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New periodic recurrences with applications

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Abstract

We develop two methods for constructing several new and explicit m-periodic difference equations. Then we apply our results to two different problems. Firstly we show that two simple natural conditions appearing in the literature are not necessary conditions for the global periodicity of the difference equations. Secondly we present the first explicit non-linear analytic potential differential system having a global isochronous center.

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1 Introduction and main results

In general, a real difference equation (or recurrence) of order n writes as

$$x_{j+n} = f(x_j, x_{j+1}, \ldots, x_{j+n-1}) \tag{1}$$

where $x_j \in \mathbb{R}$ for all $j \in \mathbb{N}$, and f is a map from an open subset of $\mathbb{R}^n$ into $\mathbb{R}$. This recurrence can be studied through the dynamical system generated by the map $F : U \to U$, where U is an open subset of $\mathbb{R}^n$ and F is

$$F(x_1, \ldots, x_n) = (x_2, \ldots, x_n, f(x_1, \ldots, x_n)). \tag{2}$$

Recurrence (1) or the dynamical system generated by (2) is m-periodic if $F^m = \mathrm{Id}$ and $m \geq n$ is the smallest natural with this property. In terms of the recurrence (1) this property reads as $x_{j+m} = x_j$ for all $j \geq 0$. Several examples of periodic recurrences are given for instance in [1, 4, 8, 9, 14, 15, 19].

The first part of the paper is devoted to give several new examples of periodic recurrences (1), defined in the whole $\mathbb{R}^n$. In our constructions we follow the next two simple ideas:

- The difference equation (1) is transformed into the new difference equation
$$y_{j+n} = \varphi^{-1}\left(f(\varphi(y_j), \varphi(y_{j+1}), \ldots, \varphi(y_{j+n-1}))\right), \tag{3}$$
with the change of variables $x_n = \varphi(y_n)$, where φ is any invertible map. Hence if (1) is m-periodic the same holds for (3). Moreover if f, φ and φ^{-1} are all elementary functions, the same happens with the function defining the new recurrence (3).

- Let G be a given m-periodic recurrence of order n and let Φ be a homeomorphism of $V \subset \mathbb{R}^n$ into $U \subset \mathbb{R}^n$. Then $F = \Phi \circ G \circ \Phi^{-1}$ gives a periodic map on U which is not necessarily a periodic recurrence. We investigate the structure of the map Φ in order that F still gives a recurrence on $\mathbb{R}^n$ and we study in more detail the case $n = 2$.

Although the first method is common knowledge, our focus has been the construction of elementary functions φ with elementary inverse function φ^{-1}. Recall that, roughly speaking, elementary functions are functions built from a finite number of (complex) exponentials, logarithms and polynomials, through composition and combinations using the four elementary operations. As far as we know the second method considered in this paper is new.

The above constructions, detailed in Sections 2.1 and 2.2, respectively, will allow to prove the main results of the paper, described in the sequel. The first one deals with the following two properties, which try to give necessary conditions for some f to define a m-periodic difference equation (1). They are:

- **P1:** For each fixed $\mathbf{y} \in \mathbb{R}$ and $w \in \mathbb{R}$ the real valuated map $f_{\mathbf{y}}(w) := f(w, \mathbf{y}, \ldots, \mathbf{y})$ is an involution.

- **P2:** The map $\sigma \circ F$ is an involution, where $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is $\sigma(x_1, x_2, \ldots, x_{n-1}, x_n) = (x_n, x_{n-1}, \ldots, x_2, x_1)$.

Note that $\sigma(F(w, x_2, \ldots, x_n)) = (f(w, x_2, \ldots, x_n), x_n, x_{n-1}, \ldots, x_2)$. So $(\sigma \circ F)^2 = \mathrm{Id}$ is equivalent to
$$f(f(w, x_2, x_3, \ldots, x_n), x_n, x_{n-1}, \ldots, x_2) = w.$$
Therefore, in general, property **P2** implies property **P1** and when $n = 2$ both properties are equivalent.

