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ON THE INTEGRABILITY OF THE DIFFERENTIAL SYSTEMS IN DIMENSION TWO AND OF THE POLYNOMIAL DIFFERENTIAL SYSTEMS IN ARBITRARY DIMENSION

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Abstract. This is a survey on recent results providing sufficient conditions for the existence of a first integral, first for vector fields defined on real surfaces, and second for polynomial vector fields in $\mathbb{R}^n$ or $\mathbb{C}^n$ with $n \geq 2$. We also provide an open question and some applications based on the existence of such first integrals.

1. Introduction

In many branches of applied mathematics, physics and, in general, in applied sciences appear nonlinear ordinary differential equations. If a differential equation or vector field defined in a real or complex manifold has a first integral, then its study can be reduced in one dimension. Therefore a natural question is: *Given a vector field on a manifold, how to recognize if this vector field has a first integral?* In general this question has no a good answer up to now.

In this survey we provide sufficient conditions for the existence of a first integral, first for vector fields defined on two–dimensional real manifolds (or surfaces), and second for polynomial vector fields in $\mathbb{R}^n$ or $\mathbb{C}^n$ with $n \geq 2$. An open question which essentially goes back to Poincaré is presented. Finally some applications on the existence of these kind of first integrals to physical problems, centers, foci, limit cycles and invariant hyperplanes are mentioned.

The table of contents of the survey is:

Sections 2, 3, 5 and 6 are dedicated to study the integrability of two–dimensional differentiable systems, and section 4 is dedicated to study the integrability of polynomial differential systems $\mathbb{R}^n$ or $\mathbb{C}^n$ with $n \geq 2$.

2. Two dimensional real integrability

For a vector field defined on a two–dimensional real manifold (or surface) the existence of a first integral determines completely its phase portrait. For such vector fields the notion of integrability is based on the existence of a first integral. In what follows we summarize some results which provide sufficient conditions for the existence of a first integral.

2.1 Two–dimensional vector fields

Let

$$\mathcal{X} = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y},$$

be a $\mathcal{C}^r$ *vector field* with $r \geq 1$, defined on a two dimensional differentiable manifold M of class $\mathcal{C}^r$. Here $r \geq 1$ means that $r = 1, 2, \cdots, \infty$, or ω. Of course, when $r = \omega$, the flow is analytic. In all this paper a *two dimensional manifold* means a two dimensional differentiable manifold of class $\mathcal{C}^r$, connected and without boundary, but not necessarily compact nor orientable.

2.2 Two–dimensional flows

Let $p \in M$ and $\phi_p(t)$ be the solution of $\mathcal{X}$ such that $\phi_p(0) = p$. The $\mathcal{C}^r$ map $\phi(t, p) = \phi_p(t)$ is the *local flow* associated to the vector field $\mathcal{X}$. This flow satisfies $d\phi_p(t)/dt = \mathcal{X}(\phi_p(t))$, and

(i) $\phi(0, p) = p$ for all $p \in M$;
(ii) $\phi(t, \phi(s, p)) = \phi(t+s, p)$ for all $p \in M$, and all s and t such that $s,\ t+s \in I_p$;
(iii) $\phi_p(-t) = \phi_p^{-1}(t)$ for all $p \in M$ such that $t,\ -t \in I_p$.

Two flows (M, ϕ) and (M', ϕ') are C^k *equivalent* with $1 \leq k \leq r$ if there is a C^k diffeomorphism of M onto M' which takes orbits of ϕ onto orbits of ϕ', preserving or reversing simultaneously the sense of all orbits.

2.3 Parallel flows

Let ϕ be a C^r local flow on the two dimensional manifold M for $r \geq 1$. The flow (M, ϕ) is C^k *parallel* if it is C^k–equivalent, with $1 \leq k \leq r$, to either the *strip*, the *annular*, the *spiral* or *radial*, or the *toral* flow. More precisely, these flows are respectively:

- $(\mathbb{R}^2, \phi)$ with the flow ϕ defined by $x' = 1$, $y' = 0$;
- $(\mathbb{R}^2 \setminus \{0\}, \phi)$ with the flow ϕ defined (in polar coordinates) by $r' = 0$, $\theta' = 1$;
- $(\mathbb{R}^2 \setminus \{0\}, \phi)$ with the flow ϕ defined by $r' = r$, $\theta' = 0$; (i.e. this flow is called spiral or radial because in $\mathbb{R} \setminus \{0\}$ the flows $r' = r$, $\theta' = 1$ and $r' = r$, $\theta' = 0$ are equivalent).
- $(\mathbb{S}^1 \times \mathbb{S}^1, \phi)$ with rational flow ϕ (i.e. the flow $x' = 1$, $y' = 0$ over the usual covering space $\mathbb{R}^2$ with rational slope; note in particular that all rational flows on the torus are equivalent).

The definition of parallel flow was given by Neumann in [40].

2.4 Separatrices

Let $p \in M$. We denote by $\gamma(p)$ the *orbit* of the flow (M, ϕ) passing through p (i.e. $\gamma(p) = \{\phi(t, p) : t \in I_p\}$), and by $\gamma^+(p)$ (respectively $\gamma^-(p)$) the *positive semiorbit* (respectively *negative semiorbit*), i.e. $\gamma^+(p) = \{\phi(t, p) : t \in I_p \text{ and } t \geq 0\}$ (respectively $\gamma^-(p) = \{\phi(t, p) : t \in I_p \text{ and } t \leq 0\}$).

