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PERIODIC ORBITS AND NON–INTEGRABILITY OF GENERALIZED CLASSICAL YANG–MILLS HAMILTONIAN SYSTEMS

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Abstract. The averaging theory of first order is applied to study a generalized Yang–Mills system with two parameters. Two main results are proved. First, we provide sufficient conditions on the two parameters of the generalized system to guarantee the existence of continuous families of isolated periodic orbits parameterized by the energy, and these families are given up to first order in a small parameter. Second, we prove that for the non–integrable classical Yang–Mills Hamiltonian systems, in the sense of Liouville–Arnold, which have the isolated periodic orbits found with averaging theory, can not exist any second first integral of class $\mathcal{C}^1$. This is important because most of the results about integrability deals with analytic or meromorphic integrals of motion.

1. Introduction

We study a *generalized classical Yang–Mills Hamiltonian* [25], which consists of a harmonic oscillator plus a homogeneous potential of fourth degree

$$H = \frac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2) + \frac{a}{4}\,x^4 + \frac{b}{2}\,x^2y^2, \tag{1}$$

with two real parameters a and b. When $a = 0$ we obtain the *Contopoulos Hamiltonian*, studied by him and coworkers during many years, see for instance [11, 12, 13]. The Contopoulos Hamiltonian describes the perturbed central part of an elliptical or barred galaxy without escapes. When in the Contopoulos Hamiltonian the quadratic part $(x^2 + y^2)/2$ is not present we have the *mechanical Yang–Mills Hamiltonian* $H_{YM} = (p_x^2+p_y^2)/2+b\,x^2y^2/4$; the term x^2y^2 characterize the Yang–Mills potential, which arises in connection with the classical Yang–Mills field with gauge group SU(2) for a homogeneous two–component field [18]. Quartic homogeneous potentials (without quadratic terms) have been studied by several authors, see for instance [2, 5, 16], and it is well known that the Hamiltonian H_{YM} with $b \neq 0$ is non–integrable and strongly chaotic. Generalizations of the mechanical Yang–Mills Hamiltonian, with three up to five quartic terms, have been considered in [9, 15, 23, 25]. Maciejewski *et. al.* [25] studied generalized Yang–Mills Hamiltonian systems which have a quadratic potential plus a homogeneous of fourth degree potential with five parameters, and they proved the existence of connected branches of non–stationary periodic trajectories emanating from the origin. Caranicolas *et. al.* [9] studied a Hamiltonian with a quartic potential of three parameters plus a quadratic harmonic potential with frequencies ω_1 and ω_2 being two extra parameters of the form

$$H = \frac{1}{2}(p_x^2 + p_y^2 + \omega_1^2 x^2 + \omega_2^2 y^2) + \varepsilon(a\,x^4 + 2b\,x^2y^2 + c\,y^4), \tag{2}$$

and they calculated numerically families of periodic orbits and its characteristic curves. In section 3 by means of a rescaling transformation we show that Hamiltonians of type (2) with $\varepsilon \neq 0$ are topologically equivalent to a similar one with $\varepsilon = 1$.

Here we study the generalization of the Yang–Mills potentials given in (1) with two real parameters a and b, in order that the problem be tractable in a two–dimensional parameter space, although these calculations can be generalized to higher dimensional parameter spaces. The Hamiltonian differential system, or simply the Hamiltonian system associated to (1) is given by

$$
\begin{aligned}
\dot{x} &= p_x, \\
\dot{y} &= p_y, \\
\dot{p}_x &= -x - ax^3 - b\,xy^2, \\
\dot{p}_y &= -y - b\,x^2 y\,.
\end{aligned}
\tag{3}
$$

As usual the dot denotes derivative with respect to the independent variable, the time t. In this paper we name (3) the *generalized classical Yang–Mills Hamiltonian system with two parameters*, or simply the *Yang–Mills (YM) systems.*

Our first aim is to compute up to first order in a small parameter periodic orbits by using the *averaging theory*, more precisely through the Averagin Theorem of section 2, and to find conditions on the two parameters a and b for their existence. The periodic orbits here studied are isolated in every energy level and they are of special interest because they are the most simple non–trivial solutions of the system, and their stability determines the kind of motion in their neighborhood. Our second objective will be to show that when these periodic orbits exist, either the non–integrability of the YM Hamiltonian system in the sense of Liouville–Arnold takes place for any second first integral of class $\mathcal{C}^1$, or the system is integrable but the two first integrals are dependent along those periodic orbits.

