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Bounding the number of zeros of certain Abelian integrals

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Abstract. In this paper we prove a criterion that provides an easy sufficient condition in order for any nontrivial linear combination of n Abelian integrals to have at most $n+k-1$ zeros counted with multiplicities. This condition involves the functions in the integrand of the Abelian integrals and it can be checked, in many cases, in a purely algebraic way.

1 Introduction and statement of the result

In a previous paper with M. Grau [12] we provided a sufficient condition in order for a collection of Abelian integrals $I_0(h), I_1(h), \ldots, I_{n-1}(h)$ to form an extended complete Chebyshev system (for short, ECT-system). This is a very good property that implies in particular that the number of real zeros of any nontrivial linear combination

$$\alpha_0 I_0(h) + \alpha_1 I_1(h) + \ldots + \alpha_{n-1} I_{n-1}(h) \tag{1}$$

counted with multiplicities is at most $n-1$. However there are situations (see for instance [9, 10, 16, 18]) in which the number of zeros of (1) is greater than $n-1$. Then one talks about being a *Chebyshev system with accuracy* k, meaning that this number is at most $n+k-1$. The present paper is addressed to the problem of finding a bound for the number of zeros in this situation.

More precisely, let $H(x,y) = A(x) + B(x)y^{2m}$ be an analytic function in some open subset of the plane that has a local minimum at the origin. Then there exists a punctured neighbourhood $\mathcal{P}$ of the origin foliated by ovals $\gamma_h \subset \{H(x,y) = h\}$. We fix that $H(0,0) = 0$ and then the set of ovals γ_h inside this, let us say, *period annulus,* is parameterized by the energy levels $h \in (0, h_0)$ for some $h_0 \in (0, +\infty]$. The projection of $\mathcal{P}$ on the x-axis is an interval (x_ℓ, x_r) with $x_\ell < 0 < x_r$. The Abelian integrals that we shall consider in this paper are

$$I_i(h) = \int_{\gamma_h} f_i(x) y^{2s-1} dx, \text{ for } h \in (0, h_0),$$

where f_i, for $i = 0, 1, \ldots, n-1$, are analytic functions on (x_ℓ, x_r) and $s \in \mathbb{N}$.

Under the above assumptions it is easy to verify that $B(x) > 0$ for all $x \in (x_\ell, x_r)$ and that $xA'(x) > 0$ for all $x \in (x_\ell, x_r) \setminus \{0\}$. Then A has a zero of even multiplicity at $x = 0$ and so there exist an analytic involution σ such that

$$A(x) = A\big(\sigma(x)\big) \text{ for all } x \in (x_\ell, x_r).$$

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Recall that a mapping $\sigma\colon I \longrightarrow I$ is an *involution* if $\sigma^2 = Id$ and $\sigma \neq Id$. Note that an involution is a diffeomorphism with a unique fixed point. In our situation we have that $0 \in I$ and $\sigma(0) = 0$. In what follows, given a function κ defined on $I \setminus \{0\}$, we denote its *balance* with respect to σ as

$$\mathscr{B}_\sigma(\kappa)(x) = \frac{\kappa(x) - \kappa(\sigma(x))}{2}.$$

For example, if $\sigma = -Id$, then the balance of a function is its odd part.

Theorem A is our main result and in its statement $W[\ell_0, \ldots, \ell_i]$ stands for the Wronskian determinant of the functions $\ell_0, \ldots, \ell_i$ (see Definition 2.2). Let us note that each ℓ_i is well defined and analytic on $(0, x_r)$.

Theorem A. *Consider the Abelian integrals*

$$I_i(h) = \int_{\gamma_h} f_i(x) y^{2s-1} dx, \quad i = 0, 1, \ldots, n-1,$$

where, for each $h \in (0, h_0)$, γ_h is the oval inside the level curve $\{A(x) + B(x)y^{2m} = h\}$. Let σ be the involution associated to A and define

$$\ell_i = \mathscr{B}_\sigma\left(\frac{f_i}{A' B^{\frac{2s-1}{2m}}}\right).$$

If the following conditions are verified:

- *(a)* $W[\ell_0, \ldots, \ell_i]$ *is non-vanishing on* $(0, x_r)$ *for* $i = 0, 1, \ldots, n-2$,
- *(b)* $W[\ell_0, \ldots, \ell_{n-1}]$ *has* k *zeros on* $(0, x_r)$ *counted with multiplicities, and*
- *(c)* $s > m(n + k - 2)$,

then any nontrivial linear combination of $I_0, I_1, \ldots, I_{n-1}$ has at most $n+k-1$ zeros on $(0, h_0)$ counted with multiplicities.

The problem of bounding the number of zeros of Abelians integrals is the subject of many recent papers (see for instance [2, 4, 5, 9, 13, 23] and references there in). The techniques and arguments to tackle the problem are usually very long and highly non-trivial. For instance, in some papers (e.g. [3, 15, 21]) the authors study the geometrical properties of the so-called *centroid curve* using that it verifies a Riccati equation (which is itself deduced from a Picard-Fuchs system). In other papers (e.g. [6, 7, 8]), the authors use complex analysis and algebraic topology (analytic continuation, argument principle, monodromy, Picard-Lefschetz formula, ...). Theorem A provides a criterion that, when it works, enables to extremely simplify the solution of the mentioned problem.

Of course the second part of *Hilbert's 16th problem* [14] is the general framework where this result is addressed to. This longstanding problem asks about the maximum number and location of limit cycles of a planar polynomial vector fields of degree d. Solving this problem, even in the case $d = 2$, seems to be out of reach at the present state of knowledge (see Ilyashenko [19] for a survey of the recent results on the subject). Our paper is concerned with a weaker version of this problem, the so-called *infinitesimal Hilbert's 16th problem*, proposed by Arnold [1]. Zeros of Abelian integrals are related to limit cycles in the following way. Consider a small deformation of a Hamiltonian vector field $X_\varepsilon = X_H + \varepsilon Y$, where

$$X_H = -H_y \partial_x + H_x \partial_x \text{ and } Y = P\partial_x + Q\partial_y.$$

Then, see [19] for details, the first approximation in ε of the displacement function of the Poincaré map of X_ε is given by $I(h) = \int_{\gamma_h} P dy - Q dx$. Hence the number of isolated zeros of $I(h)$, counted with multiplicities, provides an upper bound for the number of ovals of H that generate limit cycles of X_ε for $\varepsilon \approx 0$. (In the literature an Abelian integral is usually the integral of a rational 1-form over a continuous family of algebraic ovals. Throughout the paper, by an abuse of language, we use the name Abelian integral also in case the functions are analytic.)

To illustrate the applicability of Theorem A in this context we will reobtain a result of Dumortier, Li and Zhang [3]. In that long paper (48 pages) the authors give a complete study of the quadratic 3-parameter unfoldings of an integrable system belonging to the class Q_3^R. They obtain the bifurcation diagram and all the global phase portraits, including the number and configuration of limit cycles. It is to proving that the maximum number of limit cycles surrounding a single focus is equal to 3 where they need to study the zeros of an Abelian integral. Altogether it takes 21 pages of very technical computations to show that the upper bound for the number of zeros is 3. We shall prove this by applying Theorem A in Section 4.