We define by

$$\alpha(p) = \mathrm{cl}(\gamma^-(p)) - \gamma^-(p) \qquad (\text{respectively}\omega(p) = \mathrm{cl}(\gamma^+(p)) - \gamma^+(p)),$$

the α–*limit* of $\gamma^-(p)$ (respectively ω–*limit* of $\gamma^+(p)$). Here cl(A) denotes the closure of the subset A in M. We note that these definitions of α– and ω–limit sets do not coincide with the classical ones (see for instance [23]), but they are good for defining the notion of separatrix. More precisely the differences are that now the α– and ω–limit sets of a singular point or a periodic orbit are the empty set.

An open neighborhood U of an orbit $\gamma(p)$ of the C^r flow (M, ϕ) is said to be a C^k *parallel neighborhood* with $1 \leq k \leq r$ if (U, ϕ) is C^k equivalent to a parallel flow for some $k \geq 1$.

We define the notion of separatrix following to Neumann [40], but we remark that Markus [38] gave this notion of separatrix in the plane. An orbit $\gamma(p)$ is a *separatrix* of the flow (M, ϕ) if it is not contained in a parallel neighbourhood (U, ϕ) satisfying the following two assumptions:

(a) for any $q \in U$, $\alpha(q) = \alpha(p)$ and $\omega(q) = \omega(p)$;
(b) $\mathrm{cl}(U) \backslash U$ consists of $\alpha(p)$, $\omega(p)$ and exactly two orbits $\gamma(a)$, $\gamma(b)$ of ϕ with $\alpha(a) = \alpha(p) = \alpha(b)$ and $\omega(a) = \omega(p) = \omega(b)$.

With this definition of separatrix it is easy to check that any singular point, any limit cycle and any local separatrix of a hyperbolic sector are separatrices. We remark that an orbit of an irrational flow on the torus is not a separatrix.

2.5 Canonical regions

We denote by Σ the union of all separatrices of the flow (M, ϕ). Then Σ is a closed invariant subset of M. Every connected component of the complement of Σ in M, with the restricted flow, is called a *canonical region* of ϕ.

2.6 First integrals

We say that a non–constant C^k function $H : M\backslash\Sigma' \to \mathbb{R}$ with $k \geq 0$ and $\Sigma' \subset \Sigma$ is a *first integral* of the vector field $\mathcal{X}$ on $M\backslash\Sigma'$, if H is constant on every orbit of $\mathcal{X}$ contained in $M\backslash\Sigma'$. This definition is equivalent to

$$\mathcal{X}H = \frac{\partial H}{\partial x}P + \frac{\partial H}{\partial y}Q \equiv 0,$$

on $M\backslash\Sigma'$.

2.7 Every canonical region is integrable

The next result shows that any flow on a two–dimensional manifold M has a first integral on every canonical region of the same differentiability than the flow. Consequently in order to have a first integral of the flow on $M\backslash\Sigma'$ for some Σ' contained in the set of all separatrices of the flow, we need that that the first integrals H_i on every canonical region C_i coincide. Therefore *for a $\mathcal{C}^r$ vector field with $r \neq \omega$ we always can define a $\mathcal{C}^r$ first integral on $M \setminus \Sigma$ doing $H|_{C_i} = H_i$.*

Theorem 1. *Let ϕ be a $\mathcal{C}^r$ flow on a two dimensional manifold M with $r \in \{1, 2, \ldots, \infty, \omega\}$, and let Σ be the union of all its separatrices. Then the flow restricted to every canonical region*

(a) *is $\mathcal{C}^r$ parallel, and*
(b) *has a $\mathcal{C}^r$ first integral.*

Theorem 1 is proved in [27] by Li, Llibre, Nicolau and Zhang. Statement (a) and (b) in the case $\mathcal{C}^0$ parallel was proved by Neumann [40] and Chavarriga, Giacomini, Giné, and Llibre [4], respectively.

2.8 Planar real differential systems

In this subsection we consider a real planar differential system or simply a *differential system*

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \tag{1}$$

where P and Q are C^r functions for $r \geq 0$ defined in an open subset M of $\mathbb{R}^2$, x and y are real variables and the dot denotes derivative with respect to the independent variable t. If P and Q are polynomials in the variables x and y, we say that (1) is a *polynomial differential system.*

In the next two theorems we use that any analytic vector field on $\mathbb{S}^2$ coming from the Poincaré compactification of a polynomial vector field in $\mathbb{R}^2$ has finitely many limit cycles as it was proved by Il'Yashenko [24] and Écalle [18]. For more details on the Poincaré compactification see for instance [17]. The following result improves Theorem 1 for planar polynomial differential systems, for a proof see [27].

Theorem 2. *For every real planar polynomial differential system there exist finitely many separatrices in $\mathbb{R}^2$ namely γ_i, $i = 1, 2, \cdots, l$, such that $\mathbb{R}^2\backslash \left(\bigcup_{i=1}^{l} \gamma_i \right)$ has* *finitely many connected open sets, and on each of these connected sets the system has an analytic first integral.*

3. Planar real differential systems

Let

$$\frac{dx}{dt} = \dot{x} = P(x,y), \qquad \frac{dy}{dt} = \dot{y} = Q(x,y), \tag{2}$$

be a two dimensional *(real) differential system* where the dependent variables x and y are real, and P and Q are $\mathcal{C}^r$ functions from an open subset M of $\mathbb{R}^2$ in $\mathbb{R}$. As usual we denote the *vector field associated to differential system* (2) as

$$\mathcal{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

If P and Q are polynomials then system (2) is called a *polynomial differential system.* We denote by $m = \max\{\deg P, \deg Q\}$ the *degree* of the polynomial system, and we always assume that the polynomials P and Q are relatively prime in the ring of complex polynomials in the variables x and y.