The main reasons to introduce them are Theorem 3.1 of [2], where the author asserts that the global periodicity of (1) implies property **P1**, and the results of Lemma 10, where we prove that all real linear periodic n-th order recurrences satisfy **P2**. Moreover, surprisingly, it can be seen that many known non-linear recurrences also satisfy both properties.

Nevertheless our first main result below, Theorem 1, shows that none of these properties is a necessary condition for periodicity of (1). Our proof is based on the constructing of a 3-periodic second order recurrence for which both properties fail. Before continuing we want to comment that an example of a known recurrence not satisfying property **P2** is the 8-periodic Lyness given in [15],

$$x_{j+3} = \frac{x_{j+1} - x_{j+2} - 1}{x_j}.$$

Notice that $F(x, y, z) = (y, z, f(x, y, z))$ where $f(x, y, z) = (y-z-1)/x$. Then $\sigma \circ F(x, y, z) = (\frac{y-z-1}{x}, z, y)$. So

$$(\sigma \circ F)^2(x, y, z) = (\frac{z-y-1}{y-z-1}x, y, z) \neq (x, y, z).$$

On the other hand property **P1** holds because

$$f_{\mathbf{y}}(w) = f(w, \mathbf{y}, \mathbf{y}) = -\frac{1}{w}$$

is an involution. In any case it is a not natural example because, as can be seen by analyzing the dynamical system associated to F, it has no connected invariant region. In fact F permutes invariant regions and has no fixed point. In some sense it is natural to think that properties **P1** and **P2** are inherited from the linear part of F at the fixed points, because as we will see in Lemma 10, the recurrences associated to the linear part at them always satisfy both properties. Anyway, the example that we will present for proving next theorem has a fixed point.

Theorem 1. *None of the properties* **P1** *and* **P2** *is a necessary condition for the global periodicity of* (1). *More specifically, there are periodic recurrences, whose associated dynamical systems have a fixed point, that do not satisfy properties* **P1** *and* **P2**.

Hence, unfortunately, none of the above properties can be used to obtain necessary conditions of global periodicity. In particular our result shows that there is a gap in the proof of [2, Thm. 3.1].

As a second application of our results we will construct a 2-parameter family of non-linear explicit global isochronous potential systems. To the best of our knowledge all the known explicit examples were not globally defined. Before stating the result we will motivate the problem.

From the pioneering works of Urabe ([21, 22]) it is already known how to characterize the planar potential systems of the form

$$\dot{x} = -y, \quad \dot{y} = V'(x), \tag{4}$$

having a (local or global) isochronous center at the origin. Recall that a critical point $\mathbf{p}$ of center type is called isochronous if the periods of all the orbits in a neighborhood of $\mathbf{p}$ coincide. If moreover the center is global and all the periodic orbits have the same period then we will say the the $\mathbf{p}$ is a global isochronous center. See for instance [7, 16] and the references therein for more information about planar isochronous centers.

In [12, 23] there is a different characterization of the isochronism of a center of system (4) based on the existence of some (local or global) involution. This characterization, recalled in Theorem 12 is the one that will interest us. We only comment here that in that result it is proved that the existence of an analytic explicit global isochronous center is reduced to the construction of an analytic global explicit involution.

As usual a real involution ψ is an invertible function defined in an open subset of $\mathbb{R}$ that satisfies $\psi^2 = \mathrm{Id}$, or equivalently $\psi = \psi^{-1}$. As far as we know, apart of the trivial involution $\psi(x) = a - x$, all the other known explicit involutions, like

$$\psi(x) = \frac{ax+b}{cx-a}, \quad \psi(x) = \sqrt[2m+1]{a - x^{2m+1}}, \quad \psi(x) = 2a^4 + x - 2a^2\sqrt{a^4 + 2x}, \ldots$$

either are not defined in the whole $\mathbb{R}$, or are not differentiable. With the methods introduced in Section 2.1 we give a 2-parameter family of explicit analytic global involutions. From it we construct the first explicit non-linear analytic potential systems having a global isochronous center.