The paper is organized in the following way. Section 2 is devoted to defining the different types of Chebyshev property that we shall deal with and to proving some basic results about Wronskians. Theorem A is proved in Section 3. The idea behind the proof is simple. Indeed, we show that there exist k additional Abelian integrals, $J_i = \int_{\gamma_h} h_i(x) y^{2s-1} dx$ for $i = 1, 2, \ldots, k$, such that $(I_0, \ldots, I_{n-2}, J_1, \ldots, J_k, I_{n-1})$ is an ECT-system. The real work is to guarantee the existence of the functions h_i because then we take advantage of our previous result [12] that we mention at the beginning of the paper (see Theorem 3.8). The existence of these functions follows from Theorem 3.2, which gives a (very) partial answer to the following *embedding problem*: Given a finite dimensional space $\mathcal{E}$ of analytic functions on an interval I such that any $f \in \mathcal{E}$ has at most m zeros on I counted with multiplicities, the problem consists in finding necessary and sufficient conditions for the existence of an ECT-system of dimension $m+1$ whose linear span contains $\mathcal{E}$. Finally in Section 4 we give the example of application that we explain in the previous paragraph.

2 Definitions and basic results about wronskians

Definition 2.1 Let $f_0, f_1, \ldots, f_{n-1}$ be analytic functions on an open interval I of $\mathbb{R}$.

(*a*) $\{f_0, f_1, \ldots, f_{n-1}\}$ is a *Chebyshev system* on I if any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \ldots + \alpha_{n-1} f_{n-1}(x) = 0$$

has at most $n-1$ isolated zeros on I.

(*b*) An ordered set of n functions $(f_0, f_1, \ldots, f_{n-1})$ is a *complete Chebyshev system* (in short, CT-system) on I if $\{f_0, f_1, \ldots, f_{k-1}\}$ is a Chebyshev system on I for all $k = 1, 2, \ldots, n$.

(*c*) An ordered set of n functions $(f_0, f_1, \ldots, f_{n-1})$ is an *extended complete Chebyshev system* (in short, ECT-system) on I if, for all $k = 1, 2, \ldots, n$, any nontrivial linear combination

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \ldots + \alpha_{k-1} f_{k-1}(x) = 0$$

has at most $k-1$ isolated zeros on I counted with multiplicities.

(Let us mention that in these abbreviations "T" stands for Tchebycheff, which in some sources is the transcription of the Russian name Chebyshev.) □

Definition 2.2 Let $f_0, f_1, \ldots, f_{k-1}$ be analytic functions on an open interval L of $\mathbb{R}$. The *Wronskian* of $(f_0, f_1, \ldots, f_{k-1})$ at $x \in I$ is

$$W\big[f_0, f_1, \cdots, f_{k-1}\big](x) = \det\Big(f_j^{(i)}(x)\Big)_{0 \leqslant i,j \leqslant k-1} = \begin{vmatrix} f_0(x) & \cdots & f_{k-1}(x) \\ f_0'(x) & \cdots & f_{k-1}'(x) \\ & \vdots & \\ f_0^{(k-1)}(x) & \cdots & f_{k-1}^{(k-1)}(x) \end{vmatrix}.$$

□

For the sake of shortness sometimes we will use the notation

$$W\big[f_0, f_1, \ldots, f_{k-1}\big](x) = W\big[\mathbf{f_k}\big](x).$$

The following two lemmas are known (see, respectively, [20] and [17, 22]).

Lemma 2.3. *$(f_0, f_1, \ldots, f_{n-1})$ is an ECT-system on L if and only if, for each $k = 1, 2, \ldots, n$,*

$$W\big[\mathbf{f_k}\big](x) \neq 0 \textit{ for all } x \in L.$$

Lemma 2.4. *Let $f_0, f_1, \ldots, f_n$ be analytic functions on a open interval I such that $W[f_0, \ldots, f_{n-2}, f_{n-1}]$ does not vanish on I. Then*

$$\left(\frac{W[f_0, \ldots, f_{n-2}, f_n]}{W[f_0, \ldots, f_{n-2}, f_{n-1}]}\right)' = \frac{W[f_0, \ldots, f_n]\, W[f_0, \ldots, f_{n-2}]}{(W[f_0, \ldots, f_{n-2}, f_{n-1}])^2}.$$

Next three lemmas summarize some technical results that we will need to prove our result.

Lemma 2.5. *Let $f_0, f_1, \ldots, f_{n-1}$ be analytic functions on I. Then the following statements hold:*

(a) $W[f_0 \circ \psi, \ldots, f_{n-1} \circ \psi](x) = \big(\psi'(x)\big)^{\frac{(n-1)n}{2}} W[f_0, \ldots, f_{n-1}]\big(\psi(x)\big)$ *for any analytic diffeomorphism ψ.*

(b) $W[gf_0, \ldots, gf_{n-1}](x) = g(x)^n W[f_0, \ldots, f_{n-1}](x)$ *for any analytic function g.*

Proof. In order to prove (a) we first note that, given any analytic function f, we have

$$(f \circ \psi)^{(k)}(x) = f^{(k)}\big(\psi(x)\big)\big(\psi'(x)\big)^k + f^{(k-1)}\big(\psi(x)\big) R_{k,2}(x) + \ldots + f'\big(\psi(x)\big) R_{k,k}(x), \tag{2}$$

where $R_{k,i}$ is a product of the derivatives of ψ until order i. This can be proved easily by induction and it is left to the reader. For the sake of convenience let us fix that $W\big[\mathbf{f_n} \circ \psi\big](x) = \det(M_1)$ where $M_1 = (a_{i,j})$ with $a_{i,j} = \big(f_j \circ \psi\big)^{(i)}(x)$, and that $W\big[\mathbf{f_n}\big]\big(\psi(x)\big) = \det(M_n)$ where $M_n = (\hat{a}_{i,j})$ with $\hat{a}_{i,j} = f_j^{(i)}\big(\psi(x)\big)$. Clearly the first row of M_1 and M_n are equal by definition. In addition, for $i \geqslant 2$, from (2) it follows that the i-th row of M_1 is equal to the i-th row of M_n multiplied by $\big(\psi'(x)\big)^{i-1}$ plus a linear combination of its first $i-1$ rows. For $i = 2, 3, \ldots, n-1$, let us define M_i as the matrix that we obtain from M_1 by replacing, respectively, its first i rows by the first i rows of M_n. Taking this into account we have that $\det(M_i) = \big(\psi'(x)\big)^i \det(M_{i+1})$. Accordingly,

$$\det(M_1) = \psi'(x) \det(M_2) = \big(\psi'(x)\big)^{1+2} \det(M_3) = \ldots = \big(\psi'(x)\big)^{1+2+\ldots+n-1} \det(M_n).$$

Since $1 + 2 + \ldots + n - 1 = \frac{n(n-1)}{2}$, (a) follows. Part (b) can be found in [22]. ∎

From Definition 2.1 (or, alternatively, by Lemmas 2.3 and 2.5) it easily follows the next result:

Lemma 2.6. *Let $f_0, f_1, \ldots, f_{n-1}$ be analytic functions such that $(f_0, f_1, \ldots f_{n-1})$ is an ECT-system on I. Then the following statements hold:*

(a) If $\psi\colon J \longrightarrow I$ is a diffeomorphism, then $(f_0 \circ \psi, f_1 \circ \psi, \ldots f_{n-1} \circ \psi)$ is an ECT-system on J.

(b) If g is a non-vanishing analytic function on I, then $(gf_0, gf_1, \ldots gf_{n-1})$ is an ECT-system on I.