3.1 Hamiltonian systems

The more easiest planar differential systems having a first integral are the *Hamiltonian ones*, i.e. the differential systems of the form

$$\dot{x} = -H_y, \qquad \dot{y} = H_x,$$

where $H = H(x,y)$ is a differentiable function defined on M. Of course for this system H is a first integral. As usual H_x denotes the partial derivative of the function H with respect to the variable x. A planar differential system having a first integral is called *integrable. The integrable planar differential systems which are not Hamiltonian are in general very difficult to detect*, see for instance [4].

3.2 Integrating factors

A $\mathcal{C}^k$ non–constant function $R = R(x,y)$ defined in an open and dense subset $M \backslash \Sigma'$ of M, is called an *integrating factor* of system (2) if the *divergence*

$$\operatorname{div}(RP, RQ) = (RP)_x + (RQ)_y = 0,$$

on $M \backslash \Sigma'$. Then the differential system

$$\dot{x} = RP, \qquad \dot{y} = RQ,$$

is Hamiltonian, i.e. there exists a function H such that

$$\dot{x} = RP = -H_y, \qquad \dot{y} = RQ = H_x.$$

We say that H is the *first integral associated to the integrating factor* R, and vice versa R is the *integrating factor associated to the first integral* H.

3.3 Linearization of integrable non–Hamiltonian differential systems

The next result, due to Giné and Llibre [21], shows that *with the exception of the Hamiltonian systems the integrability of a planar differential system is essentially due to the integrability of the radial linear differential system* $\dot{u} = u$, $\dot{v} = v$.

Theorem 3. *Consider a $\mathcal{C}^r$ planar real differential system* (2) *defined in an open subset M of $\mathbb{R}^2$ having a $\mathcal{C}^k$ first integral H and a $\mathcal{C}^l$ integrating factor R defined in open and dense subsets M_H and M_R of M, respectively. Assume that the Lebesgue measure of the set $\{R(R_xH_y - R_yH_x)(P_x + Q_y) = 0\}$ in $M_H \cap M_R$ is zero. Then the change of variables $(x, y, t) \mapsto (u, v, s)$ defined by*

$$u = R(x, y), \quad v = R(x, y)H(x, y), \quad ds = -\mathrm{div}(P, Q)\, dt, \tag{3}$$

in the open and dense subset

$$(M_H \cap M_R) \setminus \{R(R_xH_y - R_yH_x)(P_x + Q_y) = 0\} \tag{4}$$

of M, transforms system (2) *restricted to the open and dense subset* (4) *into the linear differential system $du/ds = u$ and $dv/ds = v$.*

In the statement of Theorem 3 the integrating factor R may be (but it is not necessary) the one associated to the first integral H.

Note that if system (2) is Hamiltonian then $P_x + Q_y = 0$ and consequently the assumptions (3) and (4) of Theorem 3 do not hold.

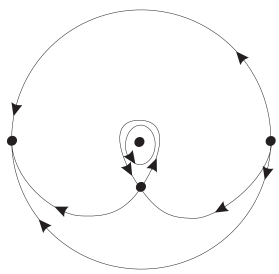

FIGURE 1. Phase portrait of system (5) in the Poincaré disc.

Example. We consider the real polynomial differential system

$$\frac{dx}{dt} = -y - b(x^2 + y^2), \qquad \frac{dy}{dt} = x, \tag{5}$$

with $b \neq 0$. See its phase portrait in Figure 1. It is easy to check that this system has the first integral $H = e^{2by}(x^2+y^2)$, and the integrating factor $R = 2e^{2by}$. Then, by Theorem 3, the change of variables $(x, y, t) \mapsto (u, v, s)$ defined by

$$u = 2e^{2by}, \quad v = 2e^{4by}(x^2 + y^2), \quad ds = 2\, b\, x\, dt,$$

in $(x, y) \in \mathbb{R}^2 \setminus \{x = 0\}$ transforms the differential system (5) into the linear differential system

$$\frac{du}{ds} = u, \qquad \frac{dv}{ds} = v.$$

4. Darboux theory of integrability in $\mathbb{R}^n$ or $\mathbb{C}^n$ with $n \geq 2$.

Since any polynomial differential system in $\mathbb{R}^n$ can be thought as a polynomial differential system inside $\mathbb{C}^n$ we shall work only in $\mathbb{C}^n$. If our initial differential system is in $\mathbb{R}^n$, once we get a complex first integral of this system inside $\mathbb{C}^n$ taking the square of the modulus of this complex integral we have a real first integral.

Moreover if that complex first integral is rational, the square of its modulus also is rational. In short in the rest of the paper we shall work in $\mathbb{C}^n$.

In this section we study the existence of first integrals for polynomial vector fields in $\mathbb{C}^n$ through the Darboux theory of integrability. The algebraic theory of integrability is a classical one, which is related with the first part of the Hilbert's 16th problem [22]. This kind of integrability is usually called Darboux integrability, and it provides a link between the integrability of polynomial vector fields and the number of invariant algebraic hypersurfaces that they have (see [14] and [44]). This theory shows the fascinating relationships between integrability (a topological phenomenon) and the existence of exact algebraic invariant hypersurfaces formed by solutions for the polynomial vector field. In fact Darboux [14] showed how we can constructed first integrals of planar polynomial differential systems in $\mathbb{R}^2$ or $\mathbb{C}^2$ possessing sufficient invariant algebraic curves. This theory is now known as the *Darboux theory of integrability*, see for more details the Chapter 8 of [17].

4.1 Polynomial vector fields in $\mathbb{C}^n$

As usual $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$ denotes the ring of all complex polynomials in the variables $x_1, \ldots, x_n$. We consider the *polynomial vector field* in $\mathbb{C}^n$

$$\mathcal{X} = \sum_{i=1}^{n} P_i(x) \frac{\partial}{\partial x_i}, \qquad x = (x_1, \ldots, x_n) \in \mathbb{C}^n,$$

where $P_i = P_i(x) \in \mathbb{C}[x]$ have no common factor for $i = 1, \ldots, n$. The integer $d = \max\{\deg P_1, \ldots, \deg P_n\}$ is the *degree* of the vector field $\mathcal{X}$. Usually for simplicity the vector field $\mathcal{X}$ will be represented by $(P_1, \ldots, P_n)$.