Theorem 2. *The 3-parameter family of analytic potential systems*

$$\dot{x} = -y, \quad \dot{y} = V'(x), \tag{5}$$

with $V(x) := K(x - \psi(x))^2$, *where*

$$\psi(x) = \frac{1}{6}\left(\sqrt[3]{A(x) + \sqrt{A^2(x) + \Delta^3}} - \frac{\Delta}{\sqrt[3]{A(x) + \sqrt{A^2(x) + \Delta^3}}} - 2r \right), \tag{6}$$

being

$$\Delta := 4(3s - r^2) > 0, \quad A(x) := 4\big((9s - 2r^2)r - 27(x^3 + rx^2 + sx)\big),$$

and $K, r, s \in \mathbb{R}$, *has a global isochronous center at the origin.*

From the proof of the above theorem it also can be seen that for some values of r and s not satisfying $\Delta > 0$, the origin of (5) is a local isochronous center.

2 New periodic recurrences

The first part of this section is devoted to give explicit new global trivially linearizable periodic recurrences based on the construction of explicit one-dimensional global homeomorphisms. In the second part we build periodic recurrences based on the construction of planar homeomorphisms which conjugate recurrences into recurrences.

2.1 Explicit global diffeomorphisms and involutions

An easy way of obtaining new difference equations having the same behavior as a given one (1) consists in constructing the difference equations (3) where φ is any given invertible map. A particularly interesting case is the one where the base difference equation is linear, that is recurrence (1) is

$$x_{j+n} = \ell(x_j, x_{j+1}, \ldots, x_{j+n-1}),$$

where $\ell(w_1, \ldots, w_n) = a_1\, w_1 + a_2\, w_2 + \cdots + a_n\, w_n$. All the recurrences that can be written as

$$y_{j+n} = \varphi^{-1}\left(\ell(\varphi(y_j), \varphi(y_{j+1}), \ldots, \varphi(y_{j+n-1}))\right),$$

for some invertible map φ, will be called *trivially linearizable*.

The above facts motivate our interest to find elements of the group with the composition:

$$\mathcal{E}_k = \{\varphi : \mathbb{R} \to \mathbb{R} \text{ with } \varphi \text{ and } \varphi^{-1} \text{ elementary functions that are } \mathcal{C}^k\text{-homeomorhisms}\}$$

where $k \in \{0, 1, \ldots, n, \infty, \omega\}$.

It is clear, that once we have some elements $\{\varphi_\lambda, \lambda \in \Lambda\}$ of $\mathcal{E}_k$, the new maps formed by composing them with their inverses are again elements of $\mathcal{E}_k$. So we center our attention to find simple elements of this set.

Lemma 3. *The following functions belong to $\mathcal{E}_k$:*

(i) The affine maps $\varphi(x) = p\,x + q, \quad p \neq 0$ with $k = \omega$.

(ii) The functions $\varphi(x) = x^{2p+1}$, being $0 < p \in \mathbb{N}$, with $k = 0$.

(iii) The function $\varphi(x) = \sinh(x) = (e^x - e^{-x})/2$, with $k = \omega$.

(iv) The function $\varphi(x) = x^3 + p\,x + q, \quad p > 0$, with $k = \omega$

Proof. The proof of the first three items is very easy: the corresponding inverse functions $\varphi^{-1}(y)$ are, $(y - q)/p$, $\sqrt[2p+1]{y}$ and $\ln\left((y + \sqrt{y^2 + 4})/2\right)$, respectively.

The proof of (iv) is a consequence of Cardano's formula. Note that when $p > 0$ the corresponding φ is a diffeomorphism and

$$\varphi^{-1}(y) = \sqrt[3]{\frac{y-q}{2} + \sqrt{\frac{(q-y)^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{\frac{q-y}{2} + \sqrt{\frac{(q-y)^2}{4} + \frac{p^3}{27}}}. \qquad (7)$$

Observe also that for $p > 0$ we do not need to pass trough complex numbers to find the real solution of the cubic equation. $\square$