By applying (b) in Lemma 2.5 with $\psi = -Id$ and taking well known properties about determinants into account one can easily obtain the following result:

Lemma 2.7. *Let $f_0, f_1, \ldots, f_{n-1}$ be analytic functions on I. Then the following holds:*

(a) If $f_0, f_1, \ldots, f_{n-1}$ are odd, then $W\big[\mathbf{f_n}\big](-x) = (-1)^{\frac{n(n+1)}{2}} W\big[\mathbf{f_n}\big](x)$.

(b) If $f_0, f_1, \ldots, f_{n-2}$ are odd and f_{n-1} is even, then $W\big[\mathbf{f_n}\big](-x) = (-1)^{\frac{n(n+1)}{2}+1} W\big[\mathbf{f_n}\big](x)$.

3 Proof of the main result

Lemma 3.1. *Let f be an analytic function on an open interval I.*

(*a*) *If f has exactly n zeros on I counted with multiplicities, then there exist $g_0, g_1, \dots, g_{n-1}$ analytic functions on I such that $(g_0, g_1, \dots, g_{n-1}, f)$ is an ECT-system on I.*

(*b*) *If f is an odd (respectively, even) function on $I = (-a, a)$ with exactly n zeros on $(0, a)$ counted with multiplicities, then there exist $g_0, g_1, \dots, g_{n-1}$ analytic odd (respectively, even) functions on I such that $(g_0, g_1, \dots, g_{n-1}, f)$ is an ECT-system on $(0, a)$.*

Proof. In order to show (*a*) let us assume that f has exactly n zeros counted with multiplicities on I, say $a_1 \leqslant a_2 \dots \leqslant a_n$. Define

$$g_i(x) = \frac{f(x)}{\prod_{j=i+1}^{n}(x - a_j)} \quad \text{for } i = 0, 1, \dots, n-1,$$

so that g_i has exactly i zeros on I counted with multiplicities. It is clear that by construction

$$\left(\frac{g_0}{g_0}, \dots, \frac{g_{n-1}}{g_0}, \frac{f}{g_0}\right) = \left(1, x - a_1, (x - a_1)(x - a_2), \dots, \prod_{i=1}^{n}(x - a_i)\right),$$

and this shows, by (*b*) in Lemma 2.6, that $(g_0, \dots, g_{n-1}, f)$ is an ECT-system on I.

Suppose next that f is odd (respectively, even) and that it has exactly n zeros on $(0, a)$ counted with multiplicities, say $a_1 \leqslant a_2 \dots \leqslant a_n$. Then $f(-a_i) = 0$ for all $i = 1, 2, \dots, n$. Define now

$$g_i(x) = \frac{f(x)}{\prod_{j=i+1}^{n}(x^2 - a_j^2)} \quad \text{for } i = 0, 1, \dots, n-1.$$

Clearly each g_i is an analytic odd (respectively, even) function on I. Then, since one can easily verify that $\left(1, x^2 - a_1^2, \dots, \prod_{i=1}^{n}(x^2 - a_i^2)\right)$ is an ECT-system on $(0, a)$, exactly as before we can conclude that so it is $(g_0, \dots, g_{n-1}, f)$. ■

Theorem 3.2. *Let $f_0, f_1, \dots, f_{n-1}$ be analytic functions such that $W[f_0, f_1, \dots, f_k]$ is non-vanishing on I for $k = 0, 1, \dots, n-1$ (i.e., such that $(f_0, f_1, \dots, f_{n-1})$ is an ECT-system on I). Assume moreover that h is an analytic function such that $W[f_0, \dots, f_{n-1}, h]$ has ℓ zeros on I counted with multiplicities. Consider ℓ functions $\widetilde{l}_1, \widetilde{l}_2, \dots, \widetilde{l}_\ell$ verifying that*

$$\left(\widetilde{l}_1, \dots, \widetilde{l}_\ell, \frac{W[f_0, \dots, f_{n-1}, h]}{W[f_0, \dots, f_{n-1}]}\right)$$

is an ECT-system on I. For each $k = 1, 2, \dots, \ell$, let l_k be an analytic function on I satisfying

$$\frac{W[f_0, \dots, f_{n-1}, l_k]}{W[f_0, \dots, f_{n-1}]} = \widetilde{l}_k. \tag{3}$$

Then $(f_0, \dots, f_{n-1}, l_1, \dots, l_\ell, h)$ is an ECT-system on I.

It is worthwhile making some remarks about Theorem 3.2. The assumptions on $f_0, f_1, \dots, f_{n-1}$ and h imply that any function in the linear span $\mathcal{E}$ of them has at most $n+\ell$ zeros on I counted with multiplicities. The result shows that there exists an ECT-system of dimension $n+\ell+1$ whose linear span contains $\mathcal{E}$. Let us also note that the existence of $\widetilde{l}_1, \widetilde{l}_2, \dots, \widetilde{l}_\ell$ is not an assumption but a consequence of Lemma 3.1. Note finally that (3) defines a n-th order linear differential equation for l_k. Since $W[f_0, \dots, f_{n-1}]$ does not vanish on I, the coefficients of $l_k^{(i)}$ for $i = 0, 1, \dots, n$ are analytic functions on I. In particular, the coefficient of $l_k^{(n)}$ is ± 1. (To show this develop the determinant $W[f_0, \dots, f_{n-1}, l_k]$ with respect to the last column and note that the minor associated to $l_k^{(n)}$ is precisely $W[f_0, \dots, f_{n-1}]$.) Accordingly the equality in (3) is verified by a well defined analytic function l_k on I.

Proof of Theorem 3.2. For the sake of convenience, let us define $l_{\ell+1} := h$ and $\widetilde{l}_{\ell+1} := \frac{W[f_0,\dots,f_{n-1},l_{\ell+1}]}{W[f_0,\dots,f_{n-1}]}$. We claim that, for all $j = 0, 1, \dots, \ell$ and $i = 1, \dots, \ell+1$, we have

$$W[f_0, \dots, f_{n-1}, l_1, \dots, l_j, l_i] = W[\widetilde{l}_1, \dots, \widetilde{l}_j, \widetilde{l}_i]\, W[f_0, \dots, f_{n-1}].$$

We prove the claim by induction on j. From (3) it follows that $W[f_0, \dots, f_{n-1}, l_i] = W[\widetilde{l}_i]\, W[f_0, \dots, f_{n-1}]$ and this shows the claim for $j = 0$. Let us show next that if the claim is true for $j \leqslant m-1$, then so it is for $j = m$. To this end note that