We shall use the averaging theory as it was established by Buică *et. al.* [8], see section 2 for a summary of this tool. This method provides periodic orbits of a perturbed periodic non–autonomous differential system depending on a small parameter ε. Such periodic orbits bifurcate from some of the periodic orbits of an invariant manifold formed by periodic orbits of the unperturbed system. Roughly speaking, the problem of finding periodic solutions of the perturbed differential system is reduced to find zeros of some convenient finite dimensional function. We check the conditions under which the averaging theory guarantees the persistence of periodic orbits under the perturbation of the harmonic oscillator, and we find them as a function of the energy. In this way we can find analytically periodic orbits in any energy level as function of the parameters a and b for the Yang–Mills systems (3). We summarize our first main result on the periodic orbits as follows.

Theorem 1. *At every energy level $H = h$ with $h \neq 0$ the Yang–Mills Hamiltonian system* (3) *has at least*

(a) *four periodic orbits if $hb(b-a) > 0$, $h(b-a)(b-2a) > 0$, $hb(b-3a) > 0$ and $h(b-6a)(b-3a) > 0$;*

(b) *two periodic orbits if either* $hb(b-a) > 0$ *and* $h(b-a)(b-2a) > 0$, *or* $hb(b-3a) > 0$ *and* $h(b-6a)(b-3a) > 0$.

Theorem 1 is proved in section 3. Then, as a consequence of the existence of these periodic orbits, we can show our second main result about the non–integrability of the Yang–Mills systems (3) in the sense of Liouville–Arnold for any second first integral of class $\mathcal{C}^1$.

Theorem 2. *Assume that the Yang–Mills Hamiltonian systems* (3) *satisfies the assumptions of one of the statements of Theorem 1, and denote by (x) this statement. Then, under the assumption of the statement (x),*

(a) *either the Yang–Mills Hamiltonian system is not Liouville–Arnold integrable with any second first integral* $\mathcal{C}^1$,
(b) *or the Yang–Mills Hamiltonian system is Liouville–Arnold integrable and the gradients of the two constants of motion are linearly dependent on almost all the points of the periodic orbits found in the statement (x) of Theorem 1.*

Theorem 2 is proved in section 4, where the definition of integrability in the sense of Liouville–Arnold is also recalled. Similar results to Theorems 1 and 2 were obtained by the authors for the Henon–Heiles Hamiltonian systems in [21], but there the results used from the averaging theory need second order, and here with first order is sufficient.

In short, in Theorem 2 we study the Hamiltonian systems (3), or (2) when $\omega_1 = \omega_2$ and $c = 0$ (or equivalently the symmetric case $a = 0$). We use a result due to Poincaré [3, 22, 28] for proving that the non Liouville–Arnold integrability, based on the existence of periodic orbits of Theorem 1 having multipliers of the monodromy matrix of the variational equations different from 1, must be for any second first integral of class $\mathcal{C}^1$.

Integrable Hamiltonian systems in the sense of Liouville–Arnold are non–generic, see [26]. The non–existence of any additional meromorphic first integral in the classical Yang–Mills Hamiltonian H_{YM} in any domain containing the origin of coordinates was proved by Ziglin [32]. The integrability of the more general Yang–Mills Hamiltonians (2) has been studied by several authors. Of course, the case $b = 0$ is integrable because it is separable. Bountis *et. al.* [7] showed that the Hamiltonian (2) has the Painlevé property when $a = c = 1$ and $b = 1, 3$ (ω_1, ω_2 arbitrary), and that these cases are integrable. Two other integrable cases have been found: $4\,\omega_1 = \omega_2$ and $a = 1$, $b = 6$, $c = 16$ [23, 14, 10], and $\omega_1 = \omega_2$ and $a = 1$, $b = 3$, $c = 8$, [15, 19, 29]. These previous results are on the existence or non–existence of *analytic or meromorphic* integrability in the sense of Liouville–Arnold. Our results are on the non–integrability in the sense of Liouville–Arnold for any second first integral of class $\mathcal{C}^1$. None of the above integrable cases satisfy the hypothesis of Theorem 1. As far as we know, all the integrable cases have $c \neq 0$, and we are studying the Hamiltonian (1) with two real parameters a, b, and $c = 0$.

