This is a preprint of: "Comet and Hill-type periodic orbits in restricted (N+1)-body problems", Jaume Llibre, Cristina Stoica, *J. Differential Equations*, vol. 250, 1747–1766, 2011.
DOI: [10.1016/j.jde.2010.08.005]

Comet- and Hill-type periodic orbits in restricted $(N+1)$-body problems

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Abstract

We consider the planar restricted $(N+1)$-body problem where the interaction potential between the particle and the primaries is taken to be a finite sum of terms of the form $(\text{distance})^{-\alpha}$ with $\alpha > 0$.

The primaries are assumed to be in a relative equilibrium, that is, they form a uniformly rotating rigid configuration.

We show two results. First, if the infinitesimal particle is far from the primaries and the long range dominant term γ/r^α of the potential is such that $\gamma < 0$ and $\alpha \neq 2$, then there exist two one-parameter families of large nearly circular periodic solutions. These solutions, called *comet solutions*, are elliptic and KAM stable for $\alpha < 2$, and unstable for $\alpha > 2$. Second, if the infinitesimal particle is close to one of the primaries and the short range dominant term γ/r^α of the potential near that primary is such that $\gamma < 0$ and $\alpha \neq 2$, then there exist two one-parameter families of nearly circular periodic solutions, called *Hill solutions*, that encircle the nearby primary. For $\alpha > 2$, these orbits are unstable.

The methodology applied involves appropriate symplectic scalings, Poincaré's continuation method and averaging theory. The KAM stability of the comet periodic orbits is decided by verifying Arnol'd's non-degeneracy conditions.

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1 Introduction

Motivated by its applicability in celestial mechanics, astronomy and astrodynamics, the dynamics of a small object (satellite) in the gravitational field of two or more mass points has been subject of numerous research papers (see [27,20,17] and references therein). Under the assumption that the evolution of the attractive mass points is not disturbed by the presence of the small mass and it is *a priori* known, the mathematical description of the planar problem is given by a non-autonomous Hamiltonian system with two degrees of freedom (see [21]).

Probably the most standard case studied is the restricted three-body problem, that is the motion of a particle of negligible mass under the Newtonian attraction of two mass points considered in a uniformly rotating rigid configuration ([27]). The extra assumption made over the motion of the mass points, which are called the *primaries*, simplifies the dynamics to the autonomous case. The analysis of this problem is non-trivial and has lead to many ideas that now lay at the foundations of modern dynamics (see [24,3]).

Looking for the objects which organize the dynamics of these systems, the periodic orbits play an important role (see [21,11,31]).

In the systems depending on a small parameter, the periodic orbits are usually determined using the Poincaré's continuation method: a periodic orbit of the unperturbed system persists in the perturbed dynamics if its characteristic multipliers are not unity (see [21]). This method is based on the finite-dimensional version of the Implicit Function Theorem, and its origin is in the work of Poincaré (see [24]) for studying the existence of periodic solutions in the restricted three-body problem.

For the differential systems with continuous symmetries the simplest class of periodic orbits are the relative equilibria, i.e. periodic solutions of the equations of motion that are also orbits of a one-dimensional symmetry subgroup.

For symmetric systems depending on a small parameter the relative equilibria may be used as the unperturbed periodic orbit that can be continued into the perturbed system.

A notable class of applications amenable to Poincaré's continuation method is given by the problems associated to the N-body problem of celestial mechanics (see [20]). In the context of the planar restricted three-body problem, two extreme situations appear: the case when the negligible mass is far away from all the primaries and in particular from its centre of mass ("comet periodic orbits"), and the case when the negligible mass is close to one of the primaries ("Hill periodic orbits"). The first case is thoroughly investigated by Meyer in [19] who demonstrates the existence of two families of periodic orbits of elliptic type (i.e. orbits with non-trivial multipliers of unit modulus and not equal to ± 1) which are close to circles of very large radii. Moreover, by using the Delaunay coordinates and verifying the KAM "twist condition", the same author proves that these orbits are stable. The second case, which is the first approximation of the lunar problem (see [12]), is analyzed by Szebehely in

[27] and Meyer in [21], wherein the existence of two families of elliptic periodic orbits that encircle a primary for all values of the mass ratio parameter is established.

In this paper we extend these previous works related to the comet and Hill solutions. First, we consider a wider class of restricted $(N+1)$-body problems, i.e. the motions of an infinitesimal particle in the potential fields of N point sources, which we call primaries. More precisely, the interaction potential between the particle and the primaries is taken to be a finite sum of terms of the form $(\text{distance})^{-\alpha}$ with $\alpha > 0$. This class of potentials includes a wide range of applications: post-Newtonian potentials such as Manev (see [6,15]) and Schwarzschild potentials (see [26,30]), truncations of the expansion of the gravitational potential of a spheroidal body (see [28]), problems with charges and classical approximations of molecular interactions (see [4]). Second, the primaries are considered in a relative equilibrium, i.e. a rigid configuration in a uniformly rotating system, of course assuming that the internal potential allows such a configuration to exist. Note that the internal potential binding the N point sources does not have to coincide with the potential exerted on the infinitesimal particle.

We prove two main results:

1. If the infinitesimal particle is far from the primaries, and the long range dominant term γ/r^α of the potential is such that $\gamma < 0$ and $\alpha \neq 2$, then there exist two one-parameter families of large nearly circular periodic solutions. These solutions are elliptic KAM stable for $0 < \alpha < 2$, and unstable for $\alpha > 2$. See Theorem 3.6.
2. If the infinitesimal particle is close to one of the primaries and the short range dominant term γ/r^α of the potential near that primary is such that $\gamma < 0$ and $0 < \alpha \neq 2$, then there exist two one-parameter families of nearly circular periodic solutions that encircle the nearby primary. For $\alpha > 2$, these orbits are unstable. See Theorem 4.4.

Each result is then applied to various physically relevant problems.

As noted in the abstract, the methodology we use involves appropriate symplectic scalings, Poincaré's continuation method and averaging theory. To settle the stability of the comet periodic orbits we verify Arnol'd's non-degeneracy conditions (see [3]). We employ the action-angle variables for the central force problem, but since these coordinates are explicitly computable only in exceptional cases (as for instance in the Newtonian case, leading to the Delaunay coordinates), we work implicitly. We thus avoid the verification of the twist condition as accomplished by Meyer in [19].

The paper is organized as follows: in Section 2 we briefly review the central force problem and introduce relative equilibria. We also define the class of quasi-homogeneous potentials and describe the generalized restricted $(N+1)$-body problems considered here. Section 3 concerns with periodic orbits at infinity. We prove the main theorem in two ways, first by applying Poincaré's continuation method and after by using the averaging theory. We further prove the KAM stability of the periodic orbits in the elliptic case. We conclude by stating a series of corollaries describing direct applications to different physical problems. In Section 4 we analyze the existence of Hill orbits and deduce analogous results to those in the previous section. There we use the Implicit Function Theorem and Poincaré's method. The Appendix contains the statement of the averaging theorem used in Section 3.

2 Preliminaries

2.1 Dynamics in a central field

Consider the planar motion of a particle in a central field described by a differentiable potential $V : (0, \infty) \to \mathbb{R}$, $V = V(r)$ where as usual r denotes the distance from the particle to the source.