Let us see a family of simple global m-periodic recurrences of order n constructed by using the diffeomorphism $\varphi^{-1}(x) = \sinh(x)$. The map

$$F(x_1,\ldots,x_n) = \left(x_2,\ldots,x_{n-1}, \frac{1}{2}\left[\prod_{i=1}^{n} \left(\frac{x_i + \sqrt{x_i^2+1}}{2} \right)^{a_i} - \prod_{i=1}^{n} \left(\frac{x_i + \sqrt{x_i^2+1}}{2} \right)^{-a_i} \right] \right),$$

where the values $a_i, i = 1\ldots,n$ are taken such that

$$L(x_1,\ldots,x_n) = (x_2,\ldots,x_{n-1}, a_1\,x_1 + a_2\,x_2 + a_3\,x_3 + \cdots + a_n\,x_n)$$

is a real linear m-periodic recurrence, gives rise to a m-periodic recurrence. In particular, when $n = 2$, $L(x,y) = (y, -x + 2\operatorname{Re}(\alpha)y)$ with α some primitive m-root of the unity (see Lemma 7 below) we obtain the global second order m-periodic recurrences

$$x_{j+2} = \frac{1}{2}\left[\left(\frac{x_{j+1} + \sqrt{x_{j+1}^2+1}}{2} \right)^{b} \left(\frac{x_j + \sqrt{x_j^2+1}}{2} \right)^{-1} - \left(\frac{x_{j+1} + \sqrt{x_{j+1}^2+1}}{2} \right)^{-b} \left(\frac{x_j + \sqrt{x_j^2+1}}{2} \right) \right],$$

where $b = 2\operatorname{Re}(\alpha)$.

There are many elementary functions that can be inverted in an open interval giving also an elementary function, but it is not possible to invert them globally. We introduce a simple trick to get from them global homeomorphisms.

Lemma 4. *Consider three $\mathcal{C}^k$-invertible real elementary functions defined on the intervals given in next diagram*

$$(-\infty,\infty) \xrightarrow{g} (r,t) \xrightarrow{\phi} (u,w) \xrightarrow{h} (-\infty,\infty).$$

If the three functions g,ϕ,h have a $\mathcal{C}^k$ elementary inverse in their corresponding intervals, then $\psi = h\circ\phi\circ g$ is an element of $\mathcal{E}_k$.

Note also that in the above result it is also possible to take some of the intervals (r,t) or (u,w) unbounded.

Simple examples of 1-parametric functions g and h are given by the expressions

$$h_v(x) = H_{u,v,w}(x) := \frac{x-v}{(x-u)(w-x)}, \quad \text{where} \quad -\infty < u < v < w < \infty,$$

and

$$g_s(x) = H^{-1}_{r,s,t}(x), \quad \text{where} \quad -\infty < r < s < t < \infty.$$

Then, once an invertible increasing elementary function $\phi : (r,t) \to (\phi(r), \phi(t))$, with elementary inverse is given, it suffices to take $u = \phi(r)$ and $w = \phi(t)$ and g and h as above. Let us give some concrete examples:

By taking

$$(-\infty, \infty) \xrightarrow{g} (-\frac{\pi}{2}, \frac{\pi}{2}) \xrightarrow{\sin} (-1,1) \xrightarrow{h} (-\infty, \infty)$$

with $g(x) = \frac{-1+\sqrt{1+\pi^2x^2}}{2x}$ and $h(x) = \frac{x}{1-x^2}$ we obtain the global diffeomorphism in $\mathcal{E}_\omega$:

$$\varphi(x) = h(\sin(g(x))) = \frac{\sin\left(\frac{-1+\sqrt{1+\pi^2x^2}}{2x}\right)}{\cos^2\left(\frac{-1+\sqrt{1+\pi^2x^2}}{2x}\right)}.$$

The functions g and h can also be taken different from the above ones. For instance, by using

$$(-\infty, \infty) \xrightarrow{g} (-\frac{\pi}{2}, \frac{\pi}{2}) \xrightarrow{\tan} (-\infty, \infty) \xrightarrow{\mathrm{Id}} (-\infty, \infty)$$

with $g(x) = \frac{\pi}{2}\frac{x}{\sqrt{1+x^2}}$ we get the element of $\mathcal{E}_\omega$:

$$\varphi(x) = \tan(g(x)) = \tan\left(\frac{\pi}{2}\frac{x}{\sqrt{1+x^2}}\right).$$

When some of the values r, t, u or w are plus or minus infinity, other functions g and h can be used. For instance, a simple one is $g(x) = e^x + r$ that sends $(-\infty, \infty)$ into (r, ∞).