2. The averaging theory

Some ingredients on averaging method are in Laplace's study on Sun-Jupiter-Saturn system. In his Mècanique Analytique, Lagrange uses averaging in the study of the planetary motion and the perturbing planets and he obtains an averaged secular equation. Poincaré [28] uses divergent series by the introduction of asymptotic series and averaging methods. An important development of the averaging theory in the XX century was made by Fatou [17], Krylov and Bogoliubov [20], and Bogoliubov and Metropolsky [6]. A general introduction to the averaging theory and more references can be found in the books of Sanders, Verhults and Murdock [30] and Verhults [31]. Now we recall the result on the averaging theory for finding periodic solutions of perturbed nonlinear periodic differential systems that we shall use here, more precisely see Theorems 1.1 and 1.2 of Buică *et. al.* in [8], and sections 2 and 3 for details and proofs of those results. In that paper the theory up to third order is explicitly presented, but in this work we only need the first order averaging theory. Instead of using the Implicit Function Theorem for proving the existence of periodic orbits, in [8] the authors used the Brouwer degree theory which needs weakened assumptions.

The first step of the averaging theory is to replace the problem of finding periodic orbits of a non–autonomous periodic differential system, to find the zeros of some finite dimensional function (the displacement function) directly related with the differential system.

Averaging Theorem (first order). *Consider the differential system*

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon), \tag{4}$$

where $F_1 : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ *are continuous functions, T-periodic in the first variable, and* D *is an open subset of* $\mathbb{R}^n$*. We define* $f_1 : D \to \mathbb{R}^n$ *as*

$$f_1(z) = \int_0^T F_1(s,z)ds\,,$$

and assume that for $a \in D$ *with* $f_1(a) = 0$*, there exist a neighborhood* $V \subset D$ *of* a *such that* $f_1(z) \neq 0$ *for all* $z \in \bar{V} \setminus \{a\}$ *and the Brouwer degree of the function* f_1 *at* a *is not zero, i.e.* $d_B(f_1\,, V,\, a) \neq 0$*.* ε_f *defines a neighborhood around the origin which depends on the function* $f = \varepsilon f_1 + \varepsilon^2 O(1)$*. Then for* $|\varepsilon| > 0$ *sufficiently small there exists a* T*–periodic solution* $\varphi(\cdot,\varepsilon)$ *of the system* (4) *such that* $\varphi(0,\varepsilon) \to a$ *when* $\varepsilon \to 0$*.*

A sufficient condition for showing that the Brouwer degree of a function f at a fixed point a is non–zero, is that the Jacobian of the function f at a (if exists) be non–zero (see [24]). If the function f_1 is identically zero, we would need to proceed to second order, but this will not be the case in this paper.

Of course, the Hamiltonian differential system (3) is not into the normal form (4) for applying the averaging theory. So we first introduce a rescaling transformation with a factor $\sqrt{\varepsilon}$ in order to have a small parameter ε in the Hamiltonian system. Second, doing convenient changes of variables and taking as new independent variable an angle coordinate instead the time , we obtain a 2π–periodic differential system. Third, fixing the energy level and omitting a redundant variable in every energy level, we will get the differential system written in the normal form for applying the averaging theorem of first order, and finally we shall prove the existence of some isolated periodic orbits in every energy level.

3. Proof of Theorem 1

Periodic orbits of a Hamiltonian system of more than one degree of freedom are generically on cylinders filled with periodic orbits in the phase space (for more details see [1]), then we will not be able to apply directly the Averaging Theorem of section 2 to a Hamiltonian system because the Jacobian of the corresponding function f_1 at the fixed point a will be always zero, and consequently the Brouwer degree can be zero. This problem will be solved by fixing an energy level, where the periodic orbits generically are isolated.

We need to introduce a small parameter ε, so in the Hamiltonian system (3) we do the following rescaling transformation from (x, y, p_x, p_y) to (X, Y, P_X, P_Y) such that $x = \sqrt{\varepsilon}X$, $y = \sqrt{\varepsilon}Y$, $p_x = \sqrt{\varepsilon}P_X$ and $p_y = \sqrt{\varepsilon}P_Y$. In the new variables the Hamiltonian system (3) becomes

$$
\begin{aligned}
\dot{X} &= P_X, \\
\dot{Y} &= P_Y, \\
\dot{P}_X &= -X - \varepsilon(a\,X^3 + b\,XY^2), \\
\dot{P}_Y &= -Y - \varepsilon\, b\, X^2 Y,
\end{aligned} \tag{5}
$$

with Hamiltonian

$$
H = \frac{1}{2}\left(P_X^2 + P_Y^2 + X^2 + Y^2\right) + \varepsilon\left(\frac{1}{4}\,a\,X^4 + \frac{1}{2}\,b\,X^2Y^2\right), \tag{6}
$$

which is a harmonic oscillator perturbed by a quartic potential. As the change to the new variables is only a rescaling transformation, the new Hamiltonian system (5) for all $\varepsilon \neq 0$ is topologically equivalent to the Yang–Mills system (3). Therefore studying the Hamiltonian system (5) for small values of $\varepsilon \neq 0$, we are also studying the original Yang–Mills Hamiltonian (3), i.e. system (3) with $\varepsilon = 1$.