It is well known that in polar coordinates $(r, \theta, p_r, p_\theta)$ the motion is determined by the Hamiltonian

$$H(r, \theta, p_r, p_\theta) = \frac{1}{2}p_r^2 + \frac{p_\theta^2}{2r^2} + V(r).$$

Moreover due to the rotational symmetry, the planar dynamics of a particle in a central field has two independent first integrals: the *total energy*

$$h = \frac{1}{2}p_r^2 + \frac{p_\theta^2}{2r^2} + V(r) = \text{const.},$$

and the *angular momentum* $c = p_\theta =$ const. Thus is an integrable problem (see [2]). Associated to the rotational symmetry there is the reduction of the problem to a system with one degree of freedom described by the Hamiltonian (see [2], [10]):

$$H_{red}(r, p_r) = \frac{1}{2}p_r^2 + \frac{c^2}{2r^2} + V(r).$$

The centrifugal term $c^2/2r^2$ added to the potential gives rise to the so-called *augmented potential* $V_{\text{aug}}(r) := (c^2/2r^2) + V(r)$ and, for non-zero values of c the dynamics is organized around the critical points $r_0(c)$ of $V_{\text{aug}}(r)$. The solution $(r_0(c), 0)$ of the reduced dynamics lifts in the phase space to the *relative equilibrium* solution

$$r(t) = r_0(c), \quad p_r(t) = 0, \quad \theta(t) = \left(\frac{c}{r_0^2(c)}\right) t + \theta(0), \quad p_\theta(t) = c.$$

Physically relative equilibria are states where the centrifugal term balances the attraction of the source and appear as circular orbits of uniform angular velocity.

2.2 Relative equilibria

For general systems with continuous symmetries the relative equilibria are solutions that are also orbits of an one-dimensional symmetry subgroup and, similarly the equilibria of the non-symmetric systems may be used as organizational centres of the dynamics (see [16]).

As in the central force problems in the case of planar rotationally symmetric systems, the relative equilibria are retrieved as critical points of the function obtained augmenting the potential with the centrifugal term. Physically they are states where the binding forces are balanced by the centrifugal term.

For planar rotationally symmetric N mass points systems the relative equilibria are solutions where the bodies are encircling with uniform angular velocity their centre of mass while the shape of the configuration formed by the bodies does not change. For example, a relative equilibrium for the two body problem is a solution where the two bodies are moving with uniform angular velocity on circles around their centre of mass; in a frame joint to the rotating system, the bodies appear in a rigid configuration (in this case a segment of fixed length).

2.3 Quasi-homogeneous potentials

In this paper we consider interactions modeled by *quasi-homogeneous* potentials, i.e. functions $V : (0, \infty) \to \mathbb{R}$ that are finite sums of homogeneous terms in $1/r$. Formally $V(r)$ is a quasi-homogeneous potential if it is of the form

$$V(r) = \frac{a_1}{r^{\alpha_1}} + \frac{a_2}{r^{\alpha_2}} + \ldots + \frac{a_n}{r^{\alpha_n}},$$

where $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n$, and $a_k \in \mathbb{R}$, $k = 1, 2 \ldots, n$ with $a_n \neq 0$. The short-range (r small) dynamics may be attractive or repulsive if the sign of the leading term a_n is negative or positive, respectively. Similarly the long-range (r large) dynamics may be attractive or repulsive if the sign of a_1 is negative or positive, respectively.

This class of potentials includes the usual Newtonian interaction with $V(r) = -1/r$, Newtonian interactions amended by post-Newtonian corrections, such as the Manev $V(r) = -a/r + b/r^2$, $a > 0$, $b > 0$ (see [6,15]) and Schwarzschild potentials $V(r) = -a/r + b/r^3$, $a > 0$, $b > 0$ (see [26,30]). It also includes truncations of the expansion of the gravitational potential of a spheroidal body taken in the equatorial plane of the body. For example the first approximation of the gravitational potential in the equatorial plane of an oblate planet is

$$V(r) = -\frac{k^2 M}{r} - \left(\frac{J_2 R^2 k^2 M}{2}\right) \frac{1}{r^3},$$

where k is the gravitational constant, M is the mass of the body, R is the radius of the equator of the planet and J_2 is a dimensionless constant related to the lengths of the spheroid's axes (see [28]).

Furthermore this class comprises classical approximations of molecular interactions, such as the "$6-12$" $V(r)=-a/r^6+b/r^{12}$ (with $a>0$, $b>0$) Lennard-Jones potential, and more general molecular attractive-repulsive potentials of the form

$$V(r)=-\frac{a}{r^\alpha}+\frac{b}{r^\beta}\,, \tag{1}$$

where $a>0$, $b>0$ and $2<\alpha<\beta$ (see [13,22]).

A particle motion with quasi-homogeneous interaction displays interesting dynamical features which are subject of numerous papers. In [7] and [23], the authors investigate motion near total collision for the collinear three-body problem where potential is taken a sum of two attractive homogenous terms. Results related with the existence of a class of special solutions for the N-body problems with attractive homogenous and molecular-type potentials of the form (1) are presented in [8,18,13]. In the same physical context the escape mechanism in collinear three point mass systems was recently analyzed in [22]. In [5] the authors study the equilibrium points and planar relative equilibria for the Lennard-Jones 2- and 3-body problems.

2.4 Restricted $(N+1)$-body problems

Consider the motion of an infinitesimal mass P in the potential field of a system formed by N punctual masses, called *primaries*, of masses $m_1, m_2, \dots m_N$. Denote by $(x,y)\in\mathbb{R}^2\times\mathbb{R}^2$, the position and momenta, respectively, of P with respect to the centre of mass of the N-body system, which is considered at the origin. While the primaries determine the dynamics of the infinitesimal P, their motion is not affected by the presence of P.

Let $q_j(t)$ be the position of the primary m_j at time t. Then the distance between P and the point mass m_j is $\|x-q_j(t)\|$, and the dynamics of P is determined by the two-degree of freedom time-dependent Hamiltonian system with Hamiltonian

$$H=\frac{1}{2}y^2+\sum_{j=1}^{N}V(\|x-q_j(t)\|)\,.$$

Consider the primaries in a relative equilibrium and denote by $(q_1^0, q_2^0, \dots q_N^0)$ the rigid configuration of the primaries in the rotating system. Again we denote by $(x,y)\in\mathbb{R}^2\times\mathbb{R}^2$, the position and momenta of P with respect to the centre of mass of the N-body system, located at the origin, but now in the rotating system. The motion of the infinitesimal P in the uniformly rotating system is then given by the Hamiltonian system

$$H(x,y)=\frac{1}{2}y^2-(x_1y_2-x_2y_1)+\sum_{j=1}^{N}V(\|x-q_j^0\|). \tag{2}$$

From now on unless otherwise stated, the interaction potential between P and the primaries is taken to be of quasi-homogeneous type and so

$$V(\|x - q_j^0\|) = \sum_{k=1}^{n} \frac{a_k^{(j)}}{\|x - q_j^0\|^{\alpha_k}}\,, \tag{3}$$

where $0 < \alpha_1 < \alpha_2 < \ldots < \alpha_n$, $n \in \mathbb{N}$, and $a_k^{(j)} \in \mathbb{R}$ with $a_n^{(j)} \neq 0$. The classical Newtonian case is retrieved for $n = 1$ and $\alpha_n = 1$.

