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ON THE BIRTH OF MINIMAL SETS FOR PERTURBED REVERSIBLE VECTOR FIELDS

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ABSTRACT. The results in this paper fit into a program to study the existence of periodic orbits, invariant cylinders and tori filled with periodic orbits in perturbed reversible systems. Here we focus on bifurcations of one-parameter families of periodic orbits for reversible vector fields in $\mathbb{R}^4$. The main used tools are normal forms theory, Lyapunov-Schmidt method and averaging theory.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

One of the main objectives of the qualitative theory of differential equations is to study the persistence or bifurcation of minimal invariant sets for a given differential equation under small perturbations. Here the unperturbed systems are reversible vector fields and symmetry breaking bifurcations are considered.

As usual $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{R}$ denotes the set of real numbers. Recall that a differential system

$$\frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}} = f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^4, \tag{1}$$

is said to be *φ-reversible* if there exists a smooth involution $\varphi : \mathbb{R}^4 \to \mathbb{R}^4$, with $\dim(\mathrm{Fix}(\varphi)) = 2$, such that

$$D\varphi(\mathbf{x})f(\mathbf{x}) = -f(\varphi(\mathbf{x})).$$

Clearly if $\mathbf{x}(t)$ is a solution of system (1), then $\varphi(\mathbf{x}(-t))$ is also a solution of (1). A *solution* $\mathbf{x}(t)$ of (1) is said to be *symmetric* if $\mathbf{x}(t) = \varphi(\mathbf{x}(-t))$. We point out that a solution is symmetric if it intersects $\mathrm{Fix}(\varphi)$ in at least one point. Moreover if $\mathbf{x}(t)$ intersects $\mathrm{Fix}(\varphi)$ in more than one point, then it is a *symmetric periodic orbit.*

We focus our attention on small perturbations of systems (1), with $\mathbf{x} = (x, y, z, w)$, of the form

$$\begin{array}{rcl} \dot{x} & = & -py - y\left(a(x^2+y^2) + b(z^2+w^2)\right) + \dots, \\ \dot{y} & = & px + x\left(a(x^2+y^2) + b(z^2+w^2)\right) + \dots, \\ \dot{z} & = & -qw - w\left(c(x^2+y^2) + d(z^2+w^2)\right) + \dots, \\ \dot{w} & = & qz + z\left(c(x^2+y^2) + d(z^2+w^2)\right) + \dots, \end{array} \tag{2}$$

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for a, b, c, $d \in \mathbb{R}$ and $p, q \in \mathbb{N}$ with $(p, q) = 1$, where the dots mean high order terms.

System (2) is a normal form up to order 3 for reversible systems around elliptic equilibria in $\mathbb{R}^4$. These terms of third degree generically cannot be eliminated from (2) using change of coordinates.

Minimal invariant sets such as periodic orbits and invariant tori generically appear in one parameter families of reversible systems, due to the symmetry of the system with respect to the set $\text{Fix}(\varphi)$ (see more details in [7]).

One of the main tools in the study of the existence of minimal invariant sets for reversible systems is the method of Lyapunov-Schmidt (LS). For a description of this method see [6]. For instance, this method was employed in [2], [8] and [10] to detect the existence of periodic orbits and invariant tori for reversible systems in dimensions 4 and 6. In [6] the LS method was considered for studying the persistence of families of periodic orbits in a linear versal deformation of a reversible vector field.

In this paper we initiate a program to study bifurcations of one-parameter families of periodic orbits and invariant tori in reversible systems, by considering polynomial perturbations.

This program consists in considering a reversible differential system (1) with $f(0) = 0$ and $A = f'(0)$ invertible. Passing this system to a normal form up to order k, we can write it as

$$\dot{\mathbf{y}} = A\mathbf{y} + f_k(\mathbf{y}) + o(|\mathbf{y}|^{k+1}), \tag{3}$$

with f_k depending on the terms of degree k of the Taylor expansion of f around 0. The existence of one-parameter families of (symmetric) periodic orbits for system (1) with $f(0) = 0$ and $A = f'(0)$ can be studied applying the LS method to the equation (3). The cases treated here generically admit such families.

Once the existence of one-parameter families of periodic orbits is obtained, we consider an auxiliary system of the form

$$\dot{\mathbf{y}} = A\mathbf{y} + \varepsilon f_k(\mathbf{y}), \tag{4}$$

and its perturbed version

$$\dot{\mathbf{y}} = A\mathbf{y} + \varepsilon f_k(\mathbf{y}) + \varepsilon P_m(\mathbf{y}), \tag{5}$$

where P_m is a polynomial vector field and ε is a small parameter, considered in (4) and (5) for technical reasons. Finally, we study the existence of minimal sets for the perturbed system (5). This analysis is done using the algorithm presented in subsection 2.1.

Although we present a study of specific perturbations of a class of reversible systems, we emphasize that this method is applicable to many more systems. In fact, this work fits into a general program in understanding in depth the dynamics of vector fields nearby reversible systems. The program presented here allows us to discuss the shape of some minimal sets and its persistence or bifurcation under small perturbations.

Our main results are the following.

Theorem 1. *Consider a reversible differential system* (2) *with* $p, q \in \mathbb{N}$, *p and q coprime. Assume that one of the following conditions is satisfied:*

$$\text{(i) } a > 0; \quad \text{(ii) } d > 0; \quad \text{(iii) } \frac{pd - qb}{ad - bc} > 0 \text{ and } \frac{pc - qa}{ad - bc} < 0.$$

Then there exists a family, parameterized by σ, *of* $\dfrac{2\pi}{1+\sigma}$*-periodic symmetric solutions of system* (2), *with* $\sigma \in (0, \sigma_0)$ *for* σ_0 *sufficiently small.*

Theorem 1 is proved in section 3.

Remark 2. *From the proof of Theorem 1 it becomes clear that if two (respectively three) of the conditions (i), (ii), (iii) are satisfied, then there exist two (respectively three) families of symmetric periodic solutions of system* (2).

Now consider the differential system

$$\begin{array}{rcl} \dot{x} & = & -py - \varepsilon y \left(a(x^2+y^2) + b(z^2+w^2)\right) + \varepsilon f_1(x,y,z,w), \\ \dot{y} & = & px + \varepsilon x \left(a(x^2+y^2) + b(z^2+w^2)\right) + \varepsilon f_2(x,y,z,w), \\ \dot{z} & = & -qw - \varepsilon w \left(c(x^2+y^2) + d(z^2+w^2)\right) + \varepsilon f_3(x,y,z,w), \\ \dot{w} & = & qz + \varepsilon z \left(c(x^2+y^2) + d(z^2+w^2)\right) + \varepsilon f_4(x,y,z,w), \end{array} \tag{6}$$

where $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ is a small parameter, $a, b, c, d \in \mathbb{R}$, $p, q \in \mathbb{N}$, p and q coprime and f_1, f_2, f_3, f_4 are arbitrary polynomials of degree 3. The unperturbed system (6) with $\varepsilon = 0$ is a linear center in $\mathbb{R}^4$ whose periodic orbits are in resonance $p : q$.

Theorem 3. *For* $\varepsilon \neq 0$ *sufficiently small the perturbed system* (6) *can have one, two or three* 2*-dimensional invariant tori filled with periodic orbits, such that when* $\varepsilon \to 0$ *the two first tori (if exist) tend to two periodic orbits of* (6) *with* $\varepsilon = 0$, *and the third torus (if exists) tends to a torus filled with periodic orbits of* (6) *with* $\varepsilon = 0$. *In addition, explicit conditions on the coefficients of the perturbed system* (6) *for the existence of one, two or three invariant tori filled with periodic orbits are provided.*

This paper is organized as follows. In section 2 we present the results of the averaging theory, normal forms, the Lyapunov-Schmidt method and some properties of reversible vector fields that we shall need, together with some results on the existence of families of periodic orbits for system (2). In section 3 we prove Theorem 1 and in section 4 we prove Theorem 3.

