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LIMIT CYCLES, INVARIANT MERIDIANS AND PARALLELS FOR POLYNOMIAL VECTOR FIELDS ON THE TORUS

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ABSTRACT. We study the polynomial vector fields of arbitrary degree in $\mathbb{R}^3$ having the 2–dimensional torus

$$\mathbb{T}^2 = \{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2 - a^2)^2 + z^2 = 1\} \text{ with } a > 1,$$

invariant by their flow. We characterize all the possible configurations of invariant meridians and parallels that these vector fields can exhibit. Furthermore we analyze when these invariant either meridians or parallels can be limit cycles.

1. Introduction and statement of the results

Polynomial vector fields or equivalently polynomial differential equations in the plane have been intensively studied since 1900 due to the second part of the 16-th Hilbert problem, which mainly states: Provide un upper bound for the maximum number of limit cycles that a given polynomial vector field of degree n can exhibit in function of n; for more details see [6, 7, 9].

There is also a big interest on the invariant algebraic curves of polynomial vector fields in the plane after 1886 when Darboux [4] showed that a sufficient number of them forces the existence of a first integral. Of course knowing a first integral of a two-dimensional differential system we can describe its phase portrait, the main objective of the qualitative theory of the differential equations. The Darboux theory of integrability has been strongly developed, see for instance the survey [10]. In particular, recently several papers have been published looking for particular invariant algebraic curves, as invariant straight lines (see for instance [17, 16, 15, 1]), or invariant circles for polynomial vector fields on the sphere (see for instance [2, 8, 12]), or invariant hyperplanes in $\mathbb{R}^n$ (see [13, 11]), ...

We consider the 2–dimensional torus

$$\mathbb{T}^2 = \{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2 - a^2)^2 + z^2 = 1\} \text{ with } a > 1.$$

Observe that $\mathbb{T}^2$ in cylindrical coordinates (r, θ, z) is

$$(r^2 - a^2)^2 + z^2 = 1, \tag{1}$$

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where $x = r\cos\theta$, $y = r\sin\theta$. Here we shall study the polynomial vector fields of arbitrary degree in $\mathbb{R}^3$ having the 2–dimensional torus $\mathbb{T}^2$ invariant by their flow. We shall characterize all the possible configurations of invariant meridians and parallels that these vector fields can exhibit. Additionally we shall consider when these invariant either meridians or parallels can be limit cycles.

As usual we denote by $\mathbb{R}[x, y, z]$ the ring of the polynomials in the variables x, y and z with real coefficients. By definition a *polynomial differential system* in $\mathbb{R}^3$ is a system of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = P_1(x, y, z), \quad \frac{\mathrm{d}y}{\mathrm{d}t} = P_2(x, y, z), \quad \frac{\mathrm{d}z}{\mathrm{d}t} = P_3(x, y, z), \tag{2}$$

where $P_i \in \mathbb{R}[x, y, z]$ for $i = 1, 2, 3$. If m_i is the degree of P_i, then we say that $m = \max_i\{m_i\}$, $i = 1, 2, 3$ is the *degree* of the polynomial differential system.

We denote by

$$\mathcal{X} = P_1(x, y, z)\frac{\partial}{\partial x_1} + P_2(x, y, z)\frac{\partial}{\partial x_2} + P_3(x, y, z)\frac{\partial}{\partial x_3} \tag{3}$$

the *polynomial vector field* associated to system (2) of degree m.

An *invariant algebraic surface* for system (2) or for the vector field (3) is an algebraic surface $f(x, y, z) = 0$ with $f \in \mathbb{R}[x, y, z]$, such that for some polynomial $K \in \mathbb{R}[x, y, z]$, we have $\mathcal{X}f = Kf$. Therefore, if a solution curve of system (2) has a point on the algebraic surface $f = 0$, then the whole solution curve is contained in $f = 0$. The polynomial K is called the *cofactor* of the invariant algebraic surface $f = 0$. We remark that if the polynomial system has degree m, then any cofactor has at most degree $m - 1$.