Let $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{S}^1$ the circle. We do the change of variables $(X, Y, P_X, P_Y) \to (r, \theta, s, \alpha) \in \mathbb{R}^+ \times \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{S}^1$ defined by

$$
X = r\cos\theta, \quad P_X = r\sin\theta, \quad Y = s\cos(\theta + \alpha), \quad P_Y = s\sin(\theta + \alpha).
$$

This change of variables is well defined when $r > 0$ and $s > 0$. Note that it is not canonical, so we lost the Hamiltonian structure of the differential equations. The differential system in the new variables become

$$
\begin{aligned}
\dot r &= -\varepsilon\, r \sin(2\theta)\left(2ar^2\cos^2\theta + bs^2\cos^2(\theta+\alpha)\right),\\
\dot\theta &= -1 - 2\varepsilon\cos^2\theta\left(bs^2\cos^2(\theta+\alpha) + 2ar^2\cos^2\theta\right),\\
\dot s &= -\varepsilon\, b\, r^2 s\cos^2\theta\sin(2(\theta+\alpha)),\\
\dot\alpha &= 2\,\varepsilon\cos^2\theta\left(b\left(s^2-r^2\right)\cos^2(\theta+\alpha) + 2ar^2\cos^2\theta\right),
\end{aligned}
\tag{7}
$$

having the first integral

$$
H = \frac{1}{2}\left(r^2+s^2\right) + \varepsilon\, r^2\cos^2\theta\left(\frac{1}{2}bs^2\cos^2(\theta+\alpha) + \frac{1}{4}ar^2\cos^2\theta\right). \tag{8}
$$

In order that the left hand side of the differential system (7) be periodic with respect to the independent variable, we change the old independent variable t by the new independent variable θ, for obtaining the periodicity necessary for applying the averaging theory. Dividing system (7) by $\dot\theta$ omitting the $\dot\theta$ equation, system (7) goes over to

$$
\begin{aligned}
r' &= \varepsilon\, r\sin(2\theta)\left(bs^2\cos^2(\theta+\alpha) + 2ar^2\cos^2\theta\right) + O(\varepsilon^2),\\
s' &= \varepsilon\, b\, r^2 s\cos^2\theta\sin(2(\theta+\alpha)) + O(\varepsilon^2),\\
\alpha' &= -2\varepsilon\cos^2\theta\left(b\left(s^2-r^2\right)\cos^2(\theta+\alpha) + 2ar^2\cos^2\theta\right) + O(\varepsilon^2),
\end{aligned}
\tag{9}
$$

where the prime denotes the derivative with respect to the new independent variable θ. System (9) is 2π–periodic in the variable θ. However as the differential system (9) comes from a Hamiltonian system, as we mentioned before, its periodic orbits are not isolated in the set of all periodic orbits of system (9). Consequently, in order to use the averaging theory for studying its periodic orbits, we restrict the differential system (9) to every fixed energy level $H(r,\theta,s,\alpha)=h$. Then in such energy levels, we can put s as a function of h, θ, r and α and substitute s in (9), and will be able to apply the Averaging Theorem. For s we get

$$s=\frac{\sqrt{h-r^2/2-\varepsilon\, a\, r^4\cos^4\theta}}{\sqrt{1/2+\varepsilon\, b\, r^2\cos^2\theta\cos^2(\theta+\alpha)}}. \tag{10}$$

Notice that

$$s\to\sqrt{2h-r^2}\quad\text{when}\quad\varepsilon\to 0. \tag{11}$$

As we will apply averaging to first order, we can substitute this zero order approximation of s in equation (9), which becomes

$$\begin{aligned}
r' = {} & \frac{\varepsilon r\sin(2\theta)\left(b\cos^2(\theta+\alpha)\left(2h-r^2+\varepsilon 2ar^4\cos^4\theta\right)+2ar^2\cos^2\theta\right)}{1+\varepsilon 2br^2\cos^2\theta\cos^2(\theta+\alpha)}+O(\varepsilon^2),\\
\alpha' = {} & -2\varepsilon\cos^2\theta\left(b\cos^2(\theta+\alpha)\left(\frac{2h-r^2-\varepsilon ar^4\cos^4\theta}{1+\varepsilon 2br^2\cos^2\theta\cos^2(\theta+\alpha)}-r^2\right)+\right.\\
& \left. 2ar^2\cos^2\theta\right)+O(\varepsilon^2).
\end{aligned}$$