2. Preliminaries

2.1. **Basic results on averaging theory.** In this subsection we present basic results on averaging theory that we shall need in the sequel of this paper.

We consider the differential system

$$\dot{\mathbf{x}}(t) = \varepsilon F(t, \mathbf{x}(t)) + \varepsilon^2 R(t, \mathbf{x}(t), \varepsilon), \tag{7}$$

with $\mathbf{x} \in D \subset \mathbb{R}^n$, D a bounded domain and $t \geq 0$. Moreover $F(t, \mathbf{x})$ and $R(t, \mathbf{x}, \varepsilon)$ are T–periodic in t.

The averaged system associated to system (7) is defined by

$$\dot{\mathbf{y}}(t) = \varepsilon f(\mathbf{y}(t)), \tag{8}$$

where

$$f(\mathbf{y}) = \frac{1}{T}\int_0^T F(s,\mathbf{y})ds. \tag{9}$$

The next theorem presents conditions for the existence of singular points of the averaged system (8) providing T–periodic orbits of system (7). For a proof see Theorem 2.6.1 of [9], Theorems 11.5 and 11.6 of [11], and Theorem 4.1.1 of [4].

Theorem 4. *We consider the differential system* (7) *and assume that the vector functions F, R, $D_{\mathbf{x}}F_1$, $D^2_{\mathbf{x}}F_1$ and $D_{\mathbf{x}}R$ are continuous and bounded by a constant M (independent of ε) in $[0,\infty)\times D$ with $-\varepsilon_0 < \varepsilon < \varepsilon_0$. Moreover we suppose that F and R are T–periodic in t with T independent of ε.*

(a) *If $a \in D$ is a singular point of the averaged system* (8) *such that* $\det(D_{\mathbf{x}}f(a)) \neq 0$ *then, for $|\varepsilon| > 0$ sufficiently small, there exists a T–periodic solution $\mathbf{x}_\varepsilon(t)$ of system* (7) *such that $\mathbf{x}_\varepsilon(0) \to a$ as $\varepsilon \to 0$.*
(b) *If the singular point a of the averaged system* (8) *is hyperbolic then, for $|\varepsilon| > 0$ sufficiently small the corresponding periodic solution $\mathbf{x}_\varepsilon(t)$ of system* (7) *is unique, hyperbolic and its return map has the same stability type as a.*

We remark that although the classical averaging method given by Theorem 4 looks for isolated periodic orbits, it is not expected that it can find symmetric isolated periodic orbit for reversible systems. The algorithm described in [8] and based in Theorem 4 allows to look for isolated cylinders and tori filled with periodic orbits. We review this algorithm in what follows.

Consider the differential system

$$\dot{\mathbf{x}} = A\mathbf{x} + \varepsilon F(\mathbf{x}), \tag{10}$$

with

$$A = \begin{pmatrix} 0 & -N & 0 & 0 \\ N & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

where $N \in \mathbb{Q}\setminus\{1\}$. Let $F = (F_1, F_2, F_3, F_4)$.

Changing the variables $\mathbf{x} = (x, y, z, w)$ to (r, θ, R, φ) through

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = R\cos\left(\theta + \frac{1-N}{N}\varphi\right), \quad w = R\sin\left(\theta + \frac{1-N}{N}\varphi\right),$$

system (10) is transformed into the system

$$\begin{aligned} \dot{r} &= \varepsilon G_1(r,\theta,R,\varphi), \\ \dot{\theta} &= N + \varepsilon G_2(r,\theta,R,\varphi), \\ \dot{R} &= \varepsilon G_3(r,\theta,R,\varphi), \\ \dot{\varphi} &= N + \varepsilon G_4(r,\theta,R,\varphi), \end{aligned} \tag{11}$$

where

$$
\begin{aligned}
G_1 &= \cos\theta\overline{F}_1(r,\theta,R,\varphi)+\sin\theta\overline{F}_2(r,\theta,R,\varphi),\\
G_2 &= \frac{1}{r}\left[\cos\theta\overline{F}_2(r,\theta,R,\varphi)-\sin\theta\overline{F}_1(r,\theta,R,\varphi)\right],\\
G_3 &= \cos\left(\theta+\frac{1-N}{N}\varphi\right)\overline{F}_3(r,\theta,R,\varphi)+\sin\left(\theta+\frac{1-N}{N}\varphi\right)\overline{F}_4(r,\theta,R,\varphi),\\
G_4 &= \frac{N}{1-N}\left[\frac{1}{r}\left(\cos\left(\theta+\frac{1-N}{N}\varphi\right)\overline{F}_4(r,\theta,R,\varphi)-\right.\right.\\
&\quad \left.\sin\left(\theta+\frac{1-N}{N}\varphi\right)\overline{F}_3(r,\theta,R,\varphi)\right)-\\
&\quad \left.\frac{1}{r}(\sin\theta\overline{F}_1(r,\theta,R,\varphi)-\cos\theta\overline{F}_2(r,\theta,R,\varphi))\right],
\end{aligned}
$$

where $\overline{F}_k(r,\theta,R,\varphi)=F_k\left(r\cos\theta, r\sin\theta, R\cos\left(\theta+\frac{1-N}{N}\varphi\right), R\sin\left(\theta+\frac{1-N}{N}\varphi\right)\right)$.

We change the independent variable t in system (11) by taking θ as the new independent variable. Thus system (11) becomes

$$
\begin{aligned}
\frac{dr}{d\theta} &= \frac{\varepsilon}{N}G_1(r,\theta,R,\varphi)+\mathcal{O}(\varepsilon^2),\\
\frac{dR}{d\theta} &= \frac{\varepsilon}{N}G_3(r,\theta,R,\varphi)+\mathcal{O}(\varepsilon^2),\\
\frac{d\varphi}{d\theta} &= 1+\frac{\varepsilon}{N}\big(G_4-G_2\big)(r,\theta,R,\varphi)+\mathcal{O}(\varepsilon^2),
\end{aligned}
\tag{12}
$$

From this last equation we have that any solution $(r(\theta),R(\theta),\varphi(\theta))$ of this system will be of the form $\varphi(\theta)=\theta+\varphi_0+\mathcal{O}(\varepsilon)$. Substituting this expression of $\varphi(\theta)$ into system (12), it reduces to the vector field $X^{\theta}_{\varphi_0}$, given by the differential system

$$
\begin{aligned}
\frac{dr}{d\theta} &= \frac{\varepsilon}{N}G_1(r,\theta,R,\theta+\varphi_0)+\mathcal{O}(\varepsilon^2),\\
\frac{dR}{d\theta} &= \frac{\varepsilon}{N}G_3(r,\theta,R,\theta+\varphi_0)+\mathcal{O}(\varepsilon^2),
\end{aligned}
\tag{13}
$$

with $\varphi_0\in\mathbb{S}^1$.

Now we shall study the periodic orbits of system (13) by means of Theorem 4. So we compute the averaged system of system (13), obtaining the vector field X_{φ_0}, given by the differential system

$$
\begin{aligned}
\frac{dr}{d\theta} &= \frac{\varepsilon}{N}g_1(r,R,\varphi_0),\\
\frac{dR}{d\theta} &= \frac{\varepsilon}{N}g_3(r,R,\varphi_0),
\end{aligned}
\tag{14}
$$

where $g_k(r,R,\varphi_0)=\dfrac{1}{2\pi p}\displaystyle\int_0^{2\pi p}G_k(r,\theta,R,\theta+\varphi_0)$ for $k=1,3$ with $N=p/q$, p and q coprime.