We consider polynomial vector fields $\mathcal{X}$ of degree m in $\mathbb{R}^3$ such that $\mathbb{T}^2$ is an invariant algebraic surface, i.e. if

$$\mathcal{T} = \mathcal{T}(x, y, z) = (x^2 + y^2 - a^2)^2 + z^2 - 1, \tag{4}$$

then $\mathcal{X}\mathcal{T} = K\mathcal{T}$, where K is a polynomial of degree at most $m - 1$. Such vector fields will be called *polynomial vector fields on* $\mathbb{T}^2$ *of degree* m.

On $\mathbb{T}^2$ we define *meridians* and *parallels* as the curves obtained by the intersection of this torus with the planes containing the $z-$axis and the planes orthogonal to the $z-$axis, respectively. More precisely, the meridians α are obtained intersecting the planes $gx+hy = 0$ (where $g, h \in \mathbb{R}$) with $\mathbb{T}^2$, or simply by $\alpha = \{gx+hy = 0\}\cap\mathbb{T}^2$. So meridians always are in pairs. We get parallels β intersecting the planes $z = k$ (where $k \in [-1, 1]$) with $\mathbb{T}^2$, or simply by $\beta = \{z = k\} \cap \mathbb{T}^2$. If $k \neq \pm 1$ then $\mathcal{X}$ has two parallels, otherwise only one parallel.

We say that a meridian α is *invariant* by the flow of the polynomial vector fields $\mathcal{X}$ on $\mathbb{T}^2$ if $\mathcal{X}(gx + hy) = K_\alpha(gx + hy)$, for some $K_\alpha \in \mathbb{R}[x, y, z]$. Note that since the torus $\mathcal{T} = 0$ and the plane $gx + hy = 0$ are invariant by the flow of $\mathcal{X}$, their intersection (the meridians α) are also invariant by the flow of $\mathcal{X}$. In a similar way we define *invariant* parallel.

Our main results are the following.

Theorem 1. *The following statements hold.*

(a) *Let $\mathcal{X}$ be a polynomial vector field on $\mathbb{T}^2$ of degree $m > 1$. Assume that $\mathcal{X}$ has finitely many invariant meridians and invariant parallels.*

(a.1) *The number of invariant meridians of $\mathcal{X}$ is at most $2(m - 1)$.*

$(a.2)$ *The number of invariant parallels of* $\mathcal{X}$ *is at most* $2(m-2)$.

(b) *Given* $k \in \{0, 1, \dots, m-1\}$ *and* $l \in \{0, 1, \dots, 2(m-2)\}$ *there is a polynomial vector field on* $\mathbb{T}^2$ *of degree* m *having exactly* $2k$ *invariant meridians and* l *invariant parallels. Other finite configurations of invariant meridians and invariant parallels are not allowed.*

In the next theorem we study when these invariant meridians and parallels are limit cycles.

Theorem 2. *There are polynomial vector field* $\mathcal{X}$ *on* $\mathbb{T}^2$ *of degree* m *having exactly*

(a) *either* $2k$ *invariant meridians which are limit cycles for* $k = 1, 2, \dots, m-1$;

(b) *or* l *invariant parallels which are limit cycles for* $l = 1, 2, \dots, 2(m-2)$.

Moreover we can realize these configurations of limit cycles in such a way that they are all the limit cycles of $\mathcal{X}$ *being stable or unstable alternately.*

Theorems 1 and 2 are proved in section 3. Theorem 1 needs for its proof the notion of extactic algebraic surface which is defined in section 2.

Theorems 1 and 2 shows the existence of polynomial vector fields of degree 2 having two invariant meridians (the maximum and the minimum number of possible invariant meridians in this case), which eventually can be limit cycles. In fact the next result shows that when a polynomial vector field on $\mathbb{T}^2$ of degree 2 has two invariant meridians always they are limit cycles.

Proposition 3. *A polynomial vector field* $\mathcal{X}$ *on* $\mathbb{T}^2$ *of degree* 2 *having nonzero finitely many invariant meridians, has exactly two invariant meridians, which are the unique two limit cycles of* $\mathcal{X}$ *on* $\mathbb{T}^2$, *one stable and the other unstable.*

2. The extactic algebraic surface

In our approach we will use the extactic algebraic surface which we define in the sequel.