$$\begin{aligned} W[f_0, \dots, f_{n-1}, l_1, \dots, l_m, l_i] &= \left(\frac{W[f_0, \dots, f_{n-1}, l_1, \dots, l_{m-1}, l_i]}{W[f_0, \dots, f_{n-1}, l_1, \dots, l_m]}\right)' \frac{(W[f_0, \dots, f_{n-1}, l_1, \dots, l_m])^2}{W[f_0, \dots, f_{n-1}, l_1, \dots, l_{m-1}]} \\ &= \left(\frac{W[\widetilde{l}_1, \dots, \widetilde{l}_{m-1}, \widetilde{l}_i]\, W[f_0, \dots, f_{n-1}]}{W[\widetilde{l}_1, \dots, \widetilde{l}_m]\, W[f_0, \dots, f_{n-1}]}\right)' \frac{(W[f_0, \dots, f_{n-1}, l_1, \dots, l_m])^2}{W[f_0, \dots, f_{n-1}, l_1, \dots, l_{m-1}]} \\ &= \frac{W[\widetilde{l}_1, \dots, \widetilde{l}_m, \widetilde{l}_i]\, W[\widetilde{l}_1, \dots \widetilde{l}_{m-1}]}{(W[\widetilde{l}_1, \dots, \widetilde{l}_m])^2} \frac{(W[f_0, \dots, f_{n-1}, l_1, \dots, l_m])^2}{W[f_0, \dots, f_{n-1}, l_1, \dots, l_{m-1}]} \\ &= \frac{W[\widetilde{l}_1, \dots, \widetilde{l}_m, \widetilde{l}_i]\, W[\widetilde{l}_1, \dots \widetilde{l}_{m-1}]}{(W[\widetilde{l}_1, \dots, \widetilde{l}_m])^2} \frac{(W[\widetilde{l}_1, \dots, \widetilde{l}_m]\, W[f_0, \dots, f_{n-1}])^2}{W[\widetilde{l}_1, \dots, \widetilde{l}_{m-1}]\, W[f_0, \dots, f_{n-1}]} \\ &= W[\widetilde{l}_1, \dots, \widetilde{l}_m, \widetilde{l}_i]\, W[f_0, \dots, f_{n-1}], \end{aligned}$$

where in the first equality we apply Lemma 2.4, in the second one the induction hypothesis for $j = m-1$, in the third one we apply Lemma 2.4 again, and finally in the fourth one we use the induction hypothesis for $j = m-1$ and $j = m-2$. Accordingly the claim is proved. (Let us note that the denominators in the above equalities do not vanish due to the fact that $(\widetilde{l}_1, \widetilde{l}_2, \dots, \widetilde{l}_{\ell+1})$ is an ECT-system and $W[f_0, \dots, f_{n-1}] \neq 0$.)

Now, by applying the claim with $i = j+1$ we have that

$$W[f_0, \dots, f_{n-1}, l_1, \dots, l_{j+1}] = W[\widetilde{l}_1, \dots, \widetilde{l}_{j+1}]\, W[f_0, \dots, f_{n-1}] \text{ for all } j = 0, 1, \dots, \ell. \tag{4}$$

Since $(\widetilde{l}_1, \widetilde{l}_2, \dots, \widetilde{l}_{\ell+1})$ is an ECT-system, $W[\widetilde{l}_1, \dots, \widetilde{l}_{j+1}]$ is non-vanishing on I for all $j = 0, 1, \dots, \ell$. On the other hand, since by assumption this is also the case of $(f_0, f_1, \dots, f_{n-1})$, the equality in (4) shows that $(f_0, \dots, f_{n-1}, l_1, \dots, l_{\ell+1})$ is an ECT-system as desired. ∎

The following result was proved by Gavrilov and Iliev for $n = 2$ (see [11, Proposition 2]). We include its generalization for completeness.

Corollary 3.3. *Consider the n-th order linear differential equation*

$$y^{(n)} + a_1(x)y^{(n-1)} + \dots + a_{n-1}(x)y' + a_n(x)y = b(x), \tag{5}$$

where a_i, for $i = 1, 2, \dots, n$, and b are analytic functions on I. Suppose that the corresponding homogeneous differential equation, i.e., (5) taking $b \equiv 0$, has a fundamental set of solutions $\{\varphi_0, \varphi_1, \dots, \varphi_{n-1}\}$ such that $(\varphi_0, \varphi_1, \dots, \varphi_{n-1})$ is an ECT-system on I. Then, if b has k zeros on I counted with multiplicities, any solution of (5) has at most $n+k$ zeros on I counted with multiplicities.

Proof. Note that (5) can be written as

$$\frac{W[\varphi_0, \dots, \varphi_{n-1}, y]}{W[\varphi_0, \dots, \varphi_{n-1}]}(x) = b(x).$$

Accordingly, if h is a solution of (5), then $W[\varphi_0, \dots, \varphi_{n-1}, h]$ has k zeros on I counted with multiplicities. Then, by Theorem 3.2, the function h belongs to an ECT-system on I of dimension $n+k+1$, and so the result follows. ∎

At this point we introduce the following definition.

Definition 3.4 Let σ be an involution on I with $\sigma(0)=0$. We say that a function f defined on $I\setminus\{0\}$ is *σ-odd* (respectively, *σ-even*) if $f\circ\sigma=-f$ (respectively, $f\circ\sigma=f$). Accordingly, f is σ-odd (respectively, σ-even) if, and only if, $\mathscr{B}_\sigma(f)=f$ (respectively, $\mathscr{B}_\sigma(f)=0$). □

Lemma 3.5. *Consider $I=(a,b)$ with $a<0<b$ and let σ be an analytic involution on I such that $\sigma(0)=0$. Define $\varphi(x)=\frac{x-\sigma(x)}{2}$, i.e., $\varphi=\mathscr{B}_\sigma(Id)$. Then φ is a diffeomorphism from I to $(\frac{a-b}{2},\frac{b-a}{2})$. Moreover an analytic function f on I is σ-odd if, and only if, $f\circ\varphi^{-1}$ is odd.*

Proof. That φ is a diffeomorphism follows from the fact that an involution is monotonous decreasing. Due to $\sigma^2=Id$, note that $\varphi(\sigma(x))=-\varphi(x)$, so that $\varphi(\sigma(\varphi^{-1}(x)))=-x$. Thus, $\sigma(\varphi^{-1}(x))=\varphi^{-1}(-x)$. Hence, if f is σ-odd, then

$$\big(f\circ\varphi^{-1}\big)(-x)=f\big(\varphi^{-1}(-x)\big)=f\big(\sigma(\varphi^{-1}(x))\big)=-\big(f\circ\varphi^{-1}\big)(x)$$

and this shows that $f\circ\varphi^{-1}$ is an odd function. Reciprocally, if $f\circ\varphi^{-1}$ is odd, then

$$\mathscr{B}_\sigma\big(f\big)(x)=\frac{f(x)-f\big(\sigma(x)\big)}{2}=\frac{f(x)-f\big(\varphi^{-1}(-\varphi(x))\big)}{2}=f(x),$$

where in the second equality we use that $\sigma(x)=\varphi^{-1}\big(-\varphi(x)\big)$ and in the third one that $f\circ\varphi^{-1}$ is odd. This proves that f is σ-odd. ■

Proposition 3.6. *Let σ be an analytic involution on $I=(a,b)$ with $a<0<b$ and $\sigma(0)=0$. Suppose that $f_0,f_1,\ldots,f_n$ are σ-odd (respectively, σ-even) analytic functions on I verifying that $\big(f_0,\ldots,f_{n-1}\big)$ is an ECT-system on $(0,b)$ and that $W[f_0,\ldots,f_n]$ has k zeros on $(0,b)$ counted with multiplicities. Then there exist $g_1,g_2,\ldots,g_k$ analytic σ-odd (respectively, σ-even) functions on I such that*

$$\big(f_0,\ldots,f_{n-1},g_1,\ldots,g_k,f_n\big)$$

is an ECT-system on $(0,b)$.