Now let $\varphi_0 \in \mathbb{S}^1$ be such that X_{φ_0} (system (14)) has a *simple* singular point $(r(\varphi_0), R(\varphi_0))$, that is, a singular point with

$$\det\left(\frac{\partial(g_1, g_3)}{\partial(r, R)}\bigg|_{r=r(\varphi_0), R=R(\varphi_0)} \right) \neq 0. \tag{15}$$

For each $\varphi_0 \in \mathbb{S}^1$ we apply Theorem 4 to system (14) to derive that for $\varepsilon \neq 0$ sufficiently small system (13) has a unique periodic orbit such that

$$\Big(r\big(\theta; (r(\varphi_0), R(\varphi_0))\big), R\big(\theta; (r(\varphi_0), R(\varphi_0))\big)\Big),$$

such that

$$\Big(r\big(0; (r(\varphi_0), R(\varphi_0))\big), R\big(0; (r(\varphi_0), R(\varphi_0))\big)\Big) \to (r(\varphi_0), R(\varphi_0)) \quad \text{when } \varepsilon \to 0.$$

Now if $I \subset \mathbb{S}^1$ is a connected subset and for every $\varphi_0 \in I \subset S^1$ the averaged system (14) has a singular point $(r(\varphi_0), R(\varphi_0))$ satisfying condition (15). Going back to system (13) we obtain that this system, for $\varepsilon \neq 0$ sufficiently small, has a continuous family of periodic orbits depending on φ_0, i.e. we get that (12) (and consequently systems (11) and (10)) has *an invariant cylinder filled with periodic orbits.* If $I = \mathbb{S}^1$ then system (12) (therefore system (11) and (10)) has *an invariant torus filled with periodic orbits.*

2.2. **Normal forms for reversible vector fields.** In this subsection we present a classical result on the normal forms for a class of vector fields. Similar results can be seen in [2] and [6].

The next lemma provides a relationship between reversible systems around an elliptic equilibria and system (2).

Lemma 5. *Let*

$$\dot{\mathbf{x}} = f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^4, \tag{16}$$

be a reversible differential system with $f(0) = 0$, such that the eigenvalues of $f'(0)$ are $\pm pi, \pm qi$, $p < q$, $p, q \in \mathbb{N}$ coprime and $(p, q) \neq (1, 3)$. Then there is a change of coordinates $\mathbf{x} = \mathbf{y} + \dots$ which writes system (16) as system (2) up to terms of order three.

Proof. Without loss of generality we can assume that $f(\mathbf{x}) = A\mathbf{x} + f_2(\mathbf{x}) + f_3(\mathbf{x}) + \dots$, where A is the real Jordan matrix

$$A = \begin{pmatrix} 0 & -p & 0 & 0 \\ p & 0 & 0 & 0 \\ 0 & 0 & 0 & -q \\ 0 & 0 & q & 0 \end{pmatrix}$$

having the eigenvalues $\pm pi, \pm qi$, and each f_k for $k = 2, 3$ is a homogeneous polynomial of degree k in the expansion in Taylor series of f around 0.

The first step consists of eliminate f_2, because system (2) does not have any quadratic terms. For this we propose a change of coordinates $\mathbf{x} = \mathbf{y} + h_2(\mathbf{y})$, where h_2 is a homogeneous polynomial mapping of degree 2 to be determined. This change of coordinates transforms system (16) into the system

$$\dot{\mathbf{y}} = A\mathbf{y} + Ah_2(\mathbf{y}) - h_2'(\mathbf{y})A\mathbf{y} + f_2(\mathbf{y}) + \tilde{f}_3(\mathbf{y}) + \dots, \tag{17}$$

where $\tilde{f}_3$ depends on f_2, f_3 and h_2. Now to eliminate the quadratic terms of (17) we must choose h_2 such that

$$h_2'(\mathbf{y})A\mathbf{y} - Ah_2(\mathbf{y}) = f_2(\mathbf{y}). \tag{18}$$

If we write $f_2 = (f_2^1, f_2^2, f_2^3, f_2^4)$ where

$$f_2^j(x,y,z,w) = \sum_{i_1+i_2+i_3+i_4=2} a_{j,i_1,i_2,i_3,i_4} x^{i_1} y^{i_2} z^{i_3} w^{i_4},$$

and $h_2 = (h_2^1, h_2^2, h_2^3, h_2^4)$ where

$$h_2^j(x,y,z,w) = \sum_{i_1+i_2+i_3+i_4=2} b_{j,i_1,i_2,i_3,i_4} x^{i_1} y^{i_2} z^{i_3} w^{i_4},$$

then equation (18) is equivalent to a linear system with 40 equations and 40 unknowns (the coefficients of b_{j,i_1,i_2,i_3,i_4}), given in the Appendix A. The determinant of the associated matrix is

$$\Delta_2 = 81p^{12}q^{12}(q+2p)^4(-q+2p)^4(2q+p)^4(-2q+p)^4.$$

That is, for $p, q \neq 0$, $p \neq \pm 2q$ and $q \neq \pm 2p$ the linear system has a unique solution. So equation (18) has a unique solution h_2, that can be used to eliminate the quadratic terms of (2).

After this change of coordinates system (17) can be rewritten as

$$\dot{\mathbf{x}} = A\mathbf{x} + \tilde{f}_3(\mathbf{x}) + \dots$$

with $\tilde{f}_3$ depending on f_2, f_3 and h_2. Now we seek a change of coordinates of the form $\mathbf{x} = \mathbf{y} + h_3(\mathbf{y})$, where h_3 is a homogeneous of degree 3 to be determined. So the last equation is rewritten as

$$\dot{\mathbf{y}} = A\mathbf{y} + Ah_3(\mathbf{y}) - h_3'(\mathbf{y})A\mathbf{y} + \tilde{f}_3(\mathbf{y}) + \dots \tag{19}$$

Now we look for h_3 so that $Ah_3(\mathbf{y}) - h_3'(\mathbf{y})A\mathbf{y} + \tilde{f}_3(\mathbf{y})$ appears as the cubic nonlinear terms in (2). We write $f_3 = (f_3^1, f_3^2, f_3^3, f_3^4)$ where

$$f_3^j(x,y,z,w) = \sum_{i_1+i_2+i_3+i_4=3} c_{j,i_1,i_2,i_3,i_4} x^{i_1} y^{i_2} z^{i_3} w^{i_4},$$

and $h_3 = (h_3^1, h_3^2, h_3^3, h_3^4)$ where

$$h_3^j(x,y,z,w) = \sum_{i_1+i_2+i_3+i_4=3} d_{j,i_1,i_2,i_3,i_4} x^{i_1} y^{i_2} z^{i_3} w^{i_4}.$$

Recall that (2) is reversible, so it is $\tilde{f}_3$ (see [3]). To determine h_3 we must solve a linear system with 80 equations and 84 unknowns (the coefficients d_{j,i_1,i_2,i_3,i_4}, a, b, c and d), given in Appendix B. After applying the Gaussian elimination procedure to this system, it becomes clear that it can be solved if and only if

$$\Delta_3 = (p-q)(p+q)(3p+q)(3p-q) \neq 0,$$

or $q \neq \pm p$, $q \neq \pm 3p$. We point out that the coefficients a, b, c and d of (2) are given by

$$\begin{aligned} a &= \frac{1}{8}(-3c_{1,0,3,0,0} - c_{1,2,1,0,0} + c_{2,1,2,0,0} + 3c_{2,3,0,0,0}),\\ b &= \frac{1}{4}(-c_{1,0,1,2,0} + c_{2,1,0,2,0} - c_{1,0,1,0,2} + c_{2,1,0,0,2}),\\ c &= \frac{1}{4}(-c_{3,2,0,0,1} - c_{3,0,2,0,1} + c_{4,0,2,1,0} + c_{4,2,0,1,0}),\\ d &= \frac{1}{8}(-3c_{3,0,0,0,3} + c_{4,0,0,1,2} - c_{3,0,0,2,1} + 3c_{4,0,0,3,0}). \end{aligned}$$

This concludes the proof of the lemma. □

System(2) is a (*Poincaré-Dulac* or *Belitskii*) *normal form* for (16). The theory of normal forms is much more deep and complete than this special case. A survey on this subject can be seen in [1].