Let $\mathcal{X}$ be a polynomial vector field on $\mathbb{R}^3$ and let W be a finite $\mathbb{R}$–vector subspace of $\mathbb{R}[x, y, z]$. The *extactic algebraic surface of* $\mathcal{X}$ *associated to* W is

$$\mathcal{E}_W(\mathcal{X}) = \det \begin{pmatrix} v_1 & v_2 & \cdots & v_l \\ \mathcal{X}(v_1) & \mathcal{X}(v_2) & \cdots & \mathcal{X}(v_l) \\ \vdots & \vdots & \cdots & \vdots \\ \mathcal{X}^{l-1}(v_1) & \mathcal{X}^{l-1}(v_2) & \cdots & \mathcal{X}^{l-1}(v_l) \end{pmatrix} = 0,$$

where $\{v_1, \dots, v_l\}$ is a basis of W, $l = \dim(W)$ is the dimension of W, and $\mathcal{X}^j(v_i) = \mathcal{X}^{j-1}(\mathcal{X}(v_i))$. It is known due to the properties of the determinant and of the derivation that the definition of extactic algebraic surface is independent of the chosen basis of W.

In fact we learn this definition from the paper of Jorge Pereira [14], but this notion goes back at least to the work of Lagutinskii at the beginning of the XX century, see the references quoted in [5]. We have used the definition of $\mathcal{E}_W(\mathcal{X})$ in different papers, see for instance [3, 12].

The notion of extactic algebraic surface $\mathcal{E}_W(\mathcal{X})$ is important in this paper because it allows to detect when an algebraic surface $f = 0$ with $f \in W$ is invariant by the polynomial vector field $\mathcal{X}$, see Proposition 4 proved in [11] and [12] for polynomial vector fields in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively.

Proposition 4. *Let $\mathcal{X}$ be a polynomial vector field in $\mathbb{R}^3$ and let W be a finite $\mathbb{R}$–vector subspace of $\mathbb{R}[x, y, z]$ with $\dim(W) > 1$. Then every algebraic invariant surface $f = 0$ for the vector field $\mathcal{X}$, with $f \in W$, is a factor of $\mathcal{E}_W(\mathcal{X})$.*

Thus, for instance, by Proposition 4 if $f = 0$ is an invariant plane of the polynomial vector field $\mathcal{X}$, then f is a factor of $\mathcal{E}_W(\mathcal{X})$ with W generated by $\{1, x, y, z\}$.

3. Vector fields of degree m

Let $\mathcal{X} = (P_1, P_2, P_3)$ be a polynomial vector field on $\mathbb{T}^2$ of degree m associated to differential system (2). For determining the invariant meridians of $\mathcal{X}$ we have to find the curves obtained by the intersection of $\mathbb{T}^2$ and the planes of the form $gx + hy = 0$ such that they are invariant by the flow of $\mathcal{X}$. By Proposition 4 it is sufficient to show that $gx + hy$ is a factor of the polynomial $\mathcal{E}_{\{x,y\}}(\mathcal{X})$. So we have to calculate

$$\mathcal{E}_{\{x,y\}}(\mathcal{X}) = \det\begin{pmatrix} x & y \\ \mathcal{X}(x) & \mathcal{X}(y) \end{pmatrix} = \det\begin{pmatrix} x & y \\ \dot{x} & \dot{y} \end{pmatrix} = x\dot{y} - y\dot{x} = xP_2 - yP_1 = 0.$$

In order to study invariant parallels we must consider the intersection of the planes $z =$ constant $\in [-1, 1]$ with the torus $\mathbb{T}^2$ such that they are invariant by the flow of $\mathcal{X}$. By Proposition 4 it is sufficient to show that $z-$ constant is a factor of the polynomial $\mathcal{E}_{\{1,z\}}(\mathcal{X})$. Therefore we must calculate

$$\mathcal{E}_{\{1,z\}}(\mathcal{X}) = \det\begin{pmatrix} 1 & z \\ \mathcal{X}(1) & \mathcal{X}(z) \end{pmatrix} = \det\begin{pmatrix} 1 & z \\ 0 & \dot{z} \end{pmatrix} = \dot{z} = P_3 = 0.$$