4.2 Invariant algebraic hypersurfaces and Darboux polynomials

Let $f = f(x) \in \mathbb{C}[x]$. We say that $\{f = 0\} \subset \mathbb{C}^n$ is an *invariant algebraic hypersurface* or a *Darboux polynomial* of the vector field $\mathcal{X}$ if there exists a polynomial $K_f \in \mathbb{C}[x]$ such that

$$\mathcal{X}(f) = \sum_{i=1}^{n} P_i \frac{\partial f}{\partial x_i} = f K_f.$$

The polynomial K_f is called the *cofactor* of $f = 0$. Note that from this definition the degree of K_f is at most $d - 1$, and also that if an orbit $x(t)$ of the vector field $\mathcal{X}$ has a point on $f = 0$, then the whole orbit is contained in $f = 0$. This justifies the name of invariant algebraic hypersurface, because it is invariant by the flow of the vector field $\mathcal{X}$.

Of course if the dimension n is equal to 2, then an invariant algebraic hypersurface is an *invariant algebraic curve*.

If the polynomial f is irreducible in $\mathbb{C}[x]$, then we say that $f = 0$ is an *irreducible* invariant algebraic hypersurface, or that f is an *irreducible* Darboux polynomial.

We remark that in the definition of invariant algebraic hypersurface $f = 0$ we always allow this curve to be complex; that is $f \in \mathbb{C}[x]$ even in the case of a real polynomial vector fields. As we will see this is due to the fact that sometimes for real polynomial vector fields the existence of a real first integral can be forced by the existence of complex invariant algebraic hypersurfaces, see for more details the chapter 8 of [17]. Of course when we look for a complex invariant algebraic hypersurface of a real polynomial system we are thinking in the real polynomial system as a complex one.

The next result shows that it is sufficient to look for the irreducible invariant algebraic hypersurfaces or the irreducible Darboux polynomials.

Proposition 4. *Suppose $f \in \mathbb{C}[x]$ and let $f = f_1^{n_1} \dots f_r^{n_r}$ be its factorization into irreducible factors over $\mathbb{C}[x]$. Then for a polynomial vector field $\mathcal{X}$, $f = 0$ is an invariant algebraic hypersurface with cofactor K_f if and only if $f_i = 0$ is an invariant algebraic hypersurface for each $i = 1, \dots, r$ with cofactor K_{f_i}. Moreover $K_f = n_1 K_{f_1} + \dots + n_r K_{f_r}$.*

4.3 Exponential factors

If $f, g \in \mathbb{C}[x]$ are coprime, we write $(f, g) = 1$. Suppose that $(f, g) = 1$, we say that $\exp(g/f)$ is an *exponential factor* of the vector field $\mathcal{X}$ if there exists a polynomial $L_e \in \mathbb{C}[x]$ of degree at most $d - 1$ such that

$$\mathcal{X}(\exp(g/f)) = \exp(g/f) L_e.$$

The polynomial L_e is called the *cofactor* of the exponential factor. It is easy to prove that if $\exp(g/f)$ is an exponential factor, then $f = 0$ is an invariant algebraic hypersurface.

4.4 Multiplicity of an invariant algebraic hypersurface or of a Darboux polynomial

Let $\mathbb{C}_m[x]$ be the $\mathbb{C}$–vector space of polynomials in $\mathbb{C}[x]$ of degree at most m. Then it has dimension $R = \begin{pmatrix} n+m \\ n \end{pmatrix}$. Let $v_1, \dots, v_R$ be a base of $\mathbb{C}_m[x]$. Denote by M_R the $R \times R$ matrix

$$\begin{pmatrix} v_1 & v_2 & \dots & v_R \\ \mathcal{X}(v_1) & \mathcal{X}(v_2) & \dots & \mathcal{X}(v_R) \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{X}^{R-1}(v_1) & \mathcal{X}^{R-1}(v_2) & \dots & \mathcal{X}^{R-1}(v_R) \end{pmatrix}, \tag{6}$$

where $\mathcal{X}^{k+1}(v_i) = \mathcal{X}(\mathcal{X}^k(v_i))$. Then $\det M_R$ is the *m–th extactic polynomial* of $\mathcal{X}$. From the properties of the determinant we note that the extactic polynomial is independent of the choice of the base of $\mathbb{C}_m[x]$. Observe that if $f = 0$ is an invariant algebraic hypersurface of degree m of $\mathcal{X}$, then f divides the polynomial $\det M_R$. This is due to the fact that if f is a member of a base of $\mathbb{C}_m[x]$, then f divides the whole column in which f is located.

An algebraic hypersurface $f = 0$ is *irreducible* if f is an irreducible polynomial in $\mathbb{C}[x]$. We say that an irreducible invariant algebraic hypersurface $f = 0$ of degree m has *defined algebraic multiplicity k* or simply *algebraic multiplicity k* if $\det M_R \not\equiv 0$ and k is the maximum positive integer such that f^k divides $\det M_R$; and it has *no defined algebraic multiplicity* if $\det M_R \equiv 0$.

We remark that the matrix (6) already appears in the work of Lagutinskii (see also Dobrovol'skii *et al* [16]). For a modern definition of the m–th extactic hypersurface and a clear geometric explanation of its meaning, the readers can look at Pereira [42]. Christopher *et al* [12] used the extactic curve to study the algebraic multiplicity of invariant algebraic curves of planar polynomial vector fields, and prove the equivalence of the algebraic multiplicity with other three ones: the infinitesimal multiplicity, the integrable multiplicity and the geometric multiplicity.