2.3. **One-parameter families of periodic orbits.** In this section we review the so called Method of Lyapunov-Schmidt. A detailed exposition can be found in [10].

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be smooth with $f(0) = 0$ and suppose that all of the eigenvalues of the matrix $f'(0)$ are (nonzero) pure imaginary numbers. We are interested in finding periodic solutions of

$$\dot{\mathbf{x}} = f(\mathbf{x}) \tag{20}$$

with period near 2π.

Denote $C^0_{2\pi}$ (respectively $C^1_{2\pi}$) the space of maps $\mathbf{x} : \mathbb{R} \to \mathbb{R}^n$ that are 2π-periodic and of class C^0 (respectively C^1). We define in $C^0_{2\pi}$ the scalar product

$$(\mathbf{x_1}, \mathbf{x_2}) = \frac{1}{2\pi}\int_0^{2\pi} \langle \mathbf{x_1}(t), \mathbf{x_2}(t)\rangle\, dt,$$

where $\langle \cdot, \cdot \rangle$ is the canonical scalar product in $\mathbb{R}^n$.

Consider the mapping $M : C^1_{2\pi} \times \mathbb{R} \to C^0_{2\pi}$ given by

$$M(\mathbf{x}, \sigma)(t) = (1 + \sigma)\dot{\mathbf{x}}(t) - f(\mathbf{x}(t)). \tag{21}$$

We shall see at the end of this subsection that if $(\mathbf{x}, \sigma)$ satisfied $M(\mathbf{x}, \sigma) = 0$ then $\mathbf{x}(\sigma t)$ is a $2\pi/(1 + \sigma)$-periodic solution of (20). So the problem of finding periodic solutions with period near 2π for (20) reduces to finding solutions for $M(\mathbf{x}, \sigma) = 0$ with σ near to zero.

While the function M is defined in a infinite-dimensional space, the method of Lyapunov-Schmidt allow to construct a function $B : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ defined in a finite-dimensional space such that the solutions for $B = 0$ are in solutions for $M = 0$.

Note that $M(0, 0) = 0$. Denote $L := M_{\mathbf{x}}(0, 0) : C^1_{2\pi} \to C^0_{2\pi}$ given by $L(\mathbf{x})(t) = \dot{\mathbf{x}}(t) - f'(0)\mathbf{x}(t)$.

Let

$$D = \{q : \mathbb{R} \to \mathbb{R}^n;\ q(t) = \exp(tf'(0))\mathbf{x},\ \mathbf{x} \in \mathbb{R}^n\}.$$

Define

$$X_1 = \{\mathbf{x} \in C^1_{2\pi};\ (\mathbf{x}, D) = 0\}$$

and

$$Y_1 = \{\mathbf{y} \in C^0_{2\pi};\ (\mathbf{y}, D) = 0\}$$

the orthogonal complements of D in $C^1_{2\pi}$ and $C^0_{2\pi}$, respectively.

Consider a basis $\{u_1, \dots, u_n\}$ of $\mathbb{R}^n$ and $q_i = \exp(tf'(0))u_i$, $i = 1, \dots, n$. Define a projection $P : C^0_{2\pi} \to C^0_{2\pi}$ by

$$P(\mathbf{x}) = \sum_{i=1}^{n} (q_i, \mathbf{x}) q_i.$$

Hence $Im(P) = D$, $Ker(P) = Y_1$; then $C^1_{2\pi} = X_1 \oplus D$ and $C^0_{2\pi} = Y_1 \oplus D$.

Finally define

$$F(\mathbf{x}, \sigma) = F(\mathbf{q} + \mathbf{x_1}, \sigma) = \hat{F}(\mathbf{q}, \mathbf{x_1}, \sigma),$$

for $\mathbf{q} \in D$ and $\mathbf{x_1} \in X_1$.

Now to solve $\hat{F}(\mathbf{q}, \mathbf{x_1}, \sigma) = 0$ we must solve the system

$$\begin{aligned} &(Id - P) \circ \hat{F}(\mathbf{q}, \mathbf{x_1}, \sigma) = 0, \\ &P \circ \hat{F}(\mathbf{q}, \mathbf{x_1}, \sigma) = 0. \end{aligned} \tag{22}$$

Lemma 6. *The first equation of system* (22) *can be solved as* $\mathbf{x_1} = \mathbf{x_1}^*(\mathbf{q}, \sigma)$.

Proof. See [5]. □

Then to solve $M = 0$ is equivalent to solve

$$\tilde{F}(\mathbf{q}, \sigma) = P \circ \hat{F}(\mathbf{q}, \mathbf{x_1}^*(\mathbf{q}, \sigma), \sigma) = 0.$$

The later equation is satisfied if and only if

$$(\mathbf{q_i}, \hat{F}(\mathbf{q}, \mathbf{x_1}^*(\mathbf{q}, \sigma), \sigma) = 0,\ i = 1, \dots, n.$$

So $M(\mathbf{x}, \sigma) = 0$ if and only if $B(\mathbf{u}, \sigma) = 0$, with $B : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ defined by

$$B(\mathbf{u}, \sigma) = \frac{1}{2\pi} \int_0^{2\pi} \exp(-tf'(0)) F(\mathbf{x}^*(\mathbf{u}, \sigma), \sigma)\, dt$$

and

$$\mathbf{x}^*(\mathbf{u}, \sigma) = \exp(tf'(0))\mathbf{u} + \mathbf{x_1}^*(\exp(tf'(0))\mathbf{u}, \sigma)$$

If (20) is R-reversible, then B inherits this property.

Lemma 7. *Suppose* (20) *is R-reversible. Then*

(a) $RB(\mathbf{x}, \sigma) = -B(R\mathbf{x}, \sigma)$,
(b) $s_\phi B(\mathbf{x}, \sigma) = B(s_\phi \mathbf{x}, \sigma)$, *with* $s_\phi \mathbf{x} = \exp(-\phi f'(0))\mathbf{x}$.

Proof. See [12]. □

Lemma 8. *Suppose that f is in the Belitskii normal form up to order k, that is,*

$$f(\mathbf{x}) = A\mathbf{x} + g_k(\mathbf{x}) + o(|\mathbf{x}|^{k+1}),$$

with $A = f'(0)$ linear and g_k sum of homogeneous terms of degrees 2 to k with $[A^t, g_k] = 0$. Then

$$B(\mathbf{x}, \sigma) = \sigma A\mathbf{x} - g_k(\mathbf{x}) + o(|\mathbf{x}|^{k+1}).$$

Proof. See [10]. □

To summarize, we have the following result that is proved in [10].

Theorem 9. *Suppose that*

$$f(\mathbf{x}) = A\mathbf{x} + g_k(\mathbf{x}) + o(|\mathbf{x}|^{k+1})$$

is in the Belitskii normal form up to order k. Define

$$B(\mathbf{x}, \sigma) = \sigma A\mathbf{x} - g_k(\mathbf{x}) + o(|\mathbf{x}|^{k+1}).$$

Then for every nonzero solution $\mathbf{x} = \mathbf{x}(\sigma)$ for $B(\mathbf{x}, \sigma) = 0$ we have a family of periodic solutions γ_σ with period $\dfrac{2\pi}{1+\sigma}$, for (20).

We shall use Theorem 9 to prove Theorem 1.

3. Proof of Theorem 1

Now we shall prove Theorem 1 that refers to system (2) and Lemma 5, but it is valid for more general systems.