Proof of Theorem 1. In cylindrical coordinates $(x, y, z) = (r\cos\theta, r\sin\theta, z)$ the vector field (3) becomes

$$\begin{aligned} \mathcal{X} = \; & \frac{1}{r}\big(P_1(r\cos\theta, r\sin\theta, z)\, r\cos\theta + P_2(r\cos\theta, r\sin\theta, z)\, r\sin\theta\big)\frac{\partial}{\partial r} + \\ & \frac{1}{r^2}\big(P_1(r\cos\theta, r\sin\theta, z)\, r\sin\theta - P_2(r\cos\theta, r\sin\theta, z)\, r\cos\theta\big)\frac{\partial}{\partial\theta} + \\ & P_3(r\cos\theta, r\sin\theta, z)\frac{\partial}{\partial z}. \end{aligned}$$

An invariant meridian of the vector field $\mathcal{X} = (P_1, P_2, P_3)$ is given by the intersection of a plane $gx + hy = 0$ with $\mathbb{T}^2$. In this case the polynomial $gx + hy$ must be a factor of $\mathcal{E}_{\{x,y\}}(\mathcal{X}) = xP_2 - yP_1 = (x^2 + y^2)\,\dot{\theta}$. Since the polynomial $xP_2 - yP_1$ has at most degree $m+1$, $\mathcal{X}$ has at most $m-1$ invariant planes of the form $gx + hy = 0$, i.e. at most $2(m-1)$ invariant meridians. This proves statement $(a.1)$ of Theorem 1.

In order to determine the invariant parallels we must consider the intersection of the planes $z =$ constant $\in [-1, 1]$ with the torus $\mathbb{T}^2$. The vector field $\mathcal{X} = (P_1, P_2, P_3)$ must satisfy $X\mathcal{T} = K\mathcal{T}$. So we get

$$4(x^2 + y^2 - a^2)(xP_1 + yP_2) + 2zP_3 = K(x, y, z)\big((x^2 + y^2 - a^2)^2 + z^2 - 1\big) \tag{5}$$

holds for all $x, y, z \in \mathbb{R}$. We write $P_3 = h(x,y,z)F_2(z) = h(x,y,z)\prod_{i=1}^{l}(z-a_i)$, in such a way that $z - z_0$ for all $z_0 \in \mathbb{R}$ is not a factor of polynomial $h(x,y,z)$. Then

$$\mathcal{E}_{\{1,z\}}(\mathcal{X}) = h(x,y,z)\prod_{i=1}^{l}(z-a_i),$$

As (5) holds for all $x, y, z \in \mathbb{R}$, in particular is true for $x = y = 0$. So we get

$$2zh(0,0,z)\prod_{i=1}^{l}(z-a_i) = \left(\sum_{i=0}^{m-1} k_i z^i\right)(a^4 + z^2 - 1).$$

From this equation it follows that $k_0 = 0$ and consequently

$$2zh(0,0,z)\prod_{i=1}^{l}(z-a_i) = z\left(\sum_{i=0}^{m-2} \tilde{k}_i z^i\right)(a^4 + z^2 - 1).$$

As $a^4 + z^2 - 1$ cannot be factored by real polynomials and $h(x,y,z)$ has no factors of the form $z - z_0$, we get that $l \le m-2$. So $\mathcal{E}_{\{1,z\}}(\mathcal{X})$ has at most $m-2$ factors of the form $z =$ constant. Hence $\mathcal{X}$ has at most $m-2$ invariant planes of the form $z =$ constant, and consequently $\mathcal{X}$ has at most $2(m-2)$ invariant parallels. This completes the proof of statement $(a.2)$ of Theorem 1.