Clearly, collecting all the above results and using the group structure of $\mathcal{E}_k$ we can obtain, for any $n > 1$ and any $m > n$, many other examples of recurrences of order n which are m-periodic on $\mathbb{R}^n$ which are trivially linearizable.

2.2 Generating non-trivially linearizable recurrences

An easy way to obtain periodic maps on $U \subset \mathbb{R}^n$ is to conjugate a known periodic recurrence of order n, G defined on V, with a homeomorphism $\Phi : V \to U$. Recall that to obtain a periodic recurrence we additionally need that $\Phi \circ G \circ \Phi^{-1}(x_1, \ldots, x_n) = (x_2, \ldots, x_n, f(x_1, \ldots, x_n))$ for some continuous map f. The next lemma characterizes the homeomorphisms Φ with this property. One of this type of examples will be used to prove Theorem 1.

Lemma 5. *Let U and V simply connected subsets of R^n. Let $G : V \to V$ be a recurrence and let $\Phi : V \to U$ be a homeomorphism. Then the map $\Phi \circ G \circ \Phi^{-1}$ is a recurrence defined in U if and only if $\Phi_i(x) = \Phi_1(G^{i-1}(x))$ for all points $x \in U$ and for any $i = 2, 3, \ldots, n$.*

Proof. Denote $\Phi \circ G \circ \Phi^{-1}$ by F. Then we will have $F \circ \Phi = \Phi \circ G$. Thus if F is a recurrence we obtain $\Phi_i = \Phi_{i-1}(G(x))$, for $i = 2, 3, \ldots, n$, which inductively shows the desired result. The converse follows by the same argument. $\square$

Next results study in more detail the case $n = 2$. First we state a classical result.

Theorem 6. *(Kerékjártó's Theorem, see [13]) Let $U \subset \mathbb{R}^2$ be homeomorphic to $\mathbb{R}^2$ and let $F : U \to U$ be a $\mathcal{C}^k$, m-periodic map, $k \geq 0$. Then F is $\mathcal{C}^0$-linearizable.*

Lemma 7. *Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be a continuous recurrence, such that $F(x, y) = (y, f(x, y))$. Then F is m-periodic if and only if there exists α, a primitive m-root of the unity, and $g : \mathbb{R}^2 \to \mathbb{R}$ continuous, such that the map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$\Phi(x, y) = (g(x, y), g(L_\alpha(x, y))),$$

where $L_\alpha(x, y) = (y, -x + 2\operatorname{Re}(\alpha)\, y)$, is a homeomorphism from $\mathbb{R}^2$ into itself and $F = \Phi \circ L_\alpha \circ \Phi^{-1}$.

Proof. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear m-periodic recurrence of the form $L(x, y) = (y, ax + by)$. The characteristic polynomial of L is $p(\lambda) = \lambda^2 - b\lambda - a$. Since L is m-periodic it follows that its eigenvalues must a pair of conjugate primitive m-roots of the unity. Thus, $a = -1$ and $b = 2\operatorname{Re}(\alpha)$ for some α a primitive m-root of the unity.

From Kerékjártó's Theorem we already know that there exists a homeomorphism $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ such that $F = \Phi \circ L_\alpha \circ \Phi^{-1}$. Then, putting $\Phi = (g, h)$ by Lemma 5 we get that $h = g \circ L_\alpha$.

Conversely, if $F = \Phi \circ L_\alpha \circ \Phi^{-1}$ then F is conjugated to L_α and hence it is m-periodic. Moreover also by Lemma 5 it follows that F gives rise to a recurrence. $\square$

Lemma 8. *Let α be primitive m-root of the unity. If $\operatorname{Re}(\alpha) \geq 0$ (respectively $\operatorname{Re}(\alpha) \leq 0$) and $d, h : \mathbb{R} \to \mathbb{R}$ are two homeomorphisms such that $d \circ h$ is increasing (respectively decreasing), then the map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\Phi(x, y) = (g(x, y), g(L_\alpha(x, y)))$, where $g(x, y) = d(x) + h(y)$, is an homeomorphism from $\mathbb{R}^2$ into itself and $\Phi \circ L_\alpha \circ \Phi^{-1}$ is a m-periodic recurrence defined in the whole plane.*