Proof. Let us consider the σ-odd case first. To this end, for the sake of convenience, we suppose first that $\sigma=-Id$, so that $f_0,f_1,\ldots,f_n$ are odd functions in the classical sense. Then, by Lemma 2.7

$$\widetilde{l}_{k+1}:=\frac{W[f_0,\ldots,f_{n-1},f_n]}{W[f_0,\ldots,f_{n-1}]}$$

is either odd or even depending on n. If $\widetilde{l}_{k+1}$ is odd (respectively, even), then by applying Lemma 3.1 there exist $\widetilde{l}_1,\widetilde{l}_2,\ldots,\widetilde{l}_k$ analytic odd (respectively, even) functions on I such that $\big(\widetilde{l}_1,\ldots,\widetilde{l}_k,\widetilde{l}_{k+1}\big)$ is an ECT-system on $(0,b)$. Now, by Theorem 3.2, we know that if for each $i=1,2,\ldots,k$ the function l_i is chosen verifying that

$$\frac{W[f_0,\ldots,f_{n-1},l_i]}{W[f_0,\ldots,f_{n-1}]}=\widetilde{l}_i,$$

then $\big(f_0,\ldots,f_{n-1},l_1,\ldots,l_k,f_n\big)$ is an ECT-system on $(0,b)$. Accordingly it only remains to check that each l_i is an odd function. To this end, let P_i and S_i be respectively the even and odd parts of l_i. Thus, since $l_i=P_i+S_i$, we get

$$\frac{W[f_0,\ldots,f_{n-1},P_i]}{W[f_0,\ldots,f_{n-1}]}+\frac{W[f_0,\ldots,f_{n-1},S_i]}{W[f_0,\ldots,f_{n-1}]}=\widetilde{l}_i. \tag{6}$$

Then, from Lemma 2.7 again, $\frac{W[f_0,\ldots,f_{n-1},S_i]}{W[f_0,\ldots,f_{n-1}]}$ has the same parity as $\frac{W[f_0,\ldots,f_{n-1},f_n]}{W[f_0,\ldots,f_{n-1}]}=\widetilde{l}_{k+1}$, which has the same parity as $\widetilde{l}_i$, while the parity of $\frac{W[f_0,\ldots,f_{n-1},P_i]}{W[f_0,\ldots,f_{n-1}]}$ is just the opposite one. Since the decomposition of a function as a sum of its even and odd parts is unique, from (6) we conclude that $W[f_0,\dots,f_{n-1},P_i]=0$. We claim that this implies $P_i=0$. To show the claim consider $W[f_0,\dots,f_{n-1},P_i]=0$ as a n-th order homogeneous linear differential equation for P_i. Note that the functions $f_0,f_1,\dots,f_{n-1}$ form a fundamental set of solutions, so that $P_i=\alpha_0 f_0+\alpha_1 f_1+\ldots+\alpha_{n-1}f_{n-1}$ for some constants α_j. Since f_j are all odd functions and P_i is even, this implies that $P_i=0$. This proves the claim and shows that l_i is odd, as desired.

Next we shall obtain the result for an arbitrary involution σ using that it is true for $\sigma=-Id$. So let us assume that $f_0,f_1,\dots,f_n$ are σ-odd functions. By applying Lemma 3.5 we have that $f_i\circ\varphi^{-1}$ is an odd analytic function on $\left(\frac{a-b}{2},\frac{b-a}{2}\right)$ for all $i=0,1,\dots,n$. Moreover, by (a) in Lemma 2.5 it follows that

$$\left(f_0\circ\varphi^{-1},\dots,f_{n-1}\circ\varphi^{-1}\right)$$

is an ECT-system on $\left(0,\frac{b-a}{2}\right)$ and that $W[f_0\circ\varphi^{-1},\dots,f_n\circ\varphi^{-1}]$ has k zeros on $\left(0,\frac{b-a}{2}\right)$ counted with multiplicities. Now, since the claim is true when $\sigma=-Id$, we can assert that there exist $l_1,l_2,\dots,l_k$ analytic odd functions on $\left(\frac{a-b}{2},\frac{b-a}{2}\right)$ such that

$$\left(f_0\circ\varphi^{-1},\dots,f_{n-1}\circ\varphi^{-1},l_1,\dots,l_k,f_n\circ\varphi^{-1}\right)$$

is an ECT system on $\left(0,\frac{b-a}{2}\right)$. Consequently, by applying (a) in Lemma 2.6,

$$\left(f_0,\dots,f_{n-1},l_1\circ\varphi,\dots,l_k\circ\varphi,f_n\right)$$

is an ECT-system on $(0,b)$. Finally, since each l_i is an odd function, by Lemma 3.5 we get that $g_i:=l_i\circ\varphi$ is σ-odd for all $i=1,2,\dots,k$. This concludes the proof of the result in the σ-odd case.

Finally we shall obtain the proof for the σ-even case using that the result is true for σ-odd functions. Thus, assume that $f_0,f_1,\dots,f_n$ are σ-even functions on I. Let κ be any σ-odd analytic function on I vanishing only at $x=0$ with multiplicity one. For instance we can choose $\kappa(x)=x-\sigma(x)$. Notice then that $\hat f_i:=\kappa f_i$ is a σ-odd analytic function for $i=0,1,\dots,n$. Moreover, on account of (b) in Lemma 2.5, $\left(\hat f_0,\dots,\hat f_{n-1}\right)$ is an ECT-system on $(0,b)$ and $W[\hat f_0,\dots,\hat f_n]$ has k zeros on $(0,b)$ counted with multiplicities. Accordingly, since the result is true for the σ-odd case, there exist $\hat g_1,\hat g_2,\dots,\hat g_k$ σ-odd analytic functions on I such that $\left(\hat f_0,\dots,\hat f_{n-1},\hat g_1,\dots,\hat g_k,\hat f_n\right)$ is an ECT-system on $(0,b)$. Therefore, since κ does not vanish on $(0,b)$, by (b) in Lemma 2.6 we can assert that

$$\left(f_0,\dots,f_{n-1},\hat g_1/\kappa,\dots,\hat g_k/\kappa,f_n\right)$$

is an ECT-system on $(0,b)$. It only remains to check that $g_i:=\hat g_i/\kappa$ is a σ-even analytic function on I. The analyticity is clear because, since $\hat g_i$ is σ-odd, it vanishes at $x=0$. The fact that g_i is σ-even is also clear because it is the quotient between two σ-odd functions. This concludes the proof of the result. ∎

Corollary 3.7. *Let σ be an analytic involution on $I=(a,b)$ with $a<0<b$ and $\sigma(0)=0$. Suppose that $f_0,f_1,\dots,f_n$ are σ-odd functions on I verifying that:*

(a) There exists $s\in\mathbb{N}$ such that $x\longmapsto x^{2s-1}f_i(x)$ is analytic on I for all $i=0,1,\dots,n$,

(b) $\left(f_0,\dots,f_{n-1}\right)$ is an ECT-system on $(0,b)$ and

(c) $W[f_0,\dots,f_n]$ has k zeros on $(0,b)$ counted with multiplicities.

Then there exist $g_1,g_2,\dots,g_k$ σ-odd functions on I such that $\left(f_0,\dots,f_{n-1},g_1,\dots,g_k,f_n\right)$ is an ECT-system on $(0,b)$. In addition $x\longmapsto x^{2s-1}g_i(x)$ is analytic on I for all $i=1,2,\dots,k$.