In short to conclude the proof of Theorem 1 we consider $\mathcal{X}$ the polynomial vector field on $\mathbb{T}^2$ of degree m associated to

$$\begin{aligned}
\dot{x} = &\ \ xzF_2(z) - yz^{m-k-1}F_1(x,y),\\
\dot{y} = &\ \ yzF_2(z) + xz^{m-k-1}F_1(x,y),\\
\dot{z} = &\ \ 2(a^2(a^2 - x^2 - y^2) + z^2 - 1)F_2(z),
\end{aligned}$$

where $F_1(x,y) = \prod_{i=1}^{k}(a_i x + b_i y)$ with $a_i, b_i \in \mathbb{R}, i = 1, \ldots, k$ and $0 \le k \le m-1$, if $k = 0$ we define $F_1(x,y) = 1$; and $F_2(z) = \prod_{i=1}^{l}(z - z_i)$ with $-1 \le z_1 < z_2 < \cdots < z_l \le 1$ and $0 \le l \le m-2$, if $l = 0$ then we take $F_2(x,y) = 1$.

For $\mathcal{X}$ we obtain that

$$\begin{aligned}
\mathcal{E}_{\{x,y\}}(\mathcal{X}) = &\ \ (x^2 + y^2)z^{m-k-1}F_1(x,y),\\
\mathcal{E}_{\{1,z\}}(\mathcal{X}) = &\ \ 2(a^2(a^2 - x^2 - y^2) + z^2 - 1)F_2(z).
\end{aligned}$$

So a polynomial of form $gx + hy$ only divides $F_1(x,y)$ in $\mathcal{E}_{\{x,y\}}(\mathcal{X})$, and for each plane $\{gx + hy = 0\} \cap \mathbb{T}^2$, $\mathcal{X}$ has two invariant meridians on $\mathbb{T}^2$. A polynomial $z - z_i$ only divides $F_2(z)$ in $\mathcal{E}_{\{1,z\}}(\mathcal{X})$, and if $z_i \in (-1,1)$ for $i = 1, \ldots, l$ then $\mathcal{X}$ has exactly $2l$ invariant parallels, if either $z_1 = -1$ or $z_l = 1$, then $\mathcal{X}$ has $2l - 1$ invariant parallels. Finally if $z_1 = -1$ and $z_l = 1$, then $\mathcal{X}$ has $2(l-1)$ invariant parallels. This concludes the proof of statement (c) of Theorem 1. $\square$

Statement (a) of Theorem 2 follows immediately from the next result.

Proposition 5. *Let $\mathcal{X}$ be the polynomial vector field on $\mathbb{T}^2$ of degree m associated to*

$$\begin{aligned}
\dot{x} = &\ \ xz - y(z+z_0)^{m-k-1}F_1(x,y),\\
\dot{y} = &\ \ yz + x(z+z_0)^{m-k-1}F_1(x,y),\\
\dot{z} = &\ \ 2(a^2(a^2 - x^2 - y^2) + z^2 - 1),
\end{aligned}$$

where $z_0 > 1$, $1 \leq k \leq m-1$ *and* $F_1(x,y) = \prod_{i=1}^{k}(a_i x + b_i y)$ *with* $a_i, b_i \in \mathbb{R}$ *for* $i = 1, \ldots, k$. *Then* $\mathcal{X}$ *has exactly* $2k$ *invariant meridians which are limit cycles. Moreover these limit cycles are stable or unstable alternately.*

Proof. We get that $\mathcal{X}$ satisfies (5) with $K(x,y,z) = 4z$, and that $\mathcal{E}_{\{x,y\}}(\mathcal{X}) = -(x^2+y^2)(z+z_0)^{m-k-1}F_1(x,y)$. We observe that $\mathcal{X}$ has no invariant parallels because $z - z_0$ is not a factor of $\dot{z}$ for any constant z_0. So $\mathcal{X}$ has $2k$ invariant meridians. In cylindrical coordinates $\mathcal{X}$ becomes

$$\begin{aligned} \dot{r} =& \quad rz, \\ \dot{\theta} =& \quad (z+z_0)^{m-k-1}F_1(r\cos\theta, r\sin\theta), \\ \dot{z} =& \quad 2(a^2(a^2-r^2)+z^2-1). \end{aligned}$$

If (r^*, θ^*, z^*) is a singular point of $\mathbb{T}^2$, then from $\dot{r} = rz = 0$ we get that $z^* = 0$, because $r^* > 0$. For $z^* = 0$, we get from (1) that $(r^*)^2 - a^2 = \pm 1$. Therefore $\dot{z} = 2(\pm a^2 - 1) = 0$. As $a > 1$, then $\dot{z} \neq 0$. So $\mathcal{X}$ has no singular points, and consequently the orbits on the invariant meridians are periodic orbits.