Proof of Theorem 1. From Lemma 5, this system is in Belitskii normal form up to order 3. Define $B : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ for system (2) as

$$B(x,y,z,w,\sigma) = \begin{pmatrix} -\sigma py + y\left(a(x^2+y^2) + b(z^2+w^2)\right) + \dots \\ \sigma px - x\left(a(x^2+y^2) + b(z^2+w^2)\right) + \dots \\ -\sigma qw + w\left(c(x^2+y^2) + d(z^2+w^2)\right) + \dots \\ \sigma qz - z\left(c(x^2+y^2) + d(z^2+w^2)\right) + \dots \end{pmatrix}.$$

From Theorem 9 if we find a solution $(x,y,z,w) = (x(\sigma), y(\sigma), z(\sigma), w(\sigma))$ for $B(x,y,z,w,\sigma) = 0$, then we obtain a family of $\dfrac{2\pi}{1+\sigma}$-periodic solutions for (2).

As (2) is R-reversible, with $R(x,y,z,w) = (x,-y,z,-w)$, we look for solutions of $B = 0$ in the plane $\mathrm{Fix}(R) = \{(x,y,z,w) \in \mathbb{R}^4;\ y = w = 0\}$.

Write the higher order terms in the expression of B as $(\Omega_1, \Omega_2, \Omega_3, \Omega_4)$, with $\Omega_i = \Omega_i(x,y,z,w)$. By Lemma 7(a), B is R-reversible, so $\Omega_1(x,y,z,w) = -y\tilde{\Omega}_1(x,y,z,w)$ and $\Omega_3(x,y,z,w) = -w\tilde{\Omega}_3(x,y,z,w)$, for some C^∞ functions $\tilde{\Omega}_1, \tilde{\Omega}_3$. Now using Lemma 7(b) with $\phi = \pi/4$, we obtain that $\Omega_2(x,y,z,w) = x\tilde{\Omega}_2(x,y,z,w)$ and $\Omega_4(x,y,z,w) = z\tilde{\Omega}_4(x,y,z,w)$, for some C^∞ functions $\tilde{\Omega}_2, \tilde{\Omega}_4$.

As (2) is R-reversible, with $R(x,y,z,w) = (x,-y,z,-w)$, we look for solutions of $B = 0$ in the plane $\mathrm{Fix}(R) = \{(x,y,z,w) \in \mathbb{R}^4;\ y = w = 0\}$.

Then $B|_{\mathrm{Fix}(R)}$ writes as

$$\bar{B}(x,z,\sigma) = \begin{pmatrix} \sigma x(p - ax^2 - bz^2 + \Omega_1(x,z)) \\ \sigma z(q - cx^2 - dz^2 + \Omega_2(x,z)) \end{pmatrix}.$$

Our aim is to solve system $B|_{\mathrm{Fix}(R)} = 0$, what can be done using the hypotheses of Theorem 1.

For $x \neq 0$ there is one solution near $(x,z) = \left(\sqrt{\dfrac{p}{a}},\ 0\right)$ (case (i) in Theorem 1).

For $z \neq 0$ there is one solutions near $(x,z) = \left(0,\ \sqrt{\dfrac{q}{d}}\right)$ (case (ii) in Theorem 1).

For $xz \neq 0$ there is one solution near $(x,z) = \left(\sqrt{\dfrac{pd-qb}{ad-bc}},\ \sqrt{-\dfrac{pc-qa}{ad-bc}}\right)$ (case (iii) in Theorem 1).

This concludes the proof of the Theorem 1. □

4. Proof of Theorem 3

In this section we shall study what happens with the families of symmetric periodic orbits when the symmetry of system (2) is broken. Here we enter in the last part of our program, that is, when we consider a perturbed version of a third order truncation of system (2). In this section we prove Theorem 3.

First we consider the time reescaling $s = qt$ in (6). We get

$$
\begin{array}{rcl}
x' & = & -Ny - \varepsilon y\left(\bar{a}(x^2+y^2) + \bar{b}(z^2+w^2)\right) + \varepsilon \bar{f}_1(x,y,z,w), \\
y' & = & Nx + \varepsilon x\left(\bar{a}(x^2+y^2) + \bar{b}(z^2+w^2)\right) + \varepsilon \bar{f}_2(x,y,z,w), \\
z' & = & -w - \varepsilon w\left(\bar{c}(x^2+y^2) + \bar{d}(z^2+w^2)\right) + \varepsilon \bar{f}_3(x,y,z,w), \\
w' & = & z + \varepsilon z\left(\bar{c}(x^2+y^2) + \bar{d}(z^2+w^2)\right) + \varepsilon \bar{f}_4(x,y,z,w),
\end{array}
\tag{23}
$$

where $N = p/q$, $\bar{k} = k/p$ for $k \in \{a,b,c,d,f\}$ and the prime denotes derivative with respecto to s.

In system (23) we write

$$
\begin{array}{rcl}
\bar{f}_1 & = & \displaystyle\sum_{j=0}^{3} \sum_{i_1+i_2+i_3+i_4=j} \alpha^j_{i_1,i_2,i_3,i_4} x^{i_1} y^{i_2} z^{i_3} w^{i_4}, \\
\bar{f}_2 & = & \displaystyle\sum_{j=0}^{3} \sum_{i_1+i_2+i_3+i_4=j} \beta^j_{i_1,i_2,i_3,i_4} x^{i_1} y^{i_2} z^{i_3} w^{i_4}, \\
\bar{f}_3 & = & \displaystyle\sum_{j=0}^{3} \sum_{i_1+i_2+i_3+i_4=j} \gamma^j_{i_1,i_2,i_3,i_4} x^{i_1} y^{i_2} z^{i_3} w^{i_4}, \\
\bar{f}_4 & = & \displaystyle\sum_{j=0}^{3} \sum_{i_1+i_2+i_3+i_4=j} \delta^j_{i_1,i_2,i_3,i_4} x^{i_1} y^{i_2} z^{i_3} w^{i_4},
\end{array}
$$

and apply the method described in subsection 2.1. Then passing system (23) to the form of system (14), we obtain

$$
\begin{array}{rcl}
\dfrac{dr}{d\theta} & = & \dfrac{\varepsilon}{N} r(a_1 r^2 + a_2 R^2 + a_3), \\
\dfrac{dR}{d\theta} & = & \dfrac{\varepsilon}{N} R(b_1 r^2 + b_2 R^2 + a_3),
\end{array}
\tag{24}
$$

where

$$
\begin{array}{rcl}
a_1 & = & 3\alpha^3_{3,0,0,0} + 3\beta^3_{0,3,0,0} + \alpha^3_{1,2,0,0} + \beta^3_{2,1,0,0}, \\
a_2 & = & 2\alpha^3_{1,0,0,2} + 2\beta^3_{0,1,2,0} + 2\beta^3_{0,1,0,2} + 2\alpha^3_{1,0,2,0}, \\
a_3 & = & 4\alpha^1_{1,0,0,0} + 4\beta^1_{0,1,0,0}, \\
b_1 & = & 2\delta^3_{2,0,0,1} + 2\gamma^3_{2,0,1,0} + 2\gamma^3_{0,2,1,0} + 2\delta^3_{0,2,0,1}, \\
b_2 & = & \gamma^3_{0,0,1,2} + 3\gamma^3_{0,0,3,0} + \delta^3_{0,0,2,1} + 3\delta^3_{0,0,0,3}, \\
b_3 & = & 4\delta^1_{0,0,0,1} + 4\gamma^1_{0,0,1,0}.
\end{array}
$$