If we write the previous system as a Taylor series of first order in ε we get

$$\begin{aligned}
r' &= \varepsilon r\sin(2\theta)\left(b\left(2h-r^2\right)\cos^2(\theta+\alpha)+2ar^2\cos^2\theta\right)+O(\varepsilon^2)\\
&= \varepsilon F_{11}+O(\varepsilon^2),\\
\alpha' &= -\varepsilon 4\cos^2\theta\left(b\left(h-r^2\right)\cos^2(\theta+\alpha)+ar^2\cos^2\theta\right)+O(\varepsilon^2)\\
&= \varepsilon F_{12}+O(\varepsilon^2).
\end{aligned} \tag{12}$$

We see that system (12) has the canonical form (4) for applying the averaging theory and satisfies the assumptions of the Averaging Theorem for $|\varepsilon| > 0$ sufficiently small, with $T = 2\pi$ and $F_1 = (F_{11}, F_{12})$ which are analytical functions.

Averaging the function F_1 with respect to the variable θ we obtain

$$f_1(r,\alpha) = (f_{11}(r,\alpha), f_{12}(r,\alpha)) = \int_0^{2\pi} (F_{11}, F_{12})\, d\theta\,, \tag{13}$$

where

$$\begin{aligned} f_{11} &= \frac{1}{2}\, b\, r \sin\alpha \cos\alpha \left(r^2 - 2h\right), \\ f_{12} &= -\frac{1}{2}\left(b(\cos(2a) + 2)\left(h - r^2\right) + 3ar^2\right). \end{aligned} \tag{14}$$

We have to find the zeros (r^*, α^*) of $f_1(r,\alpha)$, and to check that the Jacobian determinant at these points is not zero, i.e.

$$\det\left(\left.\frac{\partial(f_{11}, f_{12})}{\partial(r,\alpha)}\right|_{(r,\alpha)=(r^*,\alpha^*)} \right) \neq 0. \tag{15}$$

From $f_{11} = 0$ we obtain that either $\alpha = 0, \pm\pi/2, \pi$, or $r^2 = 2h$. This last case is not a good solution, since from (11) we get $s = 0$ when $\varepsilon \to 0$, and the Jacobian determinant of f_1 will be always zero. Then we only consider the first four solutions for $f_{11} = 0$, and we now look for the solutions of $f_{12} = 0$ at these solutions. We obtain four possible solutions (r^*, α^*) with $r^* > 0$, namely

(16)
$$\left(\sqrt{\frac{h\,b}{b-a}}, 0\right), \left(\sqrt{\frac{h\,b}{b-a}}, \pi\right), \left(\sqrt{\frac{h\,b}{b-3a}}, \pi/2\right), \left(\sqrt{\frac{h\,b}{b-3a}}, -\pi/2\right),$$

with the corresponding values of s given by (11)

$$\sqrt{\frac{h\,(b-2a)}{b-a}}, \sqrt{\frac{h\,(b-2a)}{b-a}}, \sqrt{\frac{h\,(b-6a)}{b-3a}}, \sqrt{\frac{h\,(b-6a)}{b-3a}}, \tag{17}$$

respectively.

Finally we calculate the Jacobian determinant (15), i.e.

$$\begin{vmatrix} \frac{1}{4}b\left(3r^2-2h\right)\sin(2\alpha) & \frac{1}{2}br\left(r^2-2h\right)\cos(2\alpha) \\ r(-3a+2b+b\cos(2\alpha)) & b\left(h-r^2\right)\sin(2\alpha) \end{vmatrix}$$

at the four solutions (r^*,α^*) given in (16). The determinants are respectively given by

$$\frac{3b^2h^2(2a-b)}{2(a-b)}, \frac{3b^2h^2(2a-b)}{2(a-b)}, -\frac{b^2h^2(b-6a)}{2(b-3a)}, -\frac{b^2h^2(b-6a)}{2(b-3a)}, \tag{18}$$

and for $h \neq 0$ they are defined and non–zero only when

$$b(b-2a)(b-a) \neq 0, \tag{19}$$

for the first two solutions, and when

$$b(b-6a)(b-3a) \neq 0, \tag{20}$$

for the last two solutions.