Remark 2.1 *In our context we do not need to have any knowledge about the internal potential of the N-body system; it is sufficient to consider that the primaries are in a relative equilibrium. Note that this internal potential does not have to coincide with the interaction potential between the infinitesimal and the primaries.*

3 Comet periodic orbits

We assume that the motion of the infinitesimal mass located at P is free of singularities and bounded appropriately, that is $x(t)$ is defined for all $t \in (-\infty, \infty)$, and is bounded above and below. The infinitesimal mass is considered far away from all the primaries, and consequently far away from the centre of mass of the primaries, and so we apply the scaling

$$x \to \varepsilon^{-2}\bar{x}, \qquad y \to \varepsilon^{\alpha_1}\bar{y}\,.$$

This is a symplectic transformation of multiplier $\varepsilon^{(2-\alpha_1)}$ (see [21]). The Hamiltonian (2) becomes

$$H = \frac{1}{2}\varepsilon^{2\alpha_1}\bar{y}^2 - \varepsilon^{(\alpha_1-2)}(\bar{x}_1\bar{y}_2 - \bar{x}_2\bar{y}_1) + \sum_{j=1}^{N} V\left(\frac{1}{\varepsilon^2}\|x - \varepsilon^2\, q_j^0\|\right),$$

which, after rescaling $H \to \varepsilon^{-(\alpha_1-2)}\bar{H}$, and dropping the bars, reads

$$H = -(x_1y_2 - x_2y_1) + \varepsilon^{\alpha_1+2}\frac{y^2}{2} + \varepsilon^{-(\alpha_1-2)}\sum_{j=1}^{N} V\left(\frac{1}{\varepsilon^2}\|x - \varepsilon^2 q_j^0\|\right). \tag{4}$$

For each j the potential is

$$V\left(\frac{1}{\varepsilon^2}||x - \varepsilon^2 q_j^0||\right) = \sum_{k=1}^{n} \frac{\varepsilon^{2\alpha_k}\, a_k^{(j)}}{\|x - \varepsilon^2\, q_j^0\|^{\alpha_k}}. \tag{5}$$

Expanding in powers of ε each $a_k^{(j)}/\|x - \varepsilon^2 q_j^0\|^{\alpha_k}$ we have

$$\frac{a_k^{(j)}}{\|x - \varepsilon^2\, q_j^0\|^{\alpha_k}} = \frac{a_k^{(j)}}{\|x\|^{\alpha_k}} + \mathcal{O}(\varepsilon^{2\,\alpha_k}).$$

Substituting into (5) we obtain

$$V\left(\frac{1}{\varepsilon^2}||x-\varepsilon^2\, q_j^0||\right)=\sum_{k=1}^{n}\left(\varepsilon^{2\alpha_k}\frac{a_k^{(j)}}{\|x\|^{\alpha_k}}+\mathcal{O}(\varepsilon^{4\alpha_k})\right)$$
$$=\varepsilon^{2\alpha_1}\frac{a_1^{(j)}}{\|x\|^{\alpha_1}}+\mathcal{O}(\varepsilon^{\min\{4\alpha_1,2\alpha_2\}}).$$

Using the above expansion and denoting by γ the opposite of the sum of the coefficients of the long-range dominating terms,

$$\gamma:=-\sum_{j=1}^{N}a_1^{(j)}\,,$$

the Hamiltonian (4) reads

$$H=-(x_1y_2-x_2y_1)+\varepsilon^{\alpha_1+2}\left(\frac{y^2}{2}-\frac{\gamma}{\|x\|^{\alpha_1}}\right)+\mathcal{O}\left(\varepsilon^{\min\{3\alpha_1+2,2\alpha_2-\alpha_1+2\}}\right).$$

In polar coordinates (r,θ,p_r,p_θ) this Hamiltonian becomes

$$H=-p_\theta+\varepsilon^{\alpha_1+2}\left[\frac{1}{2}\left(p_r^2+\frac{p_\theta^2}{r^2}\right)-\frac{\gamma}{r^{\alpha_1}}\right]+\mathcal{O}\left(\varepsilon^{\min\{3\alpha_1+2,2\alpha_2-\alpha_1+2\}}\right)\,,\qquad(6)$$

and the first approximation of the problem is given by

$$H=-p_\theta+\varepsilon^{\alpha_1+2}\left[\frac{1}{2}\left(p_r^2+\frac{p_\theta^2}{r^2}\right)-\frac{\gamma}{r^{\alpha_1}}\right]\,.\qquad(7)$$

Remark 3.1 *In the first approximation the infinitesimal particle feels only the accumulative effect of the attraction of the long-range dominating terms*

$$\frac{a_1^{(1)}+a_1^{(2)}+\ldots+a_1^{(N)}}{r^{\alpha_1}}=\frac{(-\gamma)}{r^{\alpha_1}},$$

as exerted by the centre of mass of the primaries.

Remark 3.2 *For $n=1$, $\alpha_1=1$ and $\gamma=1$ the Hamiltonian* (6) *coincides with the Hamiltonian of the Newtonian restricted three-body problem presented in the "Comets" example of the Chapter 9 of [21].*

The equations of motion are

$$\dot r=\varepsilon^{\alpha_1+2}p_r,\qquad(8)$$
$$\dot p_r=\varepsilon^{\alpha_1+2}\left(\frac{p_\theta^2}{r^3}-\frac{\alpha_1\,\gamma}{r^{\alpha_1+1}}\right)+\mathcal{O}\left(\varepsilon^{\min\{3\alpha_1+2,2\alpha_2-\alpha_1+2\}}\right),\qquad(9)$$
$$\dot\theta=-1+\varepsilon^{\alpha_1+2}\frac{p_\theta}{r^2},\qquad(10)$$
$$\dot p_\theta=-\mathcal{O}\left(\varepsilon^{\min\{3\alpha_1+2,2\alpha_2-\alpha_1+2\}}\right)\,.\qquad(11)$$

The standard technique for proving the existence of periodic orbits in perturbed systems is the Poincaré's continuation method: a periodic orbit of the unperturbed system persists in the perturbed dynamics if its characteristic multipliers are different from the unity (see Chapter 9 of [21]). In our case the unperturbed system is given by the first approximation of the Hamiltonian (7). Since we have an autonomous Hamiltonian system, two of the multipliers are always equal to the unity, and the restriction of the flow to an energy level is necessary (see [21]). This method is applicable in our case as well. Dropping the terms $\mathcal{O}\left(\varepsilon^{\min\{3\alpha_1+2,2\alpha_2-\alpha_1+2\}}\right)$, if $\gamma > 0$ then for every $c \neq 0$, system (8)-(11) accepts the periodic solutions

$$r_0 = \left(\frac{c^2}{\gamma\alpha_1}\right)^{1/(2-\alpha_1)}, \qquad p_r = 0, \qquad p_\theta = \pm c\,,$$

of period

$$T = \frac{2\pi}{1 \mp \varepsilon^{\alpha_1+2}\left(\frac{|c|}{r_0^2}\right)}\,.$$

Linearizing equations (8)-(9) about $r = r_0$ and $p_r = 0$, we obtain

$$\dot{r} = \varepsilon^{\alpha_1+2} p_r, \qquad \dot{p}_r = -\varepsilon^{\alpha_1+2}(2-\alpha_1)\left(\frac{|c|}{r_0^2}\right)^2 r\,.$$

The nontrivial multipliers of (8)-(11) are given by

$$1 \pm i\,\varepsilon^{\alpha_1+2} 2\pi\sqrt{2-\alpha_1}\left(\frac{|c|}{r_0^2}\right) + \mathcal{O}\left(\varepsilon^{\alpha_1+4}\right)\,.$$

Recall that for a Hamiltonian system a fixed point is *unstable* if at least one if its non-trivial multipliers has modulus greater than the unity, and *elliptic* if all its non-trivial multipliers of unit modulus and not equal to ± 1.

Assuming $\alpha_1 \neq 2$, by applying with minor modifications the arguments given in [19] and [21], we have the following result.