4.5 Multiplicity and exponential factors

The next result presents a characterization under suitable assumptions of the algebraic multiplicity of an invariant algebraic hypersurface using the number of exponential factors of $\mathcal{X}$ associated with the invariant algebraic hypersurface. This characterization due to Llibre and Zhang [34] extends the algebraic multiplicity introduced by Christopher, Llibre and Pereira in [12] for invariant algebraic curves of $\mathbb{C}^2$ to invariant algebraic hypersurfaces of $\mathbb{C}^n$. This result is a key point in the Darboux theory of integrability in arbitrary dimension.

Theorem 5. *Let $\mathcal{X}$ be a polynomial vector field. For a given irreducible invariant algebraic hypersurface $f = 0$ of $\mathcal{X}$ with f of degree m, assume that $\mathcal{X}$ restricted to $f = 0$ has no rational first integral. Then f has a defined algebraic multiplicity k if and only if the vector field $\mathcal{X}$ has $k-1$ exponential factors $\exp(g_i/f^i)$, where g_i is a polynomial of degree at most im and $(g_i, f) = 1$, for $i = 1, \ldots, k-1$.*

We remark that if $\mathcal{X}$ is a planar vector field, then clearly Theorem 5 always holds without the assumption on the non-existence of rational first integrals on $f = 0$. For higher dimensional systems the assumption is necessary as the following example shows.

The real polynomial differential system

$$\dot{x} = 1, \quad \dot{y} = y(y - 2z), \quad \dot{z} = -z(y - z),$$

has $z = 0$ as an invariant plane of multiplicity 2 and its restriction to $z = 0$ has the rational first integral $x + 1/y$. But this system has no exponential factor associated with $z = 0$ as it is proved in the appendix.

This example shows that the additional assumption on the non–existence of the rational first integral on the invariant algebraic hypersurface for polynomial vector fields of dimension larger than 2 is necessary. So if we are not in the assumptions of Theorem 5, it is possible that the number of exponential factors does not depend on the algebraic multiplicity of the invariant algebraic hypersurface.

4.6 Darboux first integral

Let M be an open subset of $\mathbb{C}^n$ having full Lebesgue measure in $\mathbb{C}^n$. A non–constant function $H : \mathcal{D} \to \mathbb{C}$ is a *first integral* of the polynomial vector field $\mathcal{X}$ on M if it is constant on all orbits $x(t)$ of $\mathcal{X}$ contained in M; i.e. $H(x(t)) =$ constant for all values of t for which the solution $x(t)$ is defined and contained in M. Clearly H is a first integral of $\mathcal{X}$ on M if and only if $\mathcal{X}(H) = 0$ on M. Of course a *rational first integral* is a first integral given by a rational function, defined in the open subset of $\mathbb{C}^n$ where its denominator does not vanish. A *Darboux first integral* is a first integral of the form

$$\left(\prod_{i=1}^{r} f_i^{l_i}\right) \exp(g/h),$$

where f_i, g and h are polynomials, and the l_i's are complex numbers.

4.7 Classical Darboux theory of integrability in $\mathbb{C}^n$

The classical Darboux theory of integrability in $\mathbb{C}^n$ with $n \geq 2$ is summarized in the next theorem.

Theorem 6. *Assume that the polynomial vector field $\mathcal{X}$ in $\mathbb{C}^n$ of degree $d > 0$ has irreducible invariant algebraic hypersurfaces $f_i = 0$ for $i = 1, \ldots, p$. Then the following statements hold.*

(a) *If $p \geq N + 1$, then the vector field $\mathcal{X}$ has a Darboux first integral, where $N = \begin{pmatrix} n+d-1 \\ n \end{pmatrix}$.*

(b) *If $p \geq N + n$, then the vector field $\mathcal{X}$ has a rational first integral.*

Statement (a) of Theorem 6 is due to Darboux [14, 15], and statement (b) of Theorem 6 was proved by Jouanolou [25]. For a short proof of statement (b) see [9, 10] for $n = 2$ and [36] for $n \geq 2$.

4.8 Darboux theory of integrability in $\mathbb{C}^n$

The following theorem improves the classical Darboux theory of integrability taking into account not only the invariant algebraic hypersurfaces but also their algebraic multiplicities.

Theorem 7. *Assume that the polynomial vector field $\mathcal{X}$ in $\mathbb{C}^n$ of degree $d > 0$ has irreducible invariant algebraic hypersurfaces.*

(i) *If some of these irreducible invariant algebraic hypersurfaces has no defined algebraic multiplicity, then the vector field $\mathcal{X}$ has a rational first integral.*

(ii) *Suppose that all the irreducible invariant algebraic hypersurfaces $f_i = 0$ has defined algebraic multiplicity q_i for $i = 1, \ldots, p$. If $\mathcal{X}$ restricted to each hypersurface $f_i = 0$ having multiplicity larger than 1 has no rational first integral, then the following statements hold.*

  (a) *If $\sum_{i=1}^{p} q_i \geq N + 1$, then the vector field $\mathcal{X}$ has a Darboux first integral, where N is the number defined in Theorem 6.*

  (b) *If $\sum_{i=1}^{p} q_i \geq N + n$, then the vector field $\mathcal{X}$ has a rational first integral.*

Statement (i) follows from the second part of Theorem 3 of Pereira [42] (see also Theorem 5.3 of [12] for dimension 2). Under the assumption (b) of Theorem 7 any orbit of the vector field $\mathcal{X}$ is contained in an invariant algebraic hypersurface. We remark that if the vector field $\mathcal{X}$ is 2–dimensional, then the assumption on the non–existence of rational first integral of $\mathcal{X}$ restricted to the invariant algebraic curves is not necessary.