We need to find the singularities of the averaged system (24), with $r \geq 0$, $R \geq 0$ and $r + R > 0$. That is, the solutions of the system

$$
r(a_1 r^2 + a_2 R^2 + a_3) = 0, \quad R(b_1 r^2 + b_2 R^2 + b_3) = 0,
$$

with $r \geq 0$, $R \geq 0$ and $r + R > 0$ are

$$(r_1, R_1) = \left(0, \sqrt{-\frac{b_3}{b_2}}\right), \text{ if } b_2 b_3 < 0;$$

$$(r_2, R_2) = \left(\sqrt{-\frac{a_3}{a_1}}, 0\right), \text{ if } a_1 a_3 < 0;$$

$$(r_3, R_3) = \left(\sqrt{\frac{a_2 b_3 - a_3 b_2}{a_1 b_2 - b_1 a_2}}, \sqrt{\frac{b_1 a_3 - b_3 a_1}{a_1 b_2 - b_1 a_2}}\right) \text{ if } \frac{a_2 b_3 - a_3 b_2}{a_1 b_2 - b_1 a_2} > 0 \,\&\, \frac{b_1 a_3 - b_3 a_1}{a_1 b_2 - b_1 a_2} > 0;$$

$$(r_4, R_4) = \left(0, \sqrt{\frac{b_1 a_3 - b_3 a_1}{a_1 b_2 - b_1 a_2}}\right) \text{ if } a_2 b_3 - a_3 b_2 = 0 \,\&\, \frac{b_1 a_3 - b_3 a_1}{a_1 b_2 - b_1 a_2} > 0;$$

$$(r_5, R_6) = \left(\sqrt{\frac{a_2 b_3 - a_3 b_2}{a_1 b_2 - b_1 a_2}}, 0\right) \text{ if } \frac{a_2 b_3 - a_3 b_2}{a_1 b_2 - b_1 a_2} > 0 \,\&\, b_1 a_3 - b_3 a_1 = 0.$$

Note that at most we have three solutions

The conditions for the solution (r_k, R_k) to be simple (that is, satisfying (15)) is

$$\frac{(a_2 b_3 - a_3 b_2) b_3}{b_2} \neq 0 \text{ for } k = 1;$$
$$\frac{a_3 (b_3 a_1 - b_1 a_3)}{a_1} \neq 0 \text{ for } k = 2;$$
$$\frac{(a_2 b_3 - a_3 b_2)(b_3 a_1 - b_1 a_3)}{a_1 b_2 - b_1 a_2} \neq 0 \text{ for } k = 3;$$
$$0 \text{ for } k = 4, 5.$$

Therefore only the first three solutions can be simple. Since the angle $\varphi_0 \in \mathbb{S}^1$ is arbitrary, clearly the three solutions (r_k, R_k) for $k = 1, 2, 3$ when they exist are tori that converge to tori of system (6) with $\varepsilon = 0$.

Hence Theorem 3 is proved.

Remark 10. *The results of Theorem 3 depend on f_1, f_2, f_3, f_4, but not on p, q, a, b, c and d in* (6)*, as the terms with these coefficients vanishes when we calculate the averaging, passing system* (23) *to the form of system* (14)*.*

In the next example we exhibit differential systems that realize the conditions on Theorem 3.

Example 11. *Consider the differential systems*

$$(25) \qquad \begin{array}{rcl} \dot{x} &=& -py - \varepsilon y \left(a(x^2 + y^2) + b(z^2 + w^2)\right) + \varepsilon \left(-\frac{1}{2}x + \frac{3}{2}xw^2\right), \\ \dot{y} &=& px + \varepsilon x \left(a(x^2 + y^2) + b(z^2 + w^2)\right) + \varepsilon \left(2x^2 y\right), \\ \dot{z} &=& -qw - \varepsilon w \left(c(x^2 + y^2) + d(z^2 + w^2)\right), \\ \dot{w} &=& qz + \varepsilon z \left(c(x^2 + y^2) + d(z^2 + w^2)\right) + \varepsilon \left(-\frac{1}{4}w + z^2 w + \frac{1}{2}y^2 w\right), \end{array}$$

$$(26) \qquad \begin{array}{rcl} \dot{x} &=& -py - \varepsilon y \left(a(x^2 + y^2) + b(z^2 + w^2)\right) + \varepsilon \left(\frac{1}{2}xw^2\right), \\ \dot{y} &=& px + \varepsilon x \left(a(x^2 + y^2) + b(z^2 + w^2)\right) + \varepsilon \left(-10x^2 y\right), \\ \dot{z} &=& -qw - \varepsilon w \left(c(x^2 + y^2) + d(z^2 + w^2)\right), \\ \dot{w} &=& qz + \varepsilon z \left(c(x^2 + y^2) + d(z^2 + w^2)\right) + \varepsilon \left(-\frac{1}{4}w + z^2 w\right) \end{array}$$

and

$$
\begin{array}{rcl}
\dot{x} & = & -py - \varepsilon y \left(a(x^2+y^2) + b(z^2+w^2) \right) + \varepsilon(x - x^3), \\
\dot{y} & = & px + \varepsilon x \left(a(x^2+y^2) + b(z^2+w^2) \right), \\
\dot{z} & = & -qw - \varepsilon w \left(c(x^2+y^2) + d(z^2+w^2) \right) + \varepsilon(z - z^3), \\
\dot{w} & = & qz + \varepsilon z \left(c(x^2+y^2) + d(z^2+w^2) \right),
\end{array} \tag{27}
$$

where $a, b, c, d \in \mathbb{R}$ *and* $p, q \in \mathbb{N}$ *with* p *and* q *coprime. Theorem 3 assures that for* $\varepsilon > 0$ *small:*

- *system* (25) *has one* 2*-dimensional invariant torus filled with periodic orbits;*
- *system* (26) *has two* 2*-dimensional invariant tori filled with periodic orbits;*
- *system* (27) *has three* 2*-dimensional invariant tori filled with periodic orbits.*

Appendix A. Linear system corresponding to equation (18)

$-a_{1,0,0,0,2} - pb_{2,0,0,0,2} + qb_{1,0,0,1,1} = 0$
$-a_{2,0,0,0,2} + pb_{1,0,0,0,2} + qb_{2,0,0,1,1} = 0$
$-a_{3,0,2,0,0} - qb_{4,0,2,0,0} + pb_{3,1,1,0,0} = 0$
$-a_{4,0,0,0,2} + qb_{3,0,0,0,2} + qb_{4,0,0,1,1} = 0$
$pb_{1,0,0,2,0} - qb_{2,0,0,1,1} - a_{2,0,0,2,0} = 0$
$pb_{2,0,0,2,0} - a_{1,0,0,2,0} - qb_{1,0,0,1,1} = 0$
$-pb_{4,1,1,0,0} - a_{4,2,0,0,0} + qb_{3,2,0,0,0} = 0$
$-qb_{4,0,0,0,2} + qb_{3,0,0,1,1} - a_{3,0,0,0,2} = 0$
$-a_{2,0,0,1,1} + pb_{1,0,0,1,1} - 2qb_{2,0,0,0,2} + 2qb_{2,0,0,2,0} = 0$
$-a_{3,1,0,1,0} - pb_{3,0,1,1,0} - qb_{4,1,0,1,0} - qb_{3,1,0,0,1} = 0$
$-a_{4,0,1,0,1} + qb_{3,0,1,0,1} + qb_{4,0,1,1,0} + pb_{4,1,0,0,1} = 0$
$pb_{1,0,1,1,0} - qb_{2,0,1,0,1} - a_{2,0,1,1,0} + pb_{2,1,0,1,0} = 0$
$-pb_{2,0,0,1,1} - a_{1,0,0,1,1} + 2qb_{1,0,0,2,0} - 2qb_{1,0,0,0,2} = 0$
$-pb_{2,0,1,1,0} - a_{1,0,1,1,0} + pb_{1,1,0,1,0} - qb_{1,0,1,0,1} = 0$
$-pb_{2,1,0,1,0} - qb_{1,1,0,0,1} - a_{1,1,0,1,0} - pb_{1,0,1,1,0} = 0$
$-2pb_{4,0,2,0,0} + 2pb_{4,2,0,0,0} - a_{4,1,1,0,0} + qb_{3,1,1,0,0} = 0$
$qb_{1,1,0,1,0} - a_{1,1,0,0,1} - pb_{1,0,1,0,1} - pb_{2,1,0,0,1} = 0$
$qb_{3,1,0,0,1} + qb_{4,1,0,1,0} - a_{4,1,0,0,1} - pb_{4,0,1,0,1} = 0$
$-qb_{4,0,1,0,1} - a_{3,0,1,0,1} + qb_{3,0,1,1,0} + pb_{3,1,0,0,1} = 0$
$-qb_{4,1,0,0,1} - pb_{4,0,1,1,0} - a_{4,1,0,1,0} + qb_{3,1,0,1,0} = 0$