Proposition 3.3 (Existence of comet periodic solutions) *In the generalized restricted $(N+1)$-body problem described by the Hamiltonian* (2) *with quasi-homogenous potential* (3), *if the sum $(-\gamma)$ of the coefficients of the long-range dominating terms is negative (i.e. if the potential near infinity is attractive) and $\alpha_1 \neq 2$, then there exist two-parameter families of nearly circular large periodic solutions located far away from the centre of mass of the primaries. If $0 < \alpha_1 < 2$ these orbits are elliptic. If $\alpha_1 > 2$ these orbits are unstable.*

An alternative and easier proof of Proposition 3.3, using the averaging theory, is presented below. From the energy integral

$$-p_\theta + \varepsilon^{\alpha_1+2}\left(\frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{\gamma}{r^{\alpha_1}}\right) = h + \mathcal{O}\left(\varepsilon^{\min\{3\alpha_1+2,2\alpha_2-\alpha_1+2\}}\right)\,,$$

we get that

$$p_\theta = -h + \mathcal{O}\left(\varepsilon^{\alpha_1+2}\right).$$

Substituting p_θ into (8) we have the system

$$\begin{aligned}
\dot{r} &= \varepsilon^{\alpha_1+2} p_r,\\
\dot{p}_r &= \varepsilon^{\alpha_1+2}\left(\frac{h^2}{r^3} - \frac{\alpha_1\,\gamma}{r^{\alpha_1+1}}\right) + \mathcal{O}\left(\varepsilon^{\min\{3\alpha_1+2,2\alpha_2-\alpha_1+2\}}\right),\\
\dot{\theta} &= -1 + \mathcal{O}\left(\varepsilon^{\min\{3\alpha_1+2,2\alpha_2-\alpha_1+2\}}\right),
\end{aligned}$$

which taking θ as independent variable becomes

$$\begin{aligned}
\frac{dr}{d\theta} &= -\varepsilon^{\alpha_1+2} p_r,\\
\frac{dp_r}{d\theta} &= -\varepsilon^{\alpha_1+2}\left(\frac{h^2}{r^3} - \frac{\alpha_1\,\gamma}{r^{\alpha_1+1}}\right) + \mathcal{O}\left(\varepsilon^{\min\{3\alpha_1+2,2\alpha_2-\alpha_1+2\}}\right).
\end{aligned} \tag{12}$$

In the variable $\mathbf{y} := (r, p_r)$ this system is of the form

$$\frac{d\mathbf{y}}{d\theta} = \varepsilon^{\alpha_1+2} f(\mathbf{y}) + \varepsilon^{\alpha_1+4} G(\mathbf{y}, \theta; \varepsilon)$$

where $G(\mathbf{y}, \theta; \varepsilon)$ is of order 0 and higher in ε and 2π-periodic in θ. Since $f(\mathbf{y})$ does not depend explicitly on θ, we may say that f is $2\pi-$periodic in θ (trivially). Thus we can "average" over θ and obtain the averaged vector field

$$f_{average}(\mathbf{y}) = \frac{1}{2\pi}\int_0^{2\pi} f(\mathbf{y})ds = f(\mathbf{y}) = -\begin{pmatrix} p_r \\ \frac{h^2}{r^3} - \frac{\alpha_1\,\gamma}{r^{\alpha_1+1}} \end{pmatrix}.$$

We apply then the averaging theorem (see [31]; also presented in the Appendix) and we calculate the zeros of the averaged equations. Assuming $\gamma > 0$ there is a unique of such zeros, given by

$$\mathbf{y}_0 = (r_0(h), 0) = \left(\left(\frac{h^2}{\gamma\alpha_1}\right)^{1/(2-\alpha_1)}, 0\right).$$

The determinant of the Jacobian matrix $Df(\mathbf{y}_0)$ of the averaged vector field at the zero is

$$\det\left(Df(\mathbf{y}_0)\right) = \frac{h^2}{r_0^4(h)}(2-\alpha_1).$$

By the averaging theorem it follows that for $\alpha_1 \neq 2$ and for sufficiently small ε, there exists a T-periodic solution $\mathbf{y}_\varepsilon(\theta)$ of (12) such that $\mathbf{y}_\varepsilon(0) \to \mathbf{y}_0$ as $\varepsilon \to 0$. Since the eigenvalues of $Df(\mathbf{y}_0)$ are given by

$$\pm\varepsilon^{\alpha_1+2}|h|/r_0^2(h)\sqrt{\alpha_1-2},$$

if $\alpha_1 > 2$ we have that the singular point $\mathbf{y}_0$ of the averaged system is hyperbolic and unstable. By the averaging theorem, it results that for sufficiently small ε, the periodic orbit $\mathbf{y}_\varepsilon(\theta)$ is also hyperbolic and unstable.

Of course, if $\alpha_1 < 2$ then the periodic orbit $\mathbf{y}_\varepsilon(\theta)$ is linearly elliptic or stable.

Remark 3.4 *Alternatively we could use $-\theta$ as the independent variable instead of θ. Then the proof follows identically, but the obtained periodic solution encircle the centre of mass of the primaries rotating in opposite direction.*

In short Proposition 3.3 follows.

3.1 Stability

In [19] it is proven using KAM theory that in case $\gamma = 1$ and $\alpha_1 = 1$, the elliptic periodic orbits at infinity are stable. We generalize this result to the case $\gamma > 0$ and $0 < \alpha_1 < 2$. To do this we follow the theory presented in Chapter 6 of [3].

Let $\gamma > 0$ and $0 < \alpha_1 < 2$ be fixed and consider the Hamiltonian (6) which, for the readers convenience, we rewrite below

$$H = -p_\theta + \varepsilon^{\alpha_1+2}\left[\frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{\gamma}{r^{\alpha_1}}\right] + \mathcal{O}\left(\varepsilon^{\min\{3\alpha_1+2, 2\alpha_2-\alpha_1+2\}}\right). \tag{13}$$

Denote

$$H_{00} := -p_\theta, \qquad H_{01} := \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{\gamma}{r^{\alpha_1}}$$

and

$$\varepsilon^{\alpha_1+2} := \tilde{\varepsilon}.$$

Then the Hamiltonian (13) becomes

$$H = H_{00}(p_\theta) + \tilde{\varepsilon} H_{01}(r, p_r, p_\theta) + \tilde{\varepsilon}^2 H_{11}(r, \theta, p_r, p_\theta; \tilde{\varepsilon}). \tag{14}$$

The first approximation $H_{00} + \tilde{\varepsilon} H_{01}$ is an integrable system, with first integrals given by the energy

$$h = H_{00}\left(p_\theta(t)\right) + \tilde{\varepsilon} H_{01}\left(r(t), p_r(t), p_\theta(t)\right)$$

and the angular momentum

$$c = p_\theta(t).$$

Let $h < 0$. Now we write H in action-angle variables $(I_r, I_\theta, \phi_r, \phi_\theta)$. Since $H_0 := H_{00} + \tilde{\varepsilon} H_{01}$ is integrable and $H_{00} = -p_\theta$, we have

$$H_{00} = H_{00}(I_\theta)$$

where

$$I_\theta = p_\theta\,.$$

The Hamiltonian H_{01} describes a central force problem. Since $0 < \alpha_1 < 2$ and $I_\theta(t) = c = \text{const.}$ on the energy level $h_{01} = H_{01}\left(r(t), p_r(t), p_\theta(t)\right) < 0$, the motion is bounded, with $r(t)$ varying in an annulus of minimum and maximum radii $r_m(c, h_{01})$ and $r_M(c, h_{01})$, respectively (for details see Chapter 1 Section 3c of [14]). The action I_r is defined by

$$I_r = \frac{\sqrt{2}}{\pi} \int\limits_{r_m(c,h_{01})}^{r_M(c,h_{01})} \sqrt{h_{01} + \frac{\gamma}{r^{\alpha_1}} - \frac{c^2}{2r^2}}\, dr\,. \tag{15}$$

Note that $r_m(c, h_{01})$ and $r_M(c, h_{01})$ are the minimum and maximum values of r, respectively, and that the integrant $\sqrt{h_{01} + \gamma/r^{\alpha_1} - I_\theta^2/(2r^2)}$ cancels for r taking these extreme values. In general the integral (15) cannot be solved by quadratures. A notable exception is $\alpha_1 = 1$ (the Kepler problem), in this case $I_r = -|I_\theta| + \gamma\sqrt{-1/(2h_{01})}$.

The Hamiltonian (14) can be written into the form

$$H = H_{00}(I_\theta) + \tilde{\varepsilon}\, H_{01}(I_r, I_\theta) + \tilde{\varepsilon}^2 H_{11}(I_r, I_\theta, \phi_r, \phi_\theta; \tilde{\varepsilon}).$$

Following the terminology of [3], Section 6.3.3, this Hamiltonian has a *proper degeneracy*, and consequently the following stability result holds.