Summarizing, from the Averaging Theorem of section 2, the two first solutions (r^*,α^*) of $f_1=0$ provide two periodic orbits of system (12) (and consequently of the Hamiltonian system (5) on the level $h \neq 0$) if $hb(b-a) > 0$ and $h(b-2a)(b-a) > 0$, where the Jacobian condition (19) is included. These conditions imply that $b(b-2a) > 0$.

Similarly, the last two solutions provide two periodic orbits of system (12) if $hb(b-3a) > 0$ and $h(b-6a)(b-3a) > 0$, where the Jacobian condition

(20) is included. These conditions imply that $b(b-6a)>0$. This completes the proof of Theorem 1.

4. Periodic orbits and the Liouville–Arnold integrability

In this section we summarize some facts on the Liouville–Arnold integrability of the Hamiltonian systems, and on the theory of the periodic orbits of the differential equations, see [1, 4] and the subsection 7.1.2 of [4]. We present these results for Hamiltonian systems of two degrees of freedom, *but these results work for an arbitrary number of degrees of freedom.*

We recall that a Hamiltonian system with Hamiltonian H of two degrees of freedom is *integrable in the sense of Liouville–Arnold* if it has a first integral C independent with H (i.e. the gradient vectors of H and C are *independent* in all the points of the phase space except perhaps in a set of zero Lebesgue measure), and in *involution* with H (i.e. the parenthesis of Poisson of H and C is zero). For Hamiltonian systems with two degrees of freedom the involution condition is redundant, because the fact that C is a first integral of the Hamiltonian system, implies that the mentioned Poisson parenthesis is always zero. A flow defined on a subspace of the phase space is *complete* if its solutions are defined for all time.

Now we shall state the Liouville–Arnold Theorem [1, 4] restricted to Hamiltonian systems of two degrees of freedom.

Liouville-Arnold Theorem. *Suppose that a Hamiltonian system with two degrees of freedom defined on the phase space M has its Hamiltonian H and the function C as two independent first integrals in involution. If $I_{hc} = \{p \in M : H(p) = h \text{ and } C(p) = c\} \neq \emptyset$ and (h, c) is a regular value of the differentiable map (H, C), then the following statements hold.*

(a) *I_{hc} is a two dimensional submanifold of M invariant under the flow of the Hamiltonian system.*

(b) *If the flow on a connected component I_{hc}^* of I_{hc} is complete, then I_{hc}^* is diffeomorphic either to the torus $\mathbb{S}^1 \times \mathbb{S}^1$, or to the cylinder $\mathbb{S}^1 \times \mathbb{R}$, or to the plane $\mathbb{R}^2$. If I_{hc}^* is compact, then the flow on it is always complete and $I_{hc}^* \approx \mathbb{S}^1 \times \mathbb{S}^1$.*

(c) *Under the hypothesis (b) the flow on I_{hc}^* is conjugated to a linear flow on $\mathbb{S}^1 \times \mathbb{S}^1$, on $\mathbb{S}^1 \times \mathbb{R}$, or on $\mathbb{R}^2$.*

The main result of this theorem is that the connected components of the invariant sets associated with the two independent first integrals in involution are generically diffeomorphic to 2–dimensional tori, cylinders or planes inside the phase space, and if the flow on them is complete then the flow restricted to these surfaces is conjugated to a linear flow.

Using the notation of the Liouville-Arnold Theorem, when a connected component I_{hc}^* is diffeomorphic to a torus, either all orbits on this torus are periodic if the rotation number associated to this torus is rational, or they are quasi–periodic (i.e. every orbit is dense in the torus) if the rotation number associated to this torus is not rational.

We consider the autonomous differential system

$$\dot{x} = f(x), \tag{21}$$

where $f\colon U \to \mathbb{R}^n$ is C^2, U is an open subset of $\mathbb{R}^n$ and the dot denotes the derivative with respect to the time t. We write its general solution as $\phi(t, x_0)$ with $\phi(0, x_0) = x_0 \in U$ and t belonging to its maximal interval of definition.