$-a_{1,2,0,0,0} - pb_{2,2,0,0,0} - pb_{1,1,1,0,0} = 0$
$-a_{2,2,0,0,0} + pb_{1,2,0,0,0} - pb_{2,1,1,0,0} = 0$
$-a_{3,2,0,0,0} - pb_{3,1,1,0,0} - qb_{4,2,0,0,0} = 0$
$-a_{4,0,2,0,0} + pb_{4,1,1,0,0} + qb_{3,0,2,0,0} = 0$
$pb_{1,0,2,0,0} + pb_{2,1,1,0,0} - a_{2,0,2,0,0} = 0$
$-pb_{2,0,2,0,0} - a_{1,0,2,0,0} + pb_{1,1,1,0,0} = 0$
$-qb_{3,0,0,1,1} - a_{3,0,0,2,0} - qb_{4,0,0,2,0} = 0$
$-qb_{4,0,0,1,1} - a_{4,0,0,2,0} + qb_{3,0,0,2,0} = 0$
$-a_{3,0,0,1,1} + 2qb_{3,0,0,2,0} - 2qb_{3,0,0,0,2} - qb_{4,0,0,1,1} = 0$
$-a_{4,0,0,1,1} + 2qb_{4,0,0,2,0} - 2qb_{4,0,0,0,2} + qb_{3,0,0,1,1} = 0$
$-a_{4,0,1,1,0} + qb_{3,0,1,1,0} - qb_{4,0,1,0,1} + pb_{4,1,0,1,0} = 0$
$pb_{1,1,1,0,0} - a_{2,1,1,0,0} - 2pb_{2,0,2,0,0} + 2pb_{2,2,0,0,0} = 0$
$-pb_{2,0,1,0,1} + qb_{2,1,0,1,0} + pb_{1,1,0,0,1} - a_{2,1,0,0,1} = 0$
$pb_{2,1,0,0,1} - a_{2,0,1,0,1} + pb_{1,0,1,0,1} + qb_{2,0,1,1,0} = 0$
$-pb_{2,1,1,0,0} - 2pb_{1,0,2,0,0} + 2pb_{1,2,0,0,0} - a_{1,1,1,0,0} = 0$
$qb_{1,0,1,1,0} - pb_{2,0,1,0,1} + pb_{1,1,0,0,1} - a_{1,0,1,0,1} = 0$
$-qb_{2,1,0,0,1} - pb_{2,0,1,1,0} + pb_{1,1,0,1,0} - a_{2,1,0,1,0} = 0$
$qb_{3,1,0,1,0} - pb_{3,0,1,0,1} - qb_{4,1,0,0,1} - a_{3,1,0,0,1} = 0$
$-qb_{4,0,1,1,0} + pb_{3,1,0,1,0} - qb_{3,0,1,0,1} - a_{3,0,1,1,0} = 0$
$-qb_{4,1,1,0,0} - a_{3,1,1,0,0} - 2pb_{3,0,2,0,0} + 2pb_{3,2,0,0,0} = 0$

Appendix B. Linear system corresponding to equation (19)