Theorem 3.5 (Arnol'd 1963; see [3]) *If a Hamiltonian system with two degrees of freedom having a proper degeneracy satisfies*

$$\frac{dH_{00}}{dI_\theta} \neq 0, \qquad \frac{\partial^2 H_{01}}{\partial I_r^2} \neq 0, \tag{16}$$

then for all initial data the action variables remain forever near their initial values.

In the context of our problem, if conditions (16) are verified, then the comet elliptic periodic orbits are stable. The first condition (16) is immediate because

$$\frac{dH_{00}}{dI_\theta} = \frac{d\,(-I_\theta)}{dI_\theta} = -1 \neq 0.$$

To verify the second condition, consider Hamiltonian $H_{01} = H_{01}(I_r, I_\theta)$ as it is defined implicitly by

$$I_r = F\left(I_\theta, H_{01}(I_r, I_\theta)\right),$$

with

$$F\left(I_\theta, H_{01}(I_r, I_\theta)\right) = \left(\frac{\sqrt{2}}{\pi} \int\limits_{r_m(c,h_{01})}^{r_M(c,h_{01})} \sqrt{h_{01} + \frac{\gamma}{r^{\alpha_1}} - \frac{c^2}{2r^2}}\, dr \right) \Bigg|_{h_{01}=H_{01},\, c=I_\theta} . \tag{17}$$

Using implicit differentiation with respect to I_r we have

$$1 = \frac{\partial F}{\partial I_\theta}\frac{\partial I_\theta}{\partial I_r} + \frac{\partial F}{\partial H_{01}}\frac{\partial H_{01}}{\partial I_r} = \frac{\partial F}{\partial H_{01}}\frac{\partial H_{01}}{\partial I_r}\,, \tag{18}$$

because I_r and I_θ are independent. Differentiating once more, we obtain

$$0 = \frac{\partial^2 F}{\partial H_{01}^2}\left(\frac{\partial H_{01}}{\partial I_r}\right)^2 + \frac{\partial F}{\partial H_{01}}\frac{\partial^2 H_{01}}{\partial I_r^2}\,.$$

From where, assuming $\partial F/\partial H_{01} \neq 0$, we get

$$\frac{\partial^2 H_{01}}{\partial I_r^2} = -\frac{\frac{\partial^2 F}{\partial H_{01}^2}\left(\frac{\partial H_{01}}{\partial I_r}\right)^2}{\frac{\partial F}{\partial H_{01}}}\,. \tag{19}$$

To calculate $\partial F/\partial H_{01}$ we differentiate under the integral sign the relation (17). Taking into account that $r_m(c, h_{01}) > 0$ and $r_M(c, h_{01}) > 0$ are only values of r for which the non-zero expression $h_{01} + \gamma/r^{\alpha_1} - c^2/(2r^2)$ vanishes, we obtain

$$\frac{\partial F}{\partial H_{01}} = \frac{1}{\sqrt{2}\pi}\left(\int_{r_m(c,h_{01})}^{r_M(c,h_{01})}\left(h_{01} + \frac{\gamma}{r^{\alpha_1}} - \frac{c^2}{2r^2}\right)^{-1/2} dr\right)\Bigg|_{h_{01}=H_{01},\, c=I_\theta} > 0. \tag{20}$$

Furthermore

$$\frac{\partial^2 F}{\partial H_{01}^2} = \frac{-1}{2\sqrt{2}\pi}\left(\int_{r_m(c,h_{01})}^{r_M(c,h_{01})}\left(h_{01} + \frac{\gamma}{r^{\alpha_1}} - \frac{I_\theta^2}{2r^2}\right)^{-3/2} dr\right)\Bigg|_{h_{01}=H_{01},\, c=I_\theta} < 0\,. \tag{21}$$

Using (20), from (18) it follows that

$$\frac{\partial H_{01}}{\partial I_r} = \frac{1}{\frac{1}{\sqrt{2}\pi}\left(\int_{r_m(c,h_{01})}^{r_M(c,h_{01})}\left(h_{01} + \frac{\gamma}{r^{\alpha_1}} - \frac{c^2}{2r^2}\right)^{-1/2} dr\right)\Big|_{h_{01}=H_{01},\, c=I_\theta}}\,. \tag{22}$$

Finally, substituting (20), (21) and (22) into (19) we obtain:

$$\frac{\partial^2 H_{01}}{\partial I_r^2} = \frac{\frac{1}{2\sqrt{2}\pi}\left(\int_{r_m(c,h_{01})}^{r_M(c,h_{01})}\left(h_{01} + \frac{\gamma}{r^{\alpha_1}} - \frac{c^2}{2r^2}\right)^{-3/2} dr\right)\Big|_{h_{01}=H_{01},\, c=I_\theta}}{\left(\frac{1}{\sqrt{2}\pi}\left(\int_{r_m(c,h_{01})}^{r_M(c,h_{01})}\left(h_{01} + \frac{\gamma}{r^{\alpha_1}} - \frac{c^2}{2r^2}\right)^{-1/2} dr\right)\Big|_{h_{01}=H_{01},\, c=I_\theta}\right)^3} > 0,$$

and thus the second condition (16) in Theorem 3.5 is satisfied.

In summary taking into account Proposition 3.3, we have the following result.

Theorem 3.6 *In the generalized restricted $(N+1)$-body problem described by the Hamiltonian* (2) *with quasi-homogenous potential* (3)*, if the sum γ of the coefficients of the long-range dominating terms is positive (i.e. if the potential near infinity is attractive) and $\alpha_1 \neq 2$, then there exist two-parameter families of nearly circular large periodic solutions of motion near infinity. For $\alpha_1 < 2$ these solution are elliptic and KAM stable, whereas for $\alpha_1 > 2$ they are hyperbolic and unstable.*

3.2 Applications

In this subsection we present some consequences of Theorem 3.6.

Corollary 3.7 *In the restricted $(N+1)$-body problem with Schwarzschild or Manev potential, there exist nearly circular periodic orbits located far away from the centre of mass of the primaries. These orbits are KAM stable.*

Corollary 3.8 *Consider a planetary system formed by $N \geq 2$ bodies of homogeneous density. The planets are taken as spinning spheroids with revolution axis perpendicular to the equatorial (ecliptic) plane. Then it can be proven that the dynamics of this system decouples into the motion of the centres of mass and the motion of each rigid body, where the interaction between any two centres of mass is given by a potential of the form $-a/r - b/r^3$ with $a > 0$ (for $N = 3$, see [9] and [29]). Assume that the system formed by the mass centres is in a relative equilibrium. Then the motion of a small object (e.g. an asteroid) includes KAM nearly circular stable periodic orbits located in the ecliptic plane and far away from the planetary centre of mass.*

Another immediate consequence is related with the molecular-type interactions.

Corollary 3.9 *In the restricted $(N+1)$-body problem with molecular potentials of the form* (1)*, there exist nearly circular periodic orbits located far away from the centre of mass of the primaries. These orbits are unstable.*

As noted in Remark 3.1, it is interesting to observe that the dynamics in our class of restricted $(N+1)$-body problems depends on the sign of the accumulative effect of the long-range dominating terms. Thus there might be situations, for example in problems with masses charged, in which the infinitesimal mass may be repelled by some of the primaries and attracted by other, and allowing the existence of nearly circular periodic orbits close to infinity.

For instance, consider the primaries as given by three point masses, say P_1, P_2 and P_3, with charge. In [1] it is shown that, under certain conditions, such systems permit the existence of planar relative equilibria. Imagine a situation in which the infinitesimal mass is repelled by P_1, P_2 and attracted by P_3. Denote the interaction potentials between the infinitesimal mass and P_1, P_2

and P_3 by $V_i = a_i/r$, $i = 1, 2, 3$, where $a_1 > 0$, $a_2 > 0$ and $a_3 < 0$. If

$$a_1 + a_2 + a_3 < 0,$$

then there exist close to infinity nearly circular periodic orbits which are KAM stable. It is easy to imagine other applications of Theorem 3.3 in the context of charged restricted $(N+1)$-body problems.