We say that $\phi(t, x_0)$ is T*–periodic* with $T > 0$ if and only if $\phi(T, x_0) = x_0$ and $\phi(t, x_0) \neq x_0$ for $t \in (0, T)$. The periodic orbit associated to the periodic solution $\phi(t, x_0)$ is $\gamma = \{\phi(t, x_0), t \in [0, T]\}$. The variational equation associated to the T–periodic solution $\phi(t, x_0)$ is

$$\dot{M} = \left(\frac{\partial f(x)}{\partial x}\Big|_{x=\phi(t,x_0)} \right) M, \tag{22}$$

where M is an $n \times n$ matrix. The *monodromy matrix* associated to the T–periodic solution $\phi(t, x_0)$ is the solution $M(T, x_0)$ of (22) satisfying that $M(0, x_0)$ is the identity matrix. The eigenvalues λ of the monodromy matrix associated to the periodic solution $\phi(t, x_0)$ are called the *multipliers* of the periodic orbit.

For an autonomous differential system one of the multipliers is always 1, the corresponding eigenvector is tangent to the periodic orbit.

A periodic solution of an autonomous Hamiltonian system always has two multipliers equal to one. One multiplier is 1 because the Hamiltonian system is autonomous, and another is 1 due to the existence of the first integral given by the Hamiltonian.

Poincaré Theorem. *If a Hamiltonian system with two degrees of freedom and Hamiltonian H is Liouville–Arnold integrable, and C is a second first integral such that the differentials of H and C are linearly independent at each point of a periodic orbit of the system, then all the multipliers of this periodic orbit are equal to* 1*.*

This theorem is due to Poincaré [28]. It gives us a method for proving that the non Liouville–Arnold integrability, based on the existence of periodic orbits of Theorem 1, must be for any second first integral of class C^1. The main problem is to find periodic orbits having multipliers different from 1.

Proof of Theorem 2. We assume that we are under the assumptions of Theorem 1 and some of the periodic orbits (16) exist, with their respective values of s given by (17). Moreover their associated Jacobians (18) are different from 0. Then playing with the energy level h since these Jacobians are the product of the multipliers of these periodic orbits, all the multipliers cannot be equal to 1 when we change the value of h. Hence under the assumptions of Theorem 1, by the Poincaré Theorem, either the Yang–Mills Hamiltonian systems cannot be Liouville–Arnold integrable with any second first integral C, or the system is Liouville–Arnold integrable and the differentials of H and C are linearly dependent at almost all the points of these periodic orbits. Therefore the theorem is proved. □

5. Conclusions

We have used two tools. First the averaging method for studying the periodic orbits of the Hamiltonian systems in their fixed energy levels, and second the Poincaré Theorem which allows to prove that the non–integrability in the sense of Liouville–Arnold of the Hamiltonian systems, is for any second first integral of class $\mathcal{C}^1$. We applied both tools to the generalized Yang–Mills Hamiltonian system (5) with two degrees of freedom, obtaining Theorems 1 and 2. We remark that these two tools can be applied to Hamiltonian systems with an arbitrary number of degrees of freedom.

An important remark is that these two tools are based on the study of the periodic orbits via the averaging method, which needs a small parameter, and this can be easily introduced if we study Hamiltonian systems near Liouville–Arnold integrable ones, in our case near the harmonic oscillator. However, the two Hamiltonian systems (3) and (5) have qualitatively the same phase portrait, since the scale transformation introduced in the section 3 does not change the topology of the system and these results are valid for the original problem (3) with $\varepsilon = 1$.

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References

[1] Abraham R. and Marsden J.E., *Foundations of Mechanics*, Benjamin, Reading, Masachusets, 1978.

[2] Almeida M., Moreira I. and Santos F, *On the Ziglin-Yoshida Analysis for some classes of homogeneous Hamiltonian systems*, Braz. J. Phys. **28** (1998), 10.1590 S0103-97331998000400022.

[3] Arnold V.I., *Forgotten and neglected theories of Poincaré*, Russian Math. Surveys **61** (2006), 1–18.

[4] Arnold V.I., Kozlov V. and Neishtadt A., *Dynamical Systems III. Mathematical Aspects of Classical and Celestial Mechanics*, Third Edition, Encyclopaedia of Mathematical Science, Springer, Berlin, 2006.

[5] Biswas D., Azam M., Lawande Q. and Lawande S., *Existence of stable periodic orbits in the x^2y^2 potential: a semiclassical approach*, J. Phys. A: Math. Gen. **25** (1992), L297–L301.

[6] Bogoliubov N. and Mitropolsky Y., *Asymptotic Methods in the theory of nonlinear oscillations*, Gordon and Breach NY (1961).

[7] Bountis T., Segur H. and Vivaldi F., *Integrable Hamiltonian systems and the Painlevé propety*, Phys. Rev. A, **25** (1982), 1257–1264.