Proof. Take any σ-odd analytic function κ vanishing only at $x=0$ with multiplicity $2s-1$. We can take for instance $\kappa(x)=x^{2s-1}-\sigma(x)^{2s-1}$. Define $\hat f_i:=\kappa f_i$. Then $\hat f_0,\hat f_1,\dots,\hat f_n$ are σ-even analytic functions on I. Moreover, on account of (b) in Lemma 2.5, $\left(\hat f_0,\dots,\hat f_{n-1}\right)$ is an ECT-system on $(0,b)$ and $W[\hat f_0,\dots,\hat f_n]$

has k zeros on $(0,b)$ counted with multiplicities. Thus, by applying Proposition 3.6, there exist $\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_k$ σ-even analytic functions on I such that $\big(\hat{f}_0, \ldots, \hat{f}_{n-1}, \hat{g}_1, \ldots, \hat{g}_k, \hat{f}_n\big)$ is an ECT-system on $(0,b)$. We define $g_i = \hat{g}_i/\kappa$ for $i = 1, 2, \ldots, k$. Therefore each g_i is a σ-odd function verifying that $x^{2s-1}g_i(x)$ is analytic on I and, from (b) in Lemma 2.6, $\big(f_0, \ldots, f_{n-1}, g_1, \ldots, g_k, f_n\big)$ is an ECT-system on $(0,b)$. ■

Finally, the following result (see [12, Theorem B]) constitutes the fundamental tool to prove Theorem A.

Theorem 3.8. *Consider the Abelian integrals*

$$I_i(h) = \int_{\gamma_h} f_i(x) y^{2s-1} dx, \quad i = 0, 1, \ldots, n-1,$$

where, for each $h \in (0, h_0)$, γ_h is the oval inside the level curve $\{A(x) + B(x)y^{2m} = h\}$. Let σ be the involution associated to A and define

$$\ell_i = \mathscr{B}_\sigma\left(\frac{f_i}{A' B^{\frac{2s-1}{2m}}}\right).$$

If the following conditions are verified:

- *(a)* $\big(\ell_0, \ell_1, \ldots, \ell_{n-1}\big)$ *is a CT-system on* $(0, x_r)$, *and*
- *(b)* $s > m(n-2)$,

then $(I_0, I_1, \ldots, I_{n-1})$ *is an ECT-system on* $(0, h_0)$.

We are now in position to show the main result of the paper.

Proof of Theorem A. Clearly the function $\ell_i = \mathscr{B}_\sigma\left(\frac{f_i}{A' B^{\frac{2s-1}{2m}}}\right)$ is σ-odd. Recall on the other hand that B does not vanish on (x_ℓ, x_r) and that A' vanishes only at $x = 0$ (with odd multiplicity, say $2r-1$). Thus, since the order of a pole at $x = 0$ does not change after computing its balance, we have that $x \longmapsto x^{2r-1}\ell_i(x)$ is analytic on (x_ℓ, x_r) for all $i = 0, 1, \ldots, n$. Hence, by Corollary 3.7, there exist $g_1, g_2, \ldots, g_k$ σ-odd functions verifying that

$$\big(\ell_0, \ldots, \ell_{n-2}, g_1, \ldots, g_k, \ell_n\big)$$

is an ECT-system on $(0, x_r)$ and that $x \longmapsto x^{2r-1}g_i(x)$ is analytic on (x_ℓ, x_r) for all $i = 1, 2, \ldots, k$. Define $h_i := g_i A' B^{\frac{2s-1}{2m}}$. Then h_i is an analytic function on (x_ℓ, x_r) such that

$$\mathscr{B}_\sigma\left(\frac{h_i}{A' B^{\frac{2s-1}{2m}}}\right) = \mathscr{B}_\sigma(g_i) = g_i.$$

Thus, setting $J_i := \displaystyle\int_{\gamma_h} h_i(x) y^{2s-1} dx$ for $i = 1, 2, \ldots, k$, by applying Theorem 3.8 we can assert that

$$\big(I_0, \ldots, I_{n-2}, J_1, \ldots, J_k, I_{n-1}\big)$$

is an ECT-system on $(0, h_0)$. Here we use, recall Definition 2.1, that an ECT-system is in particular a CT-system and that, by assumption, $s > m(n+k-2)$. This shows that any linear combination of $I_0, I_1, \ldots, I_{n-1}$ has at most $n + k - 1$ zeros on $(0, h_0)$ counted with multiplicities, as desired. ■

4 An example of application

As a toy example of application of our criterion we focus on the results obtained by Dumortier, Li and Zhang in [3]. In that paper the authors consider the family of quadratic systems

$$\text{(7)} \qquad \begin{cases} \dot{x} = y - 3x^2 + y^2 + \varepsilon(\nu_1 x + \nu_2 xy), \\ \dot{y} = x(1-2y) + \varepsilon \nu_3 x^2, \end{cases}$$

where $\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3 \setminus \{0\}$. This family is a versal unfolding of the quadratic system with $\varepsilon = 0$ among all 3-parameter unfoldings which are transversal to the stratum Q_3^R. It turns out (see [3, Lemma 2.2]) that the limit cycles that bifurcate for $\varepsilon \approx 0$ from the period annulus of the center at the origin correspond to zeros of the function

$$I(h) = \alpha I_0(h) + \beta I_1(h) + \gamma I_2(h) \text{ for } h \in (0, \tfrac{4}{3}),$$

where

$$I_i(h) = \int_{\gamma_h} x^i y dx \text{ with } H(x,y) = (x+2)y^2 + \frac{x^2(x+6)}{12} \text{ for } i = 0, 1, 2.$$

(Here α, β and γ are in linear bijective correspondence with ν_1, ν_2 and ν_3.) In this Section we will prove by applying Theorem A that any linear combination of I_0, I_1, I_2 has at most 3 zeros on $h \in (0, \frac{4}{3})$ counted with multiplicities. The proof of this fact in [3] takes 21 pages of very technical and highly non-trivial computations.

Following the notation in Theorem A, note that $m = 1$, $n = 3$, $s = 1$,

$$A(x) = \frac{x^2(x+6)}{12} \text{ and } B(x) = x + 2.$$

In this example $k = 1$, so that the condition $s > m(n + k - 2)$ is not fulfilled. To overcome this problem we shall use the next result (see [12, Lemma 4.1]) to increase the power of y in the 1-form associated to $I(h)$.

Lemma 4.1. *Let γ_h be an oval inside the level curve $\{A(x) + B(x)y^2 = h\}$ and consider a function F such that F/A' is analytic at $x = 0$. Then, for any $k \in \mathbb{N}$,*

$$\int_{\gamma_h} F(x) y^{k-2} dx = \int_{\gamma_h} G(x) y^k dx$$

where $G(x) = \frac{2}{k}\big(\frac{BF}{A'}\big)'(x) - \big(\frac{B'F}{A'}\big)(x)$.