4 Hill periodic orbits

We assume now that the infinitesimal mass is close to one of the primaries, which without loosing generality, we choose to be m_N. Let L be the position vector of m_N with respect to the centre of mass O_{N-1} of the $N-1$ primaries $m_1, m_2, \ldots, m_{N-1}$. The primaries form a rigid configuration that rotates with uniform circular velocity around their centre of mass O. In particular, in the rotating system, the vector L appears as fixed and of constant length, with one end at the centre of mass O_{N-1} of $m_1, m_2, \ldots, m_{N-1}$ and and the other one at m_N.

Without loosing generality, in the rotating system we choose coordinates with the x_1 axis along L, and take $\|L\|$ to be the unit length. We further assume that the total mass is normalized to one, i.e. $m_1 + m_2 + \ldots + m_N = 1$, and denote

$$\mu := m_1 + m_2 + \ldots + m_{N-1}\,.$$

Consequently $m_N = 1 - \mu$, $\overrightarrow{O_{N-1}O} = 1 - \mu$ and $\overrightarrow{Om_N} = -\mu$.

Under the above assumptions the Hamiltonian (2) becomes

$$H(x,y) = \frac{y^2}{2} - (x_1 y_2 - x_2 y_1) + \sum_{j=1}^{N-1} V(\|x - q_j^0\|) + V_N\left(\sqrt{(x_1-\mu)^2 + x_2^2}\right),$$

where V_N describes the potential between P and m_N. We take now the origin of coordinates at the primary m_N (see [21]). More precisely, we change $x_1 - \mu$ by x_1 and y_2 by $y_2 + \mu$. Also, each q_{jx}^0 is translated by μ and so q_j^0 is replaced by $\tilde{q}_j^0 := q_j^0 - (\mu, 0)$. We have

$$\begin{aligned} H(x,y) = \frac{y^2}{2} & -(x_1 y_2 - x_2 y_1) + \sum_{j=1}^{N-1} V(\|x - \tilde{q}_j^0\|) + \\ & V_N\left(\sqrt{x_1^2 + x_2^2}\right) - \mu x_1 - \tfrac{1}{2}\mu^2\,. \end{aligned} \tag{23}$$

We then apply the scaling

$$x \to \varepsilon^2 \bar{x}, \qquad y \to \varepsilon^{-\alpha_n} \bar{y}\,,$$

which is a symplectic transformation of multiplier $\varepsilon^{\alpha_n - 2}$ (see [21]). Ignoring the constant term, the Hamiltonian (23) becomes

$$\begin{aligned}H =& \frac{\varepsilon^{-2\alpha_n}\bar{y}^2}{2} - \varepsilon^{2-\alpha_n}(\bar{x}_1\bar{y}_2 - \bar{x}_2\bar{y}_1) - \varepsilon^2\mu\bar{x}_1 + \\ & \sum_{j=1}^{N-1} V\left(\|\varepsilon^2\bar{x} - \tilde{q}_j^0\|\right) + V_N\left(\varepsilon^2\sqrt{\bar{x}_1^2 + \bar{x}_2^2}\right),\end{aligned}$$

which, after rescaling $H \to \varepsilon^{2\alpha_n}\bar{H}$ and dropping the bars, reads

$$\begin{aligned}H =& \frac{y^2}{2} - \varepsilon^{\alpha_n+2}(x_1y_2 - x_2y_1) - \varepsilon^{2\alpha_n+2}\mu x_1 + \\ & \varepsilon^{2\alpha_n}\left(\sum_{j=1}^{N-1} V\left(\|\varepsilon^2 x - \tilde{q}_j^0\|\right) + V_N\left(\varepsilon^2\|x\|\right)\right).\end{aligned} \tag{24}$$

We have

$$V_N\left(\varepsilon^2\|x\|\right) = \sum_{k=1}^{n} \frac{a_k^{(N)}}{\varepsilon^{2\alpha_k}\|x\|^{\alpha_k}} = \frac{a_1^{(N)}}{\varepsilon^{2\alpha_1}\|x\|^{\alpha_1}} + \ldots + \frac{a_n^{(N)}}{\varepsilon^{2\alpha_n}\|x\|^{\alpha_n}}$$

and

$$\begin{aligned}V(\|\varepsilon^2 x - \tilde{q}_j^0\|) &= \sum_{k=1}^{n} \frac{a_k^{(j)}}{\|\varepsilon^2 x - \tilde{q}_j^0\|^{\alpha_k}} \\ &= \frac{a_1^{(j)}}{\|\varepsilon^2 x - \tilde{q}_j^0\|^{\alpha_2}} + \ldots + \frac{a_n^{(j)}}{\|\varepsilon^2 x - \tilde{q}_j^0\|^{\alpha_n}} \\ &= \sum_{k=1}^{n} \frac{a_k^{(j)}}{\|\tilde{q}_j^0\|^{\alpha_k}} + \mathcal{O}(\varepsilon^{2\alpha_1}).\end{aligned}$$

Substituting the above into (24), we have

$$\begin{aligned}H =& \frac{y^2}{2} - \varepsilon^{\alpha_n+2}(x_1y_2 - x_2y_1) - \varepsilon^{2\alpha_n+2}\mu x_1 + \\ & \varepsilon^{2\alpha_n}\left(\frac{a_1^{(N)}}{\varepsilon^{2\alpha_1}\|x\|^{\alpha_1}} + \ldots + \frac{a_n^{(N)}}{\varepsilon^{2\alpha_n}\|x\|^{\alpha_n}} + \sum_{j=1}^{N-1}\sum_{k=1}^{n} \frac{a_k^{(j)}}{\|\tilde{q}_j^0\|^{\alpha_k}} + \mathcal{O}(\varepsilon^{2\alpha_1})\right)\end{aligned}$$

and so

$$\begin{aligned}H =& \frac{y^2}{2} + \frac{a_n^{(N)}}{\|x\|^{\alpha_n}} + \varepsilon^{2(\alpha_n-\alpha_{n-1})}\frac{a_{n-1}^{(N)}}{\|x\|^{\alpha_{n-1}}} + \ldots + \varepsilon^{2(\alpha_n-\alpha_1)}\frac{a_1^{(N)}}{\|x\|^{\alpha_1}} - \\ & \varepsilon^{\alpha_n+2}(x_1y_2 - x_2y_1) - \varepsilon^{2\alpha_n+2}\mu x_1 + \varepsilon^{2\alpha_n}\left(\sum_{j=1}^{N-1}\sum_{k=1}^{n} \frac{a_k^{(j)}}{\|\tilde{q}_j^0\|^{\alpha_k}}\right) + \mathcal{O}\left(\varepsilon^{2(\alpha_n+\alpha_1)}\right).\end{aligned}$$

Dropping the constant term and writing the Hamiltonian in polar coordinates, up to $\mathcal{O}\left(\varepsilon^{2\alpha_n+2}\right)$ the Hamiltonian becomes

$$
\begin{aligned}
H =& \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + \frac{a_n^{(N)}}{r^{\alpha_n}} + \varepsilon^{2(\alpha_n-\alpha_{n-1})}\frac{a_{n-1}^{(N)}}{r^{\alpha_{n-1}}} + \ldots + \varepsilon^{2(\alpha_n-\alpha_1)}\frac{a_1^{(N)}}{r^{\alpha_1}} - \\
& \varepsilon^{\alpha_n+2}p_\theta + \mathcal{O}\left(\varepsilon^{2(\alpha_n+1)}\right) .
\end{aligned}
$$

