independent generating functions of $f_n^0(r)$ in expression (47) minus one. From this proposition also we can deduce that, in any case, $z(n) \le N(n) - 1$.

Proposition 18 ([16])**.** *Let $f(x)$ be a function of the form*

$$f(x) = \sum_{i=1}^{m} B_i(c_1, c_2, \ldots, c_k)\varphi_i(x),$$

where $\varphi_1, \ldots, \varphi_m$ are linearly independent solutions of a linear homogeneous ordinary differential equation of finite order and defined on an open interval J, and

$B = (B_1, \ldots, B_m)$ *is a linear map with* $\mathrm{rank}_c(B) = z(n)+1$, *where* $z(n)$ *is a positive integer number. Then,*

(i) For almost any choice of $z(n)+1$ *points in* J, *say* $x_0, x_1, \ldots, x_{z(n)}$, *and for arbitrary real numbers* $y_0, \ldots, y_{z(n)}$, *the linear system*

$$f(x_i) = y_i, \ i = 0, \ldots, z(n),$$

has exactly one solution.

(ii) There are functions $f(x)$ *not identically zero and having at least* $z(n)$ *zeros in* J. *Moreover if the function* $f(x)$ *is analytic we can take these* $z(n)$ *zeros having odd multiplicity.*

In Table 3 for each value of n we compare: $N(n) - 1$ which is the maximum number of zeroes of the function $f^0(r)$, $z(n)$ that gives the minimum number of limit cycles of system (2) obtained by using Proposition 18(ii) and $4[(n-1)/2]+4$, i.e. the lower bound that we have obtained in Theorem 1 for the number of zeroes of the function $f^0(r)$.

n	4	5	6	7	8	9
$N(n)-1$	18	19	22	23	26	27
$z(n)$	11	13	17	19	20	23
$4[(n-1)/2]+4$	8	12	12	16	16	20

Table 3. Value of $z(n)$ in comparison with the lower bound in Theorem 1.

Acknowledgements. The second author is partially supported by the grants MCI/FEDER number MTM2008-06098-C02-1 and by the CICYT grant number 2009SGR-410. The first and third authors are partially supported by the grant TIN2008-04752/TIN.

References

1. L.A. Ahlfors, *Complex analysis*, 2nd ed. McGraw'Hill Publishing Comp., Ltd. 1966.
2. V.I. Arnold and Y.S. Ilyashenko, *Dynamical systems I. Ordinary differential equations*, in Encyclopaedia Math. Sci., vol. **1**, Springer, Berlin, 1988.
3. A. Atabaigi, N. Nyamoradi and H.R.Z. Zangeneh, *The number of limit cycles of a quintic polynomial system*, Comput. Math. Appl. **57** (2009), 677–684.
4. A. Atabaigi, N. Nyamoradi and H.R.Z. Zangeneh, *The number of limit cycles of a quintic polynomial system with a center*, Nonlinear Anal., Theory Methods Appl. **71** (2009), 3008–3017.
5. A. Buică and J. Llibre, *Averaging methods for finding periodic orbits via Brouwer degree*, Bull. Sci. Math. **128** (2004), 7–22.
6. A. Buică and J. Llibre, *Limit cycles of a perturbed cubic polynomial differential center*, Chaos, Solitons and Fractals **32** (2007), 1059–1069.
7. T.R. Blows and L.M. Perko, *Bifurcation of limit cycles from centers and separatrix cycles of planar analytic systems*, SIAM Rev. **36** (1994), 341–376.
8. C. Chicone and M. Jacobs, *Bifurcation of limit cycles from quadratic isochrones*, J. Differential Equations **91** (1991), 268–326.
9. S. Chow, C. Li and D. Wang, *Normal forms and bifurcation of planar vector fields*, Cambridge University Press, Cambridge, 1994.
10. B. Coll, A. Gasull and R. Prohens, *Bifurcation of limit cycles from two families of centers*, Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. **12** (2005), 275–287.
11. J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*, in Applied Mathematical Sciences, vol. **42**, Springer-Verlag, New York, 1986.
12. H. Giacomini, J. Llibre and M. Viano, *On the nonexistence, existence and uniqueness of limit cycles*, Nonlinearity 9 (1996), 501–516.
13. H. Giacomini, J. Llibre and M. Viano, *On the shape of limit cycles that bifurcate from Hamiltonian centers*, Nonlinear Anal. 41 (2000), 523–537.

14. H. Giacomini, J. Llibre and M. Viano, *On the shape of limit cycles that bifurcate from non-Hamiltonian centers*, Nonlinear Anal. 43 (2001) 837–859.
15. A. Gasull, W. Li, J. Llibre and Z. Zhang, *Chebyshev property of complete elliptic integrals and its application to Abelian integrals*, Pacific J. Math. **202** (2002), 341–361.
16. A. Gasull, R. Prohens and J. Torregrosa, *Bifurcation of limit cycles from a polynomial non-global center*, J. Dyn. Differ. Equations **20** (2008), 945–960.
17. J. Giné and J. Llibre, *Limit cycles of cubic polynomial vector fields via the averaging theory*, Nonlinear Anal. **66** (2007), 1707–1721.
18. C. Li, J. Llibre and Z. Zhang, *Weak focus, limit cycles and bifurcations for bounded quadratic systems*, J. Differential Equations **115** (1995), 193-223.
19. J. Llibre, J.S. Pérez del Río and J.A. Rodríguez, *Averaging analysis of a perturbated quadratic center*, Nonlinear Anal. **46** (2001), 45-51.
20. J. Llibre and J. Yu, *On the upper bound for the number of limit cycles obtained by the second order averaging method*, Dynamics of Continuous, Discrete and Impulsive Systems B **14** (2007), 841–873.
21. L.S. Pontrjagin, *Über autoschwingungssysteme, die den hamiltonshen nahe liegen*, Physikalische Zeitschrift der Sowjetunion **6** (1934), 25-28.
22. J.A. Sanders and F. Verhulst, *Averaging methods in nonlinear dynamical systems*, Applied Mathematical Sci., vol. **59**, Springer-Verlag, New York, 1985.
23. F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*, Universitext, Springer-Verlag, Berlin, 1996.
24. M. Viano, J. Llibre and H. Giacomini, *Arbitrary order bifurcations for perturbed Hamiltonian planar systems via the reciprocal of an integrating factor*, Nonlinear Anal. 48 (2002) 117–136.
25. G. Xiang and M. Han, *Global bifurcation of limit cycles in a family of polynomial systems*, J. Math. Anal. Appl. **295** (2004), 633–644.
26. G. Xiang and M. Han, *Global bifurcation of limit cycles in a family of multiparameter system*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **14** (2004), 3325–3335.

[2]Dept. de Matemàtiques, Universitat Autònoma de Barcelona, Edifici Cc 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: `jllibre@mat.uab.cat`

[1]Dept. de Matemàtiques i Informàtica, Universitat de les Illes Balears, Facultat de ciències, 07071, Palma de Mallorca, Spain

E-mail address: `dmitcv0@ps.uib.cat, dmirps3@ps.uib.cat`