4.9 Darboux theory of integrability in $\mathbb{R}^n$ taking into account the multiplicity of the infinity

In the study of systems of differential equations in $\mathbb{R}^n$, if they have a first integral, then their analysis can be reduced in one dimension. But the search for a first integral of a given differential system in $\mathbb{R}^n$ is generally very difficult. Darboux theory of integrability, established by Darboux [14, 15] in 1878 and improved by Jouanolou [25] in 1979, and Llibre and Zhang [34], is a nice theory to find first integrals of a polynomial differential system in $\mathbb{C}^n$ (and consequently in $\mathbb{R}^n$) having sufficiently many invariant algebraic hypersurfaces taking into account their multiplicity.

In this subsection we show that if the hyperplane at infinity has multiplicity larger than 1, then we can go further improving the Darboux theory of integrability taking into account the multiplicity of the hyperplane at infinity.

In order to use the infinity of $\mathbb{R}^n$ as an additional invariant hyperplane for studying the integrability of the vector field $\mathcal{X}$, we need the Poincaré compactification for the vector field $\mathcal{X}$, see for more details Cima and Llibre [13]. In the chart U_1 using the change of variables

$$x_1 = \frac{1}{z}, \quad x_2 = \frac{y_2}{z}, \quad \ldots, \quad x_n = \frac{y_n}{z}. \tag{7}$$

the vector field $\mathcal{X}$ is transformed to

$$\overline{\mathcal{X}} = -z\overline{P}_1(y)\frac{\partial}{\partial z} + (\overline{P}_2(y) - y_2\overline{P}_1(y))\frac{\partial}{\partial y_2} + \ldots + (\overline{P}_n(y) - y_n\overline{P}_1(y))\frac{\partial}{\partial y_n},$$

where $\overline{P}_i = z^d P_i(1/z, y_2/z, \ldots, y_n/z)$ for $i = 1, \ldots, n$ and $y = (z, y_2, \ldots, y_n)$. We note that $z = 0$ is an invariant hyperplane of the vector field $\overline{\mathcal{X}}$ and that the infinity of $\mathbb{R}^n$ corresponds to $z = 0$ of the vector field $\overline{\mathcal{X}}$. So we can define the algebraic multiplicity of $z = 0$ for the vector field $\overline{\mathcal{X}}$.

We say that the infinity of $\mathcal{X}$ has *defined algebraic multiplicity* k or simply *algebraic multiplicity* k if $z = 0$ has defined algebraic multiplicity k for the vector field $\overline{\mathcal{X}}$; and that it has *no defined algebraic multiplicity* if $z = 0$ has no defined algebraic multiplicity for $\overline{\mathcal{X}}$.

In [47] the authors gave a definition for the algebraic multiplicity of the line at infinity for a planar vector field using a limit inside the definition. In fact the two definitions are equivalent. But ours is easier to be applied to compute the algebraic multiplicity of the line at infinity for a given planar vector field.

Similar to Theorem 5 we have the following result characterizing the existence of exponential factors associated with the hyperplane at infinity.

Theorem 8. *Assume that $\overline{\mathcal{X}}$ restricted to $z = 0$ has no rational first integral. Then $z = 0$ has algebraic multiplicity k for $\overline{\mathcal{X}}$ if and only if $\overline{\mathcal{X}}$ has $k-1$ exponential factors $\exp(\overline{g}_i/z^i)$, $i = 1, \ldots, k-1$ with $\overline{g}_i \in \mathbb{C}_i[y]$ having no factor z.*

The following result improves the Darboux theory of integrability in $\mathbb{R}^n$ taking into account the algebraic multiplicity of the hyperplane at infinity, for a proof see Llibre and Zhang [35].

Theorem 9. *Assume that the polynomial vector field $\mathcal{X}$ in $\mathbb{R}^n$ of degree $d > 0$ has irreducible invariant algebraic hypersurfaces $f_i = 0$ for $i = 1, \ldots, p$ and the invariant hyperplane at infinity.*

- (i) *If some of these irreducible invariant algebraic hypersurfaces or the invariant hyperplane at infinity has no defined algebraic multiplicity, then the vector field $\mathcal{X}$ has a rational first integral.*
- (ii) *Suppose that all the irreducible invariant algebraic hypersurfaces $f_i = 0$ have defined algebraic multiplicity q_i for $i = 1, \ldots, p$ and that the invariant hyperplane at infinity has defined algebraic multiplicity k. If the vector field restricted to each invariant hypersurface including the hyperplane at infinity having algebraic multiplicity larger than 1 has no rational first integral, then the following hold.*

(a) If $\sum_{i=1}^{p} q_i + k \geq N + 2$, then the vector field $\mathcal{X}$ has a Darboux first integral, where $N = \begin{pmatrix} n+d-1 \\ n \end{pmatrix}$.

(b) If $\sum_{i=1}^{p} q_i + k \geq N + n + 1$, then the vector field $\mathcal{X}$ has a rational first integral.

We note that if the hyperplane at infinity is not taken into account, then Theorem 9 is exactly Theorem 7. Also if the hyperplane at infinity has algebraic multiplicity 1, then it does not contribute to integrability by comparing Theorem 9 with Theorem 7.

We remark that for the moment we do not have an analogous to Theorem 9 for polynomial vector fields in $\mathbb{C}^n$ which takes into account the multiplicity of the infinity. In [35] are shown some difficulties for obtaining the extension of Theorem 9 from polynomial vector fields in $\mathbb{R}^n$ to polynomial vector fields in $\mathbb{C}^n$.