Denote

$$
U(r,\varepsilon) := \frac{a_n^{(N)}}{r^{\alpha_n}} + \varepsilon^{2(\alpha_n-\alpha_{n-1})}U_1(r,\varepsilon) ,
$$

where

$$
U_1(r,\varepsilon) :=
$$

$$
\begin{cases}
0 & \text{if} \quad \alpha_n + 2 < 2(\alpha_n - \alpha_{n-1}) , \\
\varepsilon^{2(\alpha_n-\alpha_{n-1})}\frac{a_{n-1}^{(N)}}{r^{\alpha_{n-1}}} + \ldots + \varepsilon^{2(\alpha_n-\alpha_{n-k})}\frac{a_{n-k}^{(N)}}{r^{\alpha_{n-k}}} & \text{if } 2(\alpha_n-\alpha_{n-k}) \le \alpha_n + 2 < 2(\alpha_n - \alpha_{n-k-1}), \\
& \qquad \text{for some } k = 1, \ldots, n-1, \\
\varepsilon^{2(\alpha_n-\alpha_{n-1})}\frac{a_{n-1}^{(N)}}{r^{\alpha_{n-1}}} + \ldots + \varepsilon^{2(\alpha_n-\alpha_1)}\frac{a_1^{(N)}}{r^{\alpha_1}} & \text{if} \quad 2(\alpha_n - \alpha_1) \le \alpha_n + 2 .
\end{cases}
$$

Note that $U_1(x,\varepsilon)$ is an analytic function for $r \neq 0$. Also denote

$$
\delta := \begin{cases}
2(\alpha_n - \alpha_{n-1}) & \text{if} \quad \alpha_n + 2 < 2(\alpha_n - \alpha_{n-1}) , \\
2(\alpha_n - \alpha_{n-k-1}) & \text{if} \quad 2(\alpha_n - \alpha_{n-k}) \le \alpha_n + 2 < 2(\alpha_n - \alpha_{n-k-1}) , \\
& \qquad \text{for some } k = 1, \ldots, n-1, \\
2(\alpha_n + 1) & \text{if} \quad 2(\alpha_n - \alpha_1) \le \alpha_n + 2 .
\end{cases}
$$

The Hamiltonian now reads

$$
H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + U(r,\varepsilon) - \varepsilon^{\alpha_n+2}p_\theta + \mathcal{O}\left(\varepsilon^{\delta}\right) , \tag{25}
$$

and the equations of motion are

$$
\begin{aligned}
\dot{r} &= p_r + \mathcal{O}\left(\varepsilon^{\delta}\right) , \\
\dot{p}_r &= \frac{p_\theta^2}{r^3} - U'(r,\varepsilon) + \mathcal{O}\left(\varepsilon^{\delta}\right) , \\
\dot{\theta} &= \frac{p_\theta}{r^2} - \varepsilon^{\alpha_n+2} + \mathcal{O}\left(\varepsilon^{\delta}\right) , \\
\dot{p}_\theta &= \mathcal{O}\left(\varepsilon^{\delta}\right) .
\end{aligned}
$$

From the last equation above we have

$$p_\theta = c + \mathcal{O}\left(\varepsilon^\delta\right) \,,$$

so, up to order ε^δ, the Hamiltonian is

$$H = \frac{1}{2}p_r^2 + \frac{c^2}{2r^2} + U(r,\varepsilon)\,, \tag{26}$$

where we dropped the constant term $\varepsilon^{\alpha_n+2}\,c$.

Proposition 4.1 *Let $a_n^{(N)} < 0$ and $\alpha_n \neq 2$. Then, for ε small enough, there exist circular periodic orbits in the dynamics of the first approximation Hamiltonian* (26)*.*

Proof Circular periodic orbits appear as the critical points of the augmented potential

$$U_c(r,\varepsilon) := \frac{c^2}{2r^2} + U(r,\varepsilon) = \frac{c^2}{2r^2} + \frac{a_n^{(N)}}{r^{\alpha_n}} + \varepsilon^{2(\alpha_n-\alpha_{n-1})}U_1(r,\varepsilon),$$

where $c \neq 0$.

Case $U_1(r,\varepsilon) = 0$. We have

$$U_c(r,\varepsilon) = \frac{c^2}{2r^2} + \frac{a_n^{(N)}}{r^{\alpha_n}}.$$

In this case the augmented potential does not depend on ε. The critical points of $U_c(r,\varepsilon)$ are given by the values r_0 which are solutions of $U_c'(r,\varepsilon) = 0$, that is, solutions r_0 of

$$\frac{c^2}{r^3} = -\frac{a_n^{(N)}\alpha_n}{r^{\alpha_n+1}}\,. \tag{27}$$

that is $c^2 r^{\alpha_n-2} = -a_n^{(N)}$. Since $a_n^{(N)} < 0$, this equation has one real positive root.

Case $U_1(r,\varepsilon) \neq 0$. For $\varepsilon = 0$ the critical points of $U_c(r,0)$ are given by the values r_0 which are solutions of $U_c'(r,0) = 0$, that is, solutions of (27) above. Since

$$U_c''(r_0,0) = -\frac{a_n^{(N)}\alpha_n}{r_0^{\alpha_n+2}}(\alpha_n - 2) \neq 0\,,$$

by the Implicit Function Theorem we have that for ε small enough there is a smooth function $r_0(\varepsilon)$ such that $U_c'\left(r_0(\varepsilon),\varepsilon\right) = 0$. □

Corollary 4.2 *Under the assumptions of Proposition 4.1, the circular periodic orbits in the dynamics of the first approximation Hamiltonian* (26) *are stable if $\alpha_n < 2$ and unstable if $\alpha_n > 2$.*

Proof We use the notation introduced in the proof of Proposition 4.1. Since the dynamics in radial direction r of the circular periodic orbits is described by the one-degree of freedom of the form kinetic plus augmented potential, i.e. by the Hamiltonian (26), the stability is decided by the sign of the second derivative of the potential (see [2]). The circular orbits are stable if $U_c''(r,\varepsilon) > 0$ and unstable if $U_c''(r,\varepsilon) < 0$.

We prove the stable case; the proof is analogous for the unstable one. For $\alpha_n < 2$, since $U_c''(r_0,0) > 0$, the circular orbit at $\varepsilon = 0$ is stable. For ε small enough

$$U_c''(r,\varepsilon) = \left.\frac{\partial^2 U_c(r,\varepsilon)}{\partial r^2}\right|_{r_0(\varepsilon)} > 0$$

as well, and the conclusion follows. □

Remark 4.3 *If $U_1(r,\varepsilon) \neq 0$ then the zeros r_0 of $U_c'(r,\varepsilon)$ are of the form*

$$r_0(\varepsilon) = r_0(0) + \mathcal{O}(\varepsilon^{2(\alpha_n - \alpha_{n-1})}),$$

where $r_0(0)$ is the solution of $U_c'(r;0) = 0$.

We prove now that the circular periodic orbits of (26) persist as periodic orbits in the motion of the full Hamiltonian (25) by applying the Poincaré's continuation method.

Consider the Hamiltonian obtained from (25) by dropping the terms of order ε^δ and higher:

$$H = \frac{1}{2}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + U(r,\varepsilon) - \varepsilon^{\alpha_n+2} p_\theta = \frac{p_r^2}{2} + U_c(r,\varepsilon).$$

The associated equations of motion are

$$\begin{aligned}
\dot r &= p_r, \\
\dot p_r &= -\frac{\partial U_c(r,\varepsilon)}{\partial r}, \\
\dot\theta &= \frac{c}{r^2} - \varepsilon^{\alpha_n+2}, \\
\dot p_\theta &= 0,
\end{aligned} \tag{28}$$

where we have substituted p_θ by c. The circular orbits are given by $(r_0(\varepsilon), 0)$ with period

$$T = 2\pi / \left(\frac{c}{r_0^2(\varepsilon)} \mp \varepsilon^{\alpha_n+2}\right).$$

Linearizing the r and p_r equations about these solutions, we have

$$\dot r = p_r\,, \qquad \dot p_r = \left(-\left.\frac{\partial^2 U_c(r,\varepsilon)}{\partial r^2}\right|_{r_0(\varepsilon)}\right) r.$$

Let

$$\omega^2(\varepsilon) := \left(\left. \frac{\partial^2 U_c(r,\varepsilon)}{\partial r^2} \right|_{r_0(\varepsilon)} \right) .$$

The linearized dynamics can be written as

$$\ddot{r} + \omega^2(\varepsilon) r = 0 .$$

For $\alpha_n > 2$, since $\omega^2(\varepsilon) < 0$, the non-trivial multipliers of the circular orbits $(r_0(\varepsilon), 0)$ are given by

$$\exp\left[\pm 2\pi \, \left(\frac{\omega(\varepsilon) r_0^2(\varepsilon)}{c} \right) \left(\frac{1}{1 \mp \varepsilon^{\alpha_n+2} \frac{r_0^2(\varepsilon)}{c}} \right) \right] .$$

Since the multipliers are not unity, these orbits may be continued. Moreover, the continued orbits are unstable.