Note that $I(h) = \int_{\gamma_h} F_1(x) y dx$ with $F_1(x) := \alpha + \beta x + \gamma x^2$. Since $F_1(0) \neq 0$, we can not apply Lemma 4.1 directly. To bypass this we note that

$$I(h) = \frac{1}{h^2} \int_{\gamma_h} H(x,y)^2 F_1(x) y dx = \frac{1}{h^2} \int_{\gamma_h} \big(A^2 F_1 y + 2ABF_1 y^3 + B^2 F_1 y^5\big)\, dx.$$

Thus, we first apply Lemma 4.1 with $k = 3$ and $F_2 := A^2 F_1$ to obtain[1]

$$I(h) = \frac{1}{h^2} \int_{\gamma_h} \big((G_2 + 2ABF_1) y^3 + B^2 F_1 y^5\big)\, dx.$$

Next, since one can verify that $F_3 := G_2 + 2ABF_1$ vanishes at $x = 0$, we can apply Lemma 4.1 once again to obtain $I(h) = \frac{1}{h^2} \int (G_3 + B^2 F_1) y^5 dx$, where G_3 depends on F_1, A and B and their derivatives until third order. More concretely, some computation show that

$$I(h) = \frac{1}{h^2} \int_{\gamma_h} \big(\alpha f_0(x) + \beta f_1(x) + \gamma f_2(x)\big) y^5 dx$$

[1]We remark that the computations were done with a symbolic manipulator (Maple). We include only those expressions that are essential to the exposition.

with

$$f_0(x) = \frac{8}{135}\,\frac{x^2\left(35\,x^6+732\,x^5+6354\,x^4+55296+100512\,x+74752\,x^2+29208\,x^3\right)}{(x+4)^4},$$
$$f_1(x) = \frac{8}{135}\,\frac{5134\,x^5+27\,x^7+27648+90336\,x+110640\,x^2+69512\,x^3+24842\,x^4+574\,x^6}{(x+4)^4},$$
$$f_2(x) = \frac{16}{135}\,\frac{1673\,x^4+10\,x^6+200\,x^5+13824+25008\,x+18744\,x^2+7468\,x^3}{(x+4)^4}.$$

It is clear that any linear combination of I_0, I_1, I_2 has at most 3 zeros on $(0,\frac{4}{3})$ counted with multiplicities if, and only if, so it occurs with $\widehat{I}_0, \widehat{I}_1, \widehat{I}_2$, where $\widehat{I}_i(h) = \int_{\gamma_h} f_i(x)y^5 dx$ for $i = 0,1,2$. Note that we can apply Theorem A to prove this equivalent result because now $s = 3$ (and still $m = 1$, $n = 3$ and $k = 1$), so that $s > m(n+k-2)$ holds. This is the assumption (c) in the statement of Theorem A. Next we shall prove that the assumptions (a) and (b) are verified as well. To this end let us define

$$t_i(x) := \mathscr{B}_\sigma\left(\frac{f_i}{A'B^{\frac{5}{2}}}\right)(x) = \frac{1}{2}\left(\frac{f_i}{A'B^{\frac{5}{2}}}\right)(x) - \frac{1}{2}\left(\frac{f_i}{A'B^{\frac{5}{2}}}\right)\big(\sigma(x)\big). \tag{8}$$

It is easy to show that the projection on the x-axis of the period annulus at the origin associated to $H = h$ is the interval (x_ℓ, x_r) with $x_\ell = -2$ and $x_r = 2(\sqrt{3}-1)$.

Note that, due to $t_i \circ \sigma = -t_i$, from (a) in Lemma 2.5 it follows that

$$W\big[\mathbf{t_i}\big]\big(\sigma(x)\big) = \frac{1}{\sigma'(x)^{\frac{(i-1)i}{2}}} W\big[\mathbf{t_i}\circ\sigma\big](x) = \frac{(-1)^{i-1}}{\sigma'(x)^{\frac{(i-1)i}{2}}} W\big[\mathbf{t_i}\big](x) \text{ for all } x \in (x_\ell, x_r).$$

Accordingly, since σ is a diffeomorphism with $\sigma\big((0,x_r)\big) = (x_\ell, 0)$, in order to verify the assumptions on the Wronskians we can take the interval $(x_\ell, 0)$ instead of $(0, x_r)$. We will do it for the sake of convenience. Thus our goal is to show that $W\big[\mathbf{t_1}\big]$ and $W\big[\mathbf{t_2}\big]$ do not vanish on $(-2,0)$, and that $W\big[\mathbf{t_3}\big]$ has exactly one zero on $(-2,0)$.

The involution σ is the unique analytic function with $\sigma(0) = 0$ and $\sigma'(0) = -1$ such that $A(x) = A\big(\sigma(x)\big)$ for all $x \in (x_\ell, x_r)$. One can verify that $A(x) - A(z) = \frac{1}{12}\,(x-z)\,p(x,z)$ with

$$p(x,z) = x^2 + zx + 6\,x + z^2 + 6\,z.$$

It is clear then that σ verifies $p\big(x,\sigma(x)\big) = 0$ for all $x \in (0, x_r)$. Hence, setting $z = \sigma(x)$,

$$\sigma'(x) = \frac{p_x(x,z)}{p_z(x,z)} = \frac{2\,x+z+6}{x+2\,z+6}. \tag{9}$$

Taking (8) into account, this shows in particular that $W\big[\mathbf{t_i}\big](x) = \omega_i(x,\sigma(x))$ with $\omega_i(x,z)$ being an *algebraic* function. It is clear that if $\omega_i\big(x_0,\sigma(x_0)\big) = 0$ for $x_0 \in (-2,0)$, then $p(x,z) = 0$ and $\omega_i(x,z) = 0$ have a common root, (x_0, z_0) with $z_0 = \sigma(x_0)$. This is the reason why we shall next study the common roots of $p(x,z)$ and $\omega_i(x,z)$. Let us stress however that the existence of a common (real) root is only a necessary condition for the vanishing of $W\big[\mathbf{t_i}\big]$.

Let us start by studying the third order Wronskian. Some computations show that

$$\omega_3(x,z) = \frac{A(x,z)\sqrt{x+2} - A(z,x)\sqrt{z+2}}{(x+2z+6)^3}, \tag{10}$$

where A is a rational function. Clearly, if $\omega_3(x,z) = 0$, then $r_3(x,z) := A(x,z)^2(x+2) - A(z,x)^2(z+2) = 0$. Note that if the resultant with respect to z between $p(x,z)$ and the numerator of $r_3(x,z)$ vanishes exactly once on $(-2,0)$, then $W\big[\mathbf{t_3}\big]$ has at most one zero on $(-2,0)$, as desired. By applying Sturm's theorem it follows however that it vanishes twice on $(-2,0)$, numerically at $\bar{x}_2 \approx -1.1331$ and $\bar{x}_2 \approx -1.9991$. Fortunately the latter is not a zero of $\omega_3\big(x,\sigma(x)\big)$. To prove this we first apply Sturm's theorem to check that the resultant vanishes exactly once on $(-1.9,0)$. Then we verify that $\omega_3\big(x,\sigma(x)\big)$ is negative at $x=-2$ and decreasing on $(-2,-1.9)$. This latter fact can be once again proved algebraically. Indeed, we derivate implicitly using (9) and we get a rational function on x, z, $\sqrt{x+2}$ and $\sqrt{z+2}$ as in (10) We evaluate it at $x=-2$ and $z=\sigma(-2)=2(\sqrt{3}-1)$ to verify that it is negative. The fact that it does not vanish on $(-2,-1.9)$ can be done by applying Sturm's theorem exactly as before. This shows that the assumption (b) in Theorem A holds.