[8] Buică A. and Llibre J., *Averaging methods for finding periodic orbits via Brouwer degree*, Bull. Sci. Math. **128** (2004), 7–22.

[9] Caranicolas N. and Varvoglis H., *Families of periodic orbits in a quartic potential*, Astron. Astrophys. **141** (1984), 383–388.

[10] Cimpoiasu R., Constantinescu R. and Cimpoiasu M., *Integrability of dynamical systems with polinomial Hamiltonians*, Rom. Jour. Phys. **50** (2005), 317–324.

[11] Contopoulos G., Papadaki H. and Polymilis C., *The structure of chaos in a potential without escapes*, Cel. Mech. and Dyn. Astr. **60** (1994), 249–271.

[12] Contopoulos G., Harssoula M., Voglis N. and Dvorak R., *Destruction of islands of stability*, J. Phys. A: Math. Gen. **32** (1999), 5213–5232.

[13] Contopoulos G., Efthymiopoulos C. and Giorgilli A, *Non-convergence of formal integrals of motion*, J. Phys. A: Math. Gen. **36** (2003), 8639–8660.

[14] Dorizzi B., Grammaticos B. and Ramani A., *A new class of integrable systems*, J. Math. Phys. **24** (1983), 2282–2288.

[15] Elipe A., Hietarinta J. and Tompadis S., *Comment on a paper by Kasperczuk, Cel. Mech.* **58** *(1994), 378-391*, Cel. Mech. and Dyn. Astron. **62** (1995), 191–192.

[16] Falconi M., Lacomba E. and Vidal C., *On the dynamics of mechanical systems with homogeneous polynomial potentials of degree 4*, Bull. Braz. Math. Soc., New Series **38** (2007), 301–333.

[17] Fatou P., *Sur le ,ouvement d'un système soumis a des forces a courte pèriode*, Bull. Soc. Math. **56**, (1928), 98–139.

[18] Friedberg R., Lee T. and Sirlin A., *Class of scalar-field soliton solutions in three space dimensions*, Phys. Rev. D **13** (1976), 2739–2761.

[19] Grammaticos B., Dorizzi B. and Ramani A., *Integrability with Hamiltonians with third- and foruth–degree polynomial potentials*, J. Math. Phys. **24** (1983), 2289–2295.

[20] Krylov N. and Bogoliubov N., *Introduction to non–linear mechanics (in russian)* Patent No. 1, Kiev, (1937).

[21] Jiménez-Lara L. and Llibre, J., *Periodic orbits and non–integrability of Henon-Heiles systems*, submited for publication.

[22] Kozlov V.V., *Integrability and non-integrability in Hamiltonian mechanics*, Russian Math. Surveys **38** (1983), 1–76.

[23] Kasperczuc S., *Integrability of the Yang–Mills Hamiltonian System*, Cel. Mech. and Dyn. Astr. **58** (1994), 387–391.

[24] Lloyd, N.G., *Degree Theory*, Cambridge University Press, 1978.

[25] Maciejewski A., Radzki W. and Rybicki S., *Periodic trajectories near degenerate equilibria in the Henon-Heiles and Yang-Mills Hamiltonian systems*, J. Dyn. Sys. and Diff. Eq. **17** (2005), 475–488.

[26] Markus, L. and Meyer, K.R., *Generic Hamiltonian Dynamical Systems are neither integrable nor ergodic*, Memoirs of the Amer. Math. Soc. **144**, 1974.

[27] Morales-Ruiz J.J., *Differential Galois theory and non-integrability of Hamiltonian systems*, Progress in Math. Vol. **178**, Birkhauser, Verlag, Basel, 1999.

[28] Poincaré H., *Les méthodes nouvelles de la mécanique céleste*, Vol. I, Gauthier-Villars, Paris 1899.

[29] Ramani A., Dorizzi B. and Grammaticos B, *Painlevé conjecture revisited*, Phys. Rev. Lett. **49** (1982), 1539–1541.

[30] J. A. Sanders, F. Verhultst and J. Murdock, *Averaging Methods in Nonlinear Dynamical Systems*, Applied Mathematical Sci. **59**, Springer Verlag, New York, rev. ed. 2007.

[31] F. Verhulst, *Nonlinear Differential Equations and Dynamical Systems*, Universitext Springer Verlag, 1996.

[32] Ziglin S., *Branching of Solutions and the nonexistence of first integrals in Hamiltonian Mechanics. II*, Functional Analysis and Its Applications **17** (1983), 6-17.

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