Between two consecutive invariant meridians the $\text{sign}(\dot{\theta}) = \text{sign}(F_1(r\cos\theta, r\sin\theta) \in \{-1, 1\}$ because $z_0 > 1$, and these signs change when we cross one meridian. So the periodic orbits on the invariant meridians are stable or unstable limit cycles alternately. □

Statement (b) of Theorem 2 follows immediately from the next result.

Proposition 6. *Let* $\mathcal{X}$ *be the polynomial vector field on* $\mathbb{T}^2$ *of degree* m *associated to*

$$\begin{aligned} \dot{x} =& \quad -y(z-z_0)^{m-1} + xzF_2(z), \\ \dot{y} =& \quad x(z-z_0)^{m-1} + yzF_2(z), \\ \dot{z} =& \quad 2(a^2(a^2-x^2-y^2)+z^2-1)F_2(z), \end{aligned}$$

where $1 \leq l \leq m-2$, $z_0 \notin \{0, z_1, \ldots z_k\}$ *and* $F_2(z) = \prod_{i=1}^{l}(z - z_i)$ *with* $-1 \leq z_1 < z_2 < \cdots < z_l \leq 1$. *Then the number of invariant parallels of* $\mathcal{X}$ *which are limit cycles is* $2l$ *if* $z_i \in (-1,1)$ *for all* $i = 1, 2, \ldots, l$, $2l-1$ *if either* $x_1 = -1$ *or* $x_l = 1$, $2(l-1)$ *if* $x_1 = -1$ *and* $x_l = 1$. *Moreover these limit cycles are stable or unstable alternately.*

Proof. We get that $\mathcal{X}$ satisfies (5) for $K(x,y,z) = 4zF_2(z)$. In cylindrical coordinates $\mathcal{X}$ becomes

$$\begin{aligned} \dot{r} =& \quad rzF_2(z), \\ \dot{\theta} =& \quad (z-z_0)^{m-1}, \\ \dot{z} =& \quad 2(a^2(a^2-r^2)+z^2-1)F_2(z). \end{aligned}$$

We suppose that $(r^*, \theta^*, z^*) \in \mathbb{T}^2$ is a singular point of $\mathcal{X}$, then since $\dot{r} = rzF_2(z)$ we have that $z^* \in \{0, z_1, \ldots, z_l\}$ because $r^* > 0$, but since $\dot{\theta} \neq 0$ at (r^*, θ^*, z^*) there is no singular points of $\mathcal{X}$ on $\mathbb{T}^2$. So the orbits on the invariant parallels $\{z = z_i\} \cap \mathbb{T}^2$ are periodic. As $a^2(a^2-r^2)+z^2-1 \neq 0$ for $-1 < z < 1$, the periodic orbits on the invariant parallels are either stable or unstable limit cycles according $\text{sign}(\dot{z}) = \text{sign}(F_2(z))$. Therefore except on the invariant parallels all orbits are not periodic. So the number of invariant parallels of $\mathcal{X}$ which are limit cycles is $2l$ if $z_i \in (-1,1)$ for $i = 1, 2, \ldots, l$, $2l-1$ if either $x_1 = -1$ or $x_l = 1$, $2(l-1)$ if $x_1 = -1$ and $x_l = 1$. Moreover they are stable or unstable alternately. □

4. Quadratic vector fields

Let $\mathcal{X}$ be an arbitrary polynomial vector field on $\mathbb{R}^3$ of degree 2

$$\mathcal{X}(x,y,z) = \left(\sum_{i+j+k=0}^{2} a_{ijk} x^i y^j z^k, \sum_{i+j+k=0}^{2} b_{ijk} x^i y^j z^k, \sum_{i+j+k=0}^{2} c_{ijk} x^i y^j z^k \right).$$

This vector field is a polynomial vector field on $\mathbb{T}^2$ if $\mathcal{X}\mathcal{T} = K\mathcal{T}$, where the cofactor $K(x,y,z) = \sum_{i+j+k=0}^{1} k_{ijk} x^i y^j z^k$ is a polynomial of degree one, and $\mathcal{T}$ is given in (4).