$pd_{1,0,3,0,0} + pd_{2,1,2,0,0} = 0$
$-pd_{3,2,1,0,0} - qd_{4,3,0,0,0} = 0$
$-qd_{1,0,0,2,1} - pd_{2,0,0,3,0} = 0$
$-qd_{3,0,0,2,1} - qd_{4,0,0,3,0} = 0$
$-c_{1,0,0,0,3} - pd_{2,0,0,0,3} + qd_{1,0,0,1,2} = 0$
$-c_{3,0,3,0,0} - qd_{4,0,3,0,0} + pd_{3,1,2,0,0} = 0$
$-pd_{1,0,1,0,2} - pd_{2,1,0,0,2} + qd_{1,1,0,1,1} = 0$
$pd_{1,2,1,0,0} - 2pd_{2,1,2,0,0} + 3pd_{2,3,0,0,0} = 0$
$-pd_{2,0,2,1,0} - qd_{1,0,2,0,1} + pd_{1,1,1,1,0} = 0$
$-pd_{2,1,1,0,1} + qd_{2,2,0,1,0} + pd_{1,2,0,0,1} = 0$
$-pd_{2,2,0,1,0} - qd_{1,2,0,0,1} - pd_{1,1,1,1,0} = 0$
$-pd_{3,1,1,1,0} - qd_{3,2,0,0,1} - qd_{4,2,0,1,0} = 0$
$pd_{4,1,1,0,1} + qd_{3,0,2,0,1} + qd_{4,0,2,1,0} = 0$
$-qd_{1,1,0,1,1} - pd_{1,0,1,2,0} - pd_{2,1,0,2,0} = 0$
$2qd_{3,0,0,2,1} - 3qd_{3,0,0,0,3} - qd_{4,0,0,1,2} = 0$
$qd_{3,0,1,2,0} + pd_{4,1,0,2,0} - qd_{4,0,1,1,1} = 0$
$-2qd_{4,0,0,1,2} + qd_{3,0,0,2,1} + 3qd_{4,0,0,3,0} = 0$
$-qd_{4,1,0,0,2} - pd_{3,0,1,0,2} + qd_{3,1,0,1,1} = 0$
$-c_{4,0,0,3,0} - qd_{4,0,0,2,1} + qd_{3,0,0,3,0} = b_{0,1}$
$-c_{3,0,1,2,0} - qd_{4,0,1,2,0} - qd_{3,0,1,1,1} + pd_{3,1,0,2,0}$
$pd_{1,1,2,0,0} - c_{1,0,3,0,0} - pd_{2,0,3,0,0} = a_{1,0}$
$pd_{1,3,0,0,0} - pd_{2,2,1,0,0} - c_{2,3,0,0,0} = -a_{1,0}$
$-pd_{2,1,1,0,1} - 2pd_{1,0,2,0,1} + 2pd_{1,2,0,0,1} + qd_{1,1,1,1,0} = 0$
$-2pd_{3,1,2,0,0} - qd_{4,2,1,0,0} + 3pd_{3,3,0,0,0} - c_{3,2,1,0,0} = 0$
$-3pd_{4,0,3,0,0} - c_{4,1,2,0,0} + qd_{3,1,2,0,0} + 2pd_{4,2,1,0,0} = 0$
$2qd_{2,0,0,2,1} - c_{2,0,0,1,2} - 3qd_{2,0,0,0,3} + pd_{1,0,0,1,2} = 0$
$-qd_{2,1,1,0,1} - 2pd_{2,0,2,1,0} + 2pd_{2,2,0,1,0} + pd_{1,1,1,1,0} = 0$
$-2qd_{3,0,1,0,2} + 2qd_{3,0,1,2,0} - qd_{4,0,1,1,1} + pd_{3,1,0,1,1} = 0$
$-qd_{4,1,0,1,1} + qd_{3,1,0,2,0} - c_{4,1,0,2,0} - pd_{4,0,1,2,0} = 0$
$2qd_{4,1,0,2,0} - 2qd_{4,1,0,0,2} - pd_{4,0,1,1,1} + qd_{3,1,0,1,1} = 0$
$-c_{2,1,2,0,0} + 2pd_{2,2,1,0,0} - 3pd_{2,0,3,0,0} + pd_{1,1,2,0,0} = -a_{1,0}$
$-c_{2,0,1,1,1} + 2qd_{2,0,1,2,0} - 2qd_{2,0,1,0,2} + pd_{1,0,1,1,1} + pd_{2,1,0,1,1} = 0$
$-c_{2,1,0,2,0} - qd_{2,1,0,1,1} + pd_{1,1,0,2,0} - pd_{2,0,1,2,0} = -a_{0,1}$
$-c_{3,1,1,1,0} - qd_{3,1,1,0,1} + 2pd_{3,2,0,1,0} - 2pd_{3,0,2,1,0} - qd_{4,1,1,1,0} = 0$
$-2pd_{1,0,2,1,0} - c_{1,1,1,1,0} - pd_{2,1,1,1,0} - qd_{1,1,1,0,1} + 2pd_{1,2,0,1,0} = 0$
$-pd_{2,1,0,1,1} - pd_{1,0,1,1,1} - c_{1,1,0,1,1} - 2qd_{1,1,0,0,2} + 2qd_{1,1,0,2,0} = 0$
$pd_{4,1,0,1,1} - 2qd_{4,0,1,0,2} + qd_{3,0,1,1,1} + 2qd_{4,0,1,2,0} - c_{4,0,1,1,1} = 0$
$qd_{3,0,2,1,0} - c_{4,0,2,1,0} + pd_{4,1,1,1,0} - qd_{4,0,2,0,1} = -b_{1,0}$
$-2qd_{3,1,0,0,2} - qd_{4,1,0,1,1} + 2qd_{3,1,0,2,0} - c_{3,1,0,1,1} - pd_{3,0,1,1,1} = 0$
$-3qd_{4,0,0,0,3} + qd_{3,0,0,1,2} - c_{4,0,0,1,2} + 2qd_{4,0,0,2,1} = -b_{0,1}$
$-pd_{2,3,0,0,0} - pd_{1,2,1,0,0} = 0$
$pd_{4,1,2,0,0} + qd_{3,0,3,0,0} = 0$
$qd_{2,0,0,1,2} + pd_{1,0,0,0,3} = 0$
$qd_{4,0,0,1,2} + qd_{3,0,0,0,3} = 0$
$-c_{2,0,0,3,0} + pd_{1,0,0,3,0} - qd_{2,0,0,2,1} = 0$
$pd_{1,0,0,2,1} + 3qd_{2,0,0,3,0} - 2qd_{2,0,0,1,2} = 0$
$pd_{1,0,1,0,2} + qd_{2,0,1,1,1} + pd_{2,1,0,0,2} = 0$
$-pd_{2,0,0,1,2} + 2qd_{1,0,0,2,1} - 3qd_{1,0,0,0,3} = 0$
$pd_{2,1,0,2,0} - qd_{2,0,1,1,1} + pd_{1,0,1,2,0} = 0$
$-pd_{2,1,2,0,0} + 2pd_{1,2,1,0,0} - 3pd_{1,0,3,0,0} = 0$
$-3pd_{3,0,3,0,0} - qd_{4,1,2,0,0} + 2pd_{3,2,1,0,0} = 0$
$-pd_{4,1,1,0,1} + qd_{4,2,0,1,0} + qd_{3,2,0,0,1} = 0$
$-pd_{4,2,1,0,0} - c_{4,3,0,0,0} + qd_{3,3,0,0,0} = 0$
$qd_{2,0,2,1,0} + pd_{1,0,2,0,1} + pd_{2,1,1,0,1} = 0$
$qd_{3,0,1,0,2} + qd_{4,0,1,1,1} + pd_{4,1,0,0,2} = 0$
$qd_{3,2,1,0,0} + 3pd_{4,3,0,0,0} - 2pd_{4,1,2,0,0} = 0$
$-qd_{4,0,2,1,0} - qd_{3,0,2,0,1} + pd_{3,1,1,1,0} = 0$
$-qd_{4,1,0,2,0} - qd_{3,1,0,1,1} - pd_{3,0,1,2,0} = 0$
$-c_{1,0,0,2,1} + 3qd_{1,0,0,3,0} - pd_{2,0,0,2,1} - 2qd_{1,0,0,1,2} = 0$
$pd_{1,0,2,1,0} - c_{2,0,2,1,0} - qd_{2,0,2,0,1} + pd_{2,1,1,1,0} = 0$
$pd_{1,2,0,1,0} - pd_{2,1,1,1,0} - c_{2,2,0,1,0} - qd_{2,2,0,0,1} = 0$
$-pd_{2,0,1,1,1} + pd_{1,1,0,1,1} - 2qd_{1,0,1,0,2} + 2qd_{1,0,1,2,0} = 0$
$-pd_{2,2,0,0,1} - c_{1,2,0,0,1} - pd_{1,1,1,0,1} + qd_{1,2,0,1,0} = 0$
$2pd_{3,2,0,0,1} - qd_{4,1,1,0,1} + qd_{3,1,1,1,0} - 2pd_{3,0,2,0,1} = 0$
$qd_{1,0,2,1,0} - pd_{2,0,2,0,1} + pd_{1,1,1,0,1} - c_{1,0,2,0,1} = 0$
$-2qd_{2,1,0,0,2} + 2qd_{2,1,0,2,0} + pd_{1,1,0,1,1} - pd_{2,0,1,1,1} = 0$
$qd_{3,0,0,1,2} - qd_{4,0,0,0,3} - c_{3,0,0,0,3} = b_{0,1}$
$-qd_{4,0,1,0,2} + qd_{3,0,1,1,1} + pd_{3,1,0,0,2} - c_{3,0,1,0,2} = 0$
$qd_{4,1,0,1,1} - pd_{4,0,1,0,2} - c_{4,1,0,0,2} + qd_{3,1,0,0,2} = 0$
$-qd_{4,1,1,0,1} - 2pd_{4,0,2,1,0} + qd_{3,1,1,1,0} + 2pd_{4,2,0,1,0}$
$-qd_{4,0,0,2,1} - c_{3,0,0,2,1} - 2qd_{3,0,0,1,2} + 3qd_{3,0,0,3,0} = b_{0,1}$
$-c_{2,1,0,0,2} + pd_{1,1,0,0,2} + qd_{2,1,0,1,1} - pd_{2,0,1,0,2} = -a_{0,1}$
$-c_{2,1,1,0,1} + 2pd_{2,2,0,0,1} + pd_{1,1,1,0,1} - 2pd_{2,0,2,0,1} + qd_{2,1,1,1,0} = 0$
$-c_{3,2,0,0,1} - pd_{3,1,1,0,1} + qd_{3,2,0,1,0} - qd_{4,2,0,0,1} = b_{1,0}$
$-pd_{2,0,1,2,0} - c_{1,0,1,2,0} + pd_{1,1,0,2,0} - qd_{1,0,1,1,1} = a_{0,1}$
$-pd_{2,2,1,0,0} - 2pd_{1,1,2,0,0} - c_{1,2,1,0,0} + 3pd_{1,3,0,0,0} = a_{1,0}$
$qd_{1,0,1,1,1} + pd_{1,1,0,0,2} - c_{1,0,1,0,2} - pd_{2,0,1,0,2} = a_{0,1}$
$qd_{3,0,2,1,0} - c_{3,0,2,0,1} - qd_{4,0,2,0,1} + pd_{3,1,1,0,1} = b_{1,0}$
$qd_{3,1,1,0,1} + 2pd_{4,2,0,0,1} - 2pd_{4,0,2,0,1} + qd_{4,1,1,1,0} - c_{4,1,1,0,1} = 0$
$-qd_{4,2,0,0,1} + qd_{3,2,0,1,0} - pd_{4,1,1,1,0} - c_{4,2,0,1,0} = -b_{1,0}$

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