For $\alpha_n < 2$, the non-trivial multipliers of the circular orbits $(r_0(\varepsilon), 0)$ are

$$\begin{aligned} &\exp\left[\pm 2\pi i \, \left(\frac{\omega(\varepsilon) r_0^2(\varepsilon)}{c} \right) \left(\frac{1}{1 \mp \varepsilon^{\alpha_n+2} \frac{r_0^2(\varepsilon)}{c}} \right) \right] \\ &= \exp\left[\pm 2\pi i \, \left(\frac{\omega(\varepsilon)\, r_0^2(\varepsilon)}{c} \right) \left(1 \pm \varepsilon^{\alpha_n+2} \, \left(\frac{r_0^4(\varepsilon)}{c^2} \right) + \mathcal{O}\left(\varepsilon^{2(\alpha_n+2)} \right) \right) \right] . \end{aligned}$$

We distinguish two cases:

(1) If $\left(\frac{\omega(\varepsilon)\, r_0^2(\varepsilon)}{c} \right) \notin \mathbb{Z}$ then the non-trivial multipliers are not unity and the circular orbits may be continued.

(2) If $\left(\frac{\omega(\varepsilon)\, r_0^2(\varepsilon)}{c} \right) = m \in \mathbb{Z}$, then the non-trivial multipliers receive the form

$$1 \pm \varepsilon^{\alpha_n+2} \, 2\pi i \, \left(\frac{r_0^4(\varepsilon)}{c^2} \right) + \mathcal{O}\left(\varepsilon^{2(\alpha_n+2)} \right) .$$

Taking into account Remark 4.3, the multipliers can be written as

$$1 \pm \varepsilon^{\alpha_n+2} \, 2\pi i \, \left(\frac{r_0^4(0)}{c^2} \right) + \mathcal{O}\left(\varepsilon^{\eta} \right) ,$$

where

$$\eta = \begin{cases} 2\alpha_n + 2 & \text{if } \; G_1(r,\varepsilon) = 0, \\ (\alpha_n+2) + 2(\alpha_n - \alpha_{n-1}) & \text{if } \; G_1(r,\varepsilon) \neq 0 . \end{cases}$$

The continuation of the periodic orbits is achieved by applying essentially the argument presented in Section 9.4. of [21]. Indeed, consider the period map in an energy level of the Hamiltonian about the circular orbit, and let u be the coordinate on this energy level, with $u = 0$ corresponding to the circular orbit when $\varepsilon = 0$. The period map is of the form $P(u) = u + \varepsilon^{\alpha_n+2}p(u) + \mathcal{O}(\varepsilon^{\eta})$, where $p(0) = 0$ and $p'(0)$ is nonsingular, its eigenvalues being $\pm 2\pi i\, r_0^4(0)/c^2$. Let $G(u, \varepsilon) := (P(u) - u)\,/\varepsilon^{\alpha_n+2} = p(u) + \mathcal{O}(\varepsilon^{\eta-\alpha_n-2})$. We have $G(0, 0) = 0$. Since $\partial G(0, 0)/\partial u = p'(0)$ is nonsingular, by the Implicit Function Theorem, there is a smooth function $\bar{u}(\varepsilon)$ such that $G\,(\bar{u}(\varepsilon), \varepsilon) = 0$ for ε small enough. Thus the circular orbit can be continued from the truncated equations (28) to the full equations.

So we have proved:

Theorem 4.4 *In the generalized restricted $(N + 1)$-body problem described by the Hamiltonian (2) with quasi-homogenous potential (3), if $a_n^{(N)} < 0$ and $\alpha_n \neq 2$, then there exist two one-parameter families of nearly circular periodic solutions that encircle primary m_N. These orbits are unstable if $\alpha_n > 2$.*

4.1 Applications

Corollary 4.5 *There exist nearly circular periodic orbits close to a primary in the restricted $(N+1)$-body problem with Manev or Schwarzschild potential.*

Corollary 4.6 *Consider a planetary system formed by $N \geq 2$ bodies of homogeneous density. The planets are taken as spinning spheroids with revolution axis perpendicular to the equatorial (ecliptic) plane. Then it can be proven that the dynamics of this system decouples into the motion of the centres of mass and the motion of each rigid body, where the interaction between any two centres of mass is given by a potential of the form $-a/r - b/r^3$ with $a > 0$ (for $N = 3$, see [9] and [29]). Assume that the system formed by the mass centres is in a relative equilibrium. Then the motion of a small object (e.g. a satellite) includes nearly circular periodic orbits located close to a primary.*

Corollary 4.7 *In the restricted $(N + 1)$-body problem with molecular potentials of the form (1), there exist nearly circular periodic orbits close to a primary. These orbits are unstable.*

Acknowledgements

This work was supported by a MCYT/FEDER grant MTM2008-03437, by a CIRIT grant number 2009SGR-410, by ICREA Academia, and by the Centre de Recerca Matemàtica (CRM), Bellaterra, Barcelona. CS wishes to thank the Department of Mathematics at Universitat Autònoma de Barcelona and CRM for their hospitality.

5 Appendix

For readers convenience we state the averaging theorem that we apply in Section 3; for proof and details, see [31] and [11].

Theorem 5.1 *Consider a differential system in the form*

$$\dot{\mathbf{x}} = \varepsilon F\left(t, \mathbf{x}(t)\right) + \varepsilon^2 R\left(t, \mathbf{x}(t), \varepsilon\right), \tag{29}$$

with $\mathbf{x} \in U \subset \mathbb{R}^n$, U *a bounded domain and* $t \geq 0$ *and* $F\left(t, \mathbf{x}\right)$ *and* $R\left(t, \mathbf{x}, \varepsilon\right)$ *periodic functions in* t *of period* T. *Consider the averaged system*

$$\dot{\mathbf{y}} = \varepsilon f(\mathbf{y})$$

where

$$f(\mathbf{y}) = \frac{1}{T} \int\limits_0^T F\left(s, \mathbf{y}\right) ds,$$

and assume that the vector functions $F, R, D_{\mathbf{x}}F, D^2_{\mathbf{x}}F$ *and* $D_{\mathbf{x}}R$ *and continuous and uniformly bounded by a constant* M *in* $[0, \infty) \times U$ *for* ε *in a neighborhood of zero. Then*

(1) If $a \in U$ *is a singular point of the averaged system with* $detD_{\mathbf{x}}F(a) \neq 0$, *then for* $\varepsilon > 0$ *sufficiently small there exists a unique* T*-periodic solution* $\mathbf{x}_\varepsilon(t)$ *of system (29) such that* $\mathbf{x}_\varepsilon(0) \to a$ *as* $\varepsilon \to 0$.

(2) If the singular point a *of the averaged system is hyperbolic then, for* $|\varepsilon| > 0$ *sufficiently small, the corresponding solution* $\mathbf{x}_\varepsilon(t)$ *of (29) is hyperbolic and has the same stability as* a.

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