In the previous subsection we showed with an example that the assumption on the non–existence of rational first integral of $\mathcal{X}$ restricted to an invariant algebraic hypersurface with multiplicity larger than 1 is necessary for the vector field in $\mathbb{R}^n$ with $n > 2$. There are also examples showing that if $n > 2$, the additional assumption is also necessary for the infinity having multiplicity larger than 1, see [35]. If $\mathcal{X}$ is a planar vector field, then this additional assumption about the rational first integral is not necessary. We have the following

Corollary 10. *Assume that the polynomial vector field $\mathcal{X}$ in $\mathbb{R}^2$ of degree $d > 0$ has irreducible invariant algebraic curves $f_i = 0$ with defined algebraic multiplicity q_i for $i = 1, \ldots, p$ and that the invariant straight line at infinity has defined algebraic multiplicity k. Then the following hold.*

(a) If $\sum_{i=1}^{p} q_i + k \geq \begin{pmatrix} d+1 \\ 2 \end{pmatrix} + 2$, then the vector field $\mathcal{X}$ has a Darboux first integral.

(b) If $\sum_{i=1}^{p} q_i + k \geq \begin{pmatrix} d+1 \\ 2 \end{pmatrix} + 3$, then the vector field $\mathcal{X}$ has a rational first integral.

4.10 Construction of the first integrals

From the previous main results of this section it follows easily statements (ii) and (iii) of the next theorem. Statements (i) and (iv) without taking into account the cofactors of the exponential factors are essentially due to Darboux [14]. The rest of the statements come from Christopher and Llibre [9, 10].

Theorem 11. *Suppose that a polynomial vector field $\mathcal{X}$ of degree d in $\mathbb{C}^n$ admits p irreducible invariant algebraic hypersurfaces $f_i = 0$ with cofactors K_i for $i = 1, \ldots, p$ and q exponential factors $\exp(g_j/h_j)$ with cofactors L_j for $j = 1, \ldots, q$.*

(i) *There exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$, if and only if the (multi–valued) function*

$$f_1^{\lambda_1} \ldots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1} \right) \right)^{\mu_1} \ldots \left(\exp\left(\frac{g_q}{h_q} \right) \right)^{\mu_q} \tag{8}$$

is a first integral of $\mathcal{X}$.

(ii) *If* $p+q \geq N+1$, *then there exist* $\lambda_i, \mu_j \in \mathbb{C}$ *not all zero such that* $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$.

(iii) *If* $p+q \geq N+n$, *then* $\mathcal{X}$ *has a rational first integral.*

In the particular case that $n = 2$ *the following statements also hold.*

(iv) *There exist* $\lambda_i, \mu_j \in \mathbb{C}$ *not all zero such that* $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\mathrm{div}(P,Q)$, *if and only if function* (8) *is an integrating factor of* $\mathcal{X}$.

(v) *If* $p+q = N$ *then function* (8) *is a first integral if* $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0$, *or an integrating factor if* $\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\mathrm{div}(P,Q)$, *under the condition that not all* $\lambda_i, \mu_j \in \mathbb{C}$ *are zero.*

5. First integrals obtained by the Darboux theory of integrability in dimension two

In this section we summarize the effectiveness of the Darboux theory of integrability in dimension two, i.e. which sort of integrals does it capture. For additional details on this section see [17, 45, 48].

The idea of calculating what sort of functions can arise as the result of evaluating an indefinite integral or solving a differential equation goes back to Liouville. The modern formulation of these ideas is usually done through differential algebra. Some of the advantages over an analytic approach are first that the messy details of branch points etc, is hidden completely, and second the Darboux theory of integrability can be studied using symbolic computation.

5.1 Elementary and Liouvillian functions

We assume that the set of functions we are interested form a field together with a number of *derivations*. We call such an object a *differential field*. The process of adding more functions to a given set of functions is described by a tower of such fields:

$$F_0 \subset F_1 \subset \cdots \subset F_n.$$

Of course, we must also specify how the derivations of F_0 are extended to derivations on each F_i.

The fields we are interested in arise by adding exponentials, logarithms or the solutions of algebraic equations based on the previous set of functions. That is we take

$$F_i = F_0(\theta_1, \ldots \theta_i),$$

where one of the following holds:

(i) $\delta\theta_i = \theta_i \delta g$, for some $g \in F_{i-1}$ and for each derivation δ;
(ii) $\delta\theta_i = g^{-1}\delta g$, for some $g \in F_{i-1}$ and for each derivation δ;
(iii) θ_i is algebraic over F_{i-1}. If we have such a tower of fields, F_n is called an *elementary* extension of F_0.

This is essentially what we mean by a function being expressible in closed form. We call the set of all elements of a differential field which are annihilated by all the derivations of the field the *field of constants*. We shall always assume that the field of constants is algebraically closed.

We say that our system (1) has an *elementary* first integral if there is an element u in an elementary extension field of the field of rational functions $\mathbb{C}(x, y)$ with the same field of constants such that $Du = 0$. The derivations on $\mathbb{C}(x, y)$ are of course d/dx and d/dy.

Another class of integrals we are interested in are the *Liouvillian* ones. Here we say that an extension F_n is a *Liouvillian* extension of F_0 if there is a tower of differential fields as above which satisfies conditions (i), (iii) or

$(ii)'$ $\delta_\alpha \theta_i = h_\alpha$ for some elements $h_\alpha \in F_{i-1}$ such that $\delta_\alpha h_\beta = \delta_\beta h_\alpha$.

This last condition, mimics the introduction of line integrals into the class of functions. Clearly (ii) is included in $(ii)'$.

This class of functions represents those functions which are obtainable "by quadratures". An element u of a Liouvillian extension field of $\mathbb{C}(x, y)$ with the same field of constants is said to be a *Liouvillian* first integral.