Equalling to zero the coefficients of the monomials $x^i y^j z^k$ of the polynomial $X\mathcal{T} - K\mathcal{T} = 0$, we get a nonlinear system with unknowns the variables $a_{ijk}, b_{ijk}, c_{ijk}, k_{ijk}$. Solving this system and renaming the variables, we get that the polynomial vector fields $\mathcal{X}_2$ of degree 2 on $\mathbb{T}^2$ are the ones associated to the differential systems

$$\begin{aligned} \dot{x} &= a_0 xz + y(a_1 + a_2 x + a_3 y + a_4 z), \\ \dot{y} &= a_0 yz - x(a_1 + a_2 x + a_3 y + a_4 z), \\ \dot{z} &= 2a_0(-1 + a^4 - a^2 x^2 - a^2 y^2 + z^2). \end{aligned} \tag{6}$$

Moreover the cofactor of $\mathcal{T} = 0$ is $K(x,y,z) = 4a_0 z$. We remark that if $a_0 = 0$ then system (6) has two independent polynomial first integrals, namely $\mathcal{T}$ and z. So all the parallels are invariant when $a_0 = 0$, and we do not consider such vector fields in this paper. Hence *in what follows* $a_0 \neq 0$.

Proof of Proposition 3. We have seen that a polynomial vector field $\mathcal{X}$ on $\mathbb{T}^2$ of degree 2 can be written as the vector field $\mathcal{X}_2$ given in (6). Since by assumptions $\mathcal{X}_2$ has invariant meridians, then there exist $g, h \in \mathbb{R}$ such that the polynomial $gx + hy$ divides

$$\mathcal{E}_{\{x,y\}}(\mathcal{X}_2) = -(x^2 + y^2)(a_1 + a_2 x + a_3 y + a_4 z).$$

So $a_1 = a_4 = 0$, $a_2 = \lambda g$ and $a_3 = \lambda h$, with $|a_2|^2 + |a_3|^2 \neq 0$. If $a_3 \neq 0$ these meridians $\alpha_\pm$ are given by

$$\begin{aligned} ((1 + a_2^2/a_3^2)x^2 - a^2)^2 + z^2 - 1 &= 0, \\ a_2 x + a_3 y &= 0. \end{aligned}$$

In cylindrical coordinates $\mathcal{X}_2$ with $a_1 = a_4 = 0$ has the form

$$(\dot{r}, \dot{\theta}, \dot{z}) = (rz, r(a_2 \cos\theta + a_3 \sin\theta), 2a_0(-1 + a^2(a^2 - r^2) + z^2)).$$

Assume that $p = (r_0, \theta_0, z_0) \in \mathbb{T}^2$ is a singular point of $\mathcal{X}_2$. From $\dot{r} = rz = 0$ and since $r_0 \neq 0$, we get that $z_0 = 0$. So $\dot{z} = 2a_0(-1 + a^2(a^2 - r_0^2))$. Since $p \in \mathbb{T}^2$ and $z_0 = 0$, we get that $(r_0^2 - a^2)^2 = 1$, and this implies that $\dot{z} = 2a_0(-1 \pm a^2) \neq 0$. So $\mathcal{X}_2$ has no singular points on $\mathbb{T}^2$.

Observe that $\operatorname{sign}(\dot{\theta}) = \operatorname{sign}(a_2 x + a_3 y)$, therefore $\dot{\theta}$ is zero only on the two invariant meridians contained in the plane $a_2 x + a_3 y = 0$, and that it changes its sign according the plane $a_2 x + a_3 y = 0$.

As $\mathcal{X}_2$ does not have singular points and the sign of $\dot{\theta}$ is different of zero, except on the two invariant meridians, all orbits outside these two meridians are not periodic. Moreover as $\operatorname{sign}(\dot{\theta})$ change of sign according with the plane $a_2 x + a_3 y = 0$, we get that one of invariant meridians is a stable limit cycle and the other is an unstable limit cycle. □

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