Let us consider next the assumption in (a). We begin by studying the second order Wronskian. Unfortunately it turns out that $W\big[\mathbf{t_1}\big]=W[t_0,t_1]$ vanishes once on $(-2,0)$. Accordingly we can not apply Theorem A directly to $\widehat{I}_0,\widehat{I}_1,\widehat{I}_2$, at least in this order. Alternatively, we could try with $\widehat{I}_0,\widehat{I}_2,\widehat{I}_1$ or $\widehat{I}_1,\widehat{I}_2,\widehat{I}_0$. However these two options lead to a dead end as well because $W[t_0,t_2]$ and $W[t_1,t_2]$ both have a zero on $(-2,0)$. After several attempts with other linear combinations, we found out numerically that $W[t_2,t_0+t_1]$ does not vanish on $(-2,0)$. To prove it algebraically we first note that $W[t_2,t_0+t_1](x)=\widetilde{\omega}_2\big(x,\sigma(x)\big)$ where

$$\widetilde{\omega}_2(x,z)=A(x,z)\sqrt{x+2}\sqrt{z+2}+B(x,z),$$

with A and B rational functions. Hence it suffices to show that $A(x,z)^2(x+2)(z+2)-B(x,z)^2$ and $p(x,z)$ does not have common roots when $x\in(-2,0)$. This fact can be studied by computing the resultant with respect to z and applying Sturm's theorem. Unfortunately the resultant has one zero on $(-2,0)$, numerically at $\bar{x}_3\approx -1.9980$. Exactly as before we will prove that this is not a "true" zero of $\widetilde{\omega}_2\big(x,\sigma(x)\big)$. Firstly, by applying Sturm's theorem, we check that the resultant does not vanish on $(-1.9,0)$. Secondly we verify that $\widetilde{\omega}_2\big(x,\sigma(x)\big)$ is positive at $x=-2$ and increasing on $(-2,-1.9)$.

Finally we will prove that $W[t_2]$ does not vanish on $(-2,0)$. Note that in fact

$$W[t_2](x)=t_2(x)=\mathscr{B}_\sigma\left(\frac{f_2}{A'B^{\frac{5}{2}}}\right)(x)=\frac{1}{2}\left(\frac{f_2}{A'B^{\frac{5}{2}}}\right)(x)-\frac{1}{2}\left(\frac{f_2}{A'B^{\frac{5}{2}}}\right)\big(\sigma(x)\big).$$

Thus, since $x<0<\sigma(x)$ for all $x\in(-2,0)$, it suffices to verify that $x\longmapsto\left(\frac{f_2}{A'B^{\frac{5}{2}}}\right)(x)$ is a monotonic function. Some computations show that $\left(\frac{f_2}{A'B^{\frac{5}{2}}}\right)'(x)=\frac{16S(x)}{135(x+2)^{7/2}(x+4)^6}$ with

$$S(x)=-35\,x^8-656\,x^7-4426\,x^6-7104\,x^5+69216\,x^4+464672\,x^3+1261056\,x^2+1668096\,x+884736.$$

By Sturm's theorem it follows that this polynomial does not vanish on $(-2,0)$.

Accordingly, taking $W[t_2,t_0+t_1,t_0]=-W[t_0,t_1,t_2]$ into account, by applying Theorem A we can assert that any linear combination of $\widehat{I}_2,\widehat{I}_0+\widehat{I}_1,\widehat{I}_0$ has at most 3 zeros on $(0,\frac{4}{3})$ counted with multiplicities. Obviously, this proves that so it occurs with $\widehat{I}_0,\widehat{I}_1,\widehat{I}_2$, as desired.

References

[1] V.I. Arnold, "Arnold's problems", Springer-Verlag, Berlin, 2004.

[2] G. Binyamini, D. Novikov and S. Yakovenko, *On the number of zeros of Abelian integrals: A constructive solution of the infinitesimal Hilbert sixteenth problem,* Invent. Math. **181** (2010), 227–289.

[3] F. Dumortier, Chengzhi Li and Zifen Zhang, *Unfolding of a quadratic integrable system with two centers and two unbounded heteroclinic loops,* J. Differential Equations **139** (1997), 146–193.

[4] F. Dumortier and R. Roussarie, *Abelian integrals and limit cycles,* J. Differential Equations **227** (2006), 116–165.

[5] A. Gasull, Weigu Li, J. Llibre and Zhifen Zhang, *Chebyshev property of complete elliptic integrals and its application to Abelian integrals,* Pacific J. Math. **202** (2002), 341–361.

[6] S. Gautier, L. Gavrilov and I. Iliev, *Perturbations of quadratic centers of genus one,* Discrete Contin. Dyn. Syst. **25** (2009), 511–535.

[7] L. Gavrilov, *The infinitesimal 16th Hilbert problem in the quadratic case,* Invent. Math. **143** (2001), 449–497.

[8] L. Gavrilov and I. Iliev, *Bifurcations of limit cycles from infinity in quadratic systems,* Canad. J. Math. **54** (2002), 1038–1064.

[9] L. Gavrilov and I. Iliev, *Two-dimensional Fuchsian systems and the Chebyshev property,* J. Differential Equations **191** (2003), 105–120.

[10] L. Gavrilov and I. Iliev, *Complete hyperelliptic integrals of the first kind and their non-oscillation,* Trans. Amer. Math. Soc. **356** (2003), 1185–1207.

[11] L. Gavrilov and I. Iliev, *Quadratic perturbations of quadratic codimension-four centers,* J. Math. Anal. Appl. **357** (2009), 69–76.

[12] M. Grau, F. Mañosas and J. Villadelprat, *A Chebyshev criterion for Abelian integrals,* Trans. Amer. Math. Soc. **363** (2011), 109–129.

[13] Maoan Han, *Existence of at most 1, 2, or 3 zeros of a Melnikov function and limit cycles,* J. Differential Equations **170** (2001), 325–343.

[14] D. Hilbert, *Mathematische Problem* (*lecture*), Second Internat. Congress Math. Paris 1900, Nachr. Ges. Wiss. Gottingen Math.-Phys. Kl. 1900, 253–297.

[15] E. Horozov and I. Iliev, *On the number of limit cycles in perturbations of quadratic Hamiltonian systems,* Proc. London Math. Soc. **69** (1994), 198–224.

[16] E. Horozov and I. Iliev, *Linear estimate for the number of zeros of Abelian integrals with cubic Hamiltonians,* Nonlinearity **11** (1998), 1521–1537.

[17] L. A. Howland, *Note on the derivative of the quotient of two wronskians,* Amer. Math. Monthly **28** (1911), 219–221.

[18] I. Iliev, *Perturbations of quadratic centers,* Bull. Sci. Math. **122** (1998), 107–161.

[19] Yu. Ilyashenko, *Centennial history of Hilbert's 16th problem,* Bull. Amer. Math. Soc. (N.S.) **39** (2002), 301–354.

[20] S. Karlin and W. Studden, "Tchebycheff systems: with applications in analysis and statistics", Interscience Publishers, 1966.

[21] Lin Ping Peng, *Unfolding of a quadratic integrable system with a homoclinic loop,* Acta Mathematica Sinica **18** (2002), 737—754.

[22] G. Polya, *On the mean-value theorem corresponding to a given linear homogeneous differential equation,* Trans. Amer. Math. Soc. **24** (1922), 312–324.

[23] Yulin Zhao, Zhaojun Liang and Gang Lu, *The cyclicity of the period annulus of the quadratic Hamiltonian systems with non-Morsean point,* J. Differential Equations **162** (2000), 199–223.