A function of the form $e^{w_0 + \sum c_i \ln(w_i)}$, where c_i are constants and w_i are rational functions is called *Darboux* function.

As we shall see the Darboux theory of integrability finds all Liouvillian first integrals of the planar polynomial vector fields.

5.2 The relation between a first integral and its associated integrating factor

We consider the following classes of functions, polynomial, rational, Darboux, elementary and Liouvillian. We note that each of these classes of functions is contained in the one which follows it in the previous list. The more easy functions which can be first integrals of polynomial differential systems are functions of one of the mentioned classes. The Darboux theory of integrability allows to compute all the first integrals belonging to one of these classes, see [17, 45, 48]. In the next result and for polynomial differential systems in $\mathbb{R}^2$ or $\mathbb{C}^2$ we summarize the explicit relationships between the functions defining the first integrals and their integrating factors.

Theorem 12. *Let $\mathcal{X}$ be a planar polynomial vector field.*

(a) *If $\mathcal{X}$ has a Liouvillian first integral, then it has a Darboux integrating factor.*
(b) *If $\mathcal{X}$ has an elementary first integral, then it has an integrating factor of the form a rational function to power $1/n$ for positive integer n.*
(c) *If $\mathcal{X}$ has a Darboux first integral, then it has a rational integrating factor.*
(d) *If $\mathcal{X}$ has a rational first integral, then it has a rational integrating factor.*
(e) *If $\mathcal{X}$ has a polynomial first integral, then it has a polynomial integrating factor*

Statement (a) is due to Singer [48], see also Christopher [8] and Pereira [43]. Statement (b) was proved by Prelle and Singer [45]. Statement (c) was shown by Chavarriga, Giacomini, Giné and Llibre in [5]. The proof of statement (d) follows easily. Finally the proof of statement (e) follows from [5] and [19], this last paper is due to Ferragut, Llibre and Mahdi.

6. An open question for planar polynomial vector fields

In all this section $n = 2$.

From Jouanolou's result (see Theorem 7 (ii.b)) it follows that for a given planar polynomial differential system of degree d the maximum degree of its irreducible invariant algebraic curves is bounded, since either it has a finite number $p < [d(d+1)/2]+2$ of invariant algebraic curves, or all its trajectories are contained in invariant algebraic curves and the system admits a rational first integral. Thus for each polynomial differential system there is a natural number N which bounds the degree of all its *irreducible* invariant algebraic curves. A natural question which goes back to Poincaré [44] is: *to give an effective procedure to find* N. Partial answers to this question were given by Cerveau and Lins Neto [3], Carnicer [2], Campillo and Carnicer [1], and Walcher [50]. These results depend on either restricting the nature of the polynomial differential system, or more specifically on the singularities of its invariant algebraic curves.

Of course, given such a bound for N, it is then easy to compute the invariant algebraic curves of the system and also describe its elementary or Liouvillian first integrals (modulo any exponential factors) see for instance [37, 7, 41].

Unfortunately for the class of polynomial differential systems with fixed degree d, there does not exist a uniform upper bound $N(d)$ for N as shown by the polynomial differential system of degree 1:

$$\dot{x} = rx, \qquad \dot{y} = sy,$$

with r and s be positive integers. This system has a rational first integral

$$H = \frac{y^r}{x^s}.$$

and hence invariant algebraic curves $x^s - hy^r = 0$ for all $h \in \mathbb{C}$.

A common suggestion (coming from Poincaré) was that the following open question would have a positive answer:

> *For a given* $d \geq 2$ *is there a positive integer* $M(d)$ *such that if a polynomial vector field of degree* d *has an irreducible invariant algebraic curve of degree* $\geq M(d)$*, then it has a rational first integral.*

See for instance the open question 2 of [10], or the question at the end of the introduction of [26].

The previous question has a negative answer, two counterexamples appeared at the same time, one due to Moulin Ollagnier [39], and another due to Christopher and Llibre [11]. Later on other counterexamples appeared see for instance Chavarriga and Grau [6]. But all these counterexamples exhibit a Darboux first integral or a Darboux integrating factor. So we think that the following **open question** would have a positive answer(see [28]):

> *There is some number* $D(d)$ *for which any polynomial differential system of degree* d *having some irreducible invariant algebraic curve of degree* $\geq D(d)$ *has a Darboux first integral or Darboux integrating factor.*

7. Some applications

The Darboux theory of integrability has been successfully applied to the study of some physical models. Thus, for instance, for the classical Bianchi IX system, and for the Einstein-Yang-Mills differential equations, Llibre and Valls in [31] and [32] provided a complete description of its Darboux polynomials, exponential factors, rational first integrals and Darboux first integrals. Similar studies was done by Valls [49] for the Rikitake system, and for the Lorenz system by Llibre and Zhang [33] and Zhang [51].

The proof of the classification of all centers of the polynomial differential systems of degree 2 can be strongly simplified using the Darboux first integrals, see Schlomiuk [46] and the chapter 8 of Dumortier, Llibre and Artés [17].

The Darboux first integrals can be used for obtaining new classes of integrable planar polynomial differential systems having a focus, see Giné and Llibre [21].

Using the Darboux theory of integrability Llibre and Rodríguez in [30] proved that every finite configuration of disjoint simple closed curves of the plane is topologically realizable as the set of limit cycles of a polynomial vector field. Moreover the realization can be made by algebraic limit cycles, and explicit polynomial vector fields exhibiting any given finite configuration of limit cycles are given.

In [29] Llibre and Medrado gave the best upper bounds of the maximal number of invariant hyperplanes, of the maximal number of parallel invariant hyperplanes, and of the maximal number of invariant hyperplanes that pass through a single point for the polynomial vector fields in $\mathbb{C}^n$ with a given degree.

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