Proof. We will prove that Φ has a global inverse. Taking $(u, v) \in \mathbb{R}^2$ we need to show that there exists one and only one $(x, y) \in \mathbb{R}^2$ such that $\Phi(x, y) = (u, v)$. So for each $(u, v) \in \mathbb{R}^2$

we have to solve the system:

$$\begin{aligned} d(x) + h(y) &= u, \\ d(y) + h(-x + 2\operatorname{Re}(\alpha)y) &= v. \end{aligned}$$

Assume for instance $\operatorname{Re}(\alpha) \geq 0$ (the other case follows in a similar way). From the first equation we get $y = h^{-1}(u - d(x))$. Substituting in the second equation we obtain

$$d(h^{-1}(u - d(x))) + h(-x + 2\operatorname{Re}(\alpha)h^{-1}(u - d(x))) = v.$$

Fixed $u \in \mathbb{R}$ consider the map $M_u : \mathbb{R} \to \mathbb{R}$ defined by

$$M_u(x) = d(h^{-1}(u - d(x))) + h(-x + 2\operatorname{Re}(\alpha)h^{-1}(u - d(x))).$$

From the facts that that $\operatorname{Re}(\alpha) \leq 0$ and $d \circ h$ is increasing it follows that M_u is a homeomorphism from $\mathbb{R}$ to itself. Then $x = M_u^{-1}(v)$ and $y = h^{-1}(u - d(M_u^{-1}(v)))$ gives the continuous global inverse for Φ. The fact that F is a m-periodic recurrence follows from Lemma 7. $\square$

The case $m = 4$ is very special because $Re(\alpha) = 0$. In this situation we can go further in the above computations. We have got the following result which gives a simple method to generate new 4-periodic second order recurrences.

Lemma 9. *Let $P : \mathbb{R} \to \mathbb{R}$ be a $\mathcal{C}^k$ even map be such that $\operatorname{Id} + P$ is a $\mathcal{C}^k$ diffeomorphism from $\mathbb{R}$ into itself and $P(0) = 0$. Then the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by*

$$F(x, y) = (y, y - (\operatorname{Id} + P)^{-1}(y + x - P(y - x))$$

is 4-periodic.

Proof. Denote by L the linear map $L(x, y) = (y, -x)$. Consider $h(x) = x + \frac{P(2x)}{2}$ which is also a $\mathcal{C}^k$ diffeomorfism of $\mathbb{R}$ into itself and the map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\Phi(x, y) = (h(x) + h(-y), h(y) + h(x)).$$

From Lemma 8 it follows that Φ is a $\mathcal{C}^k$ diffeomorfism of $\mathbb{R}^2$ and the map F defined by $\Phi \circ L \circ \Phi^{-1}$ is a 4-periodic recurrence. Easy computations show that

$$\Phi^{-1}(x, y) = \left(h^{-1}\left(y - h\left(\frac{y - x}{2}\right)\right), \frac{y - x}{2}\right).$$

Thus we will have

$$F(x, y) = \left(y, h\left(-h^{-1}\left(y - h\left(\frac{y - x}{2}\right)\right)\right) + h\left(\frac{y - x}{2}\right)\right).$$

Now using that $h(-x) = h(x) - 2x$ we obtain

$$
\begin{aligned}
F(x,y) &= (y, y - 2h^{-1}(y - h\left(\tfrac{y-x}{2}\right))) \\
&= (y, y - 2h^{-1}(\tfrac{y+x-P(y-x)}{2})) \\
&= (y, y - (\mathrm{Id} + P)^{-1}(y + x - P(y-x))).
\end{aligned}
$$

$\square$

Clearly if $P : \mathbb{R} \to \mathbb{R}$ is an even function such that $\mathrm{Id} + P$ is a $\mathcal{C}^k$-diffeomorphism of $\mathbb{R}$, $k \geq 1$, then $|P'(x)| < 1$ for all $x \in \mathbb{R}$. Moreover it is easy to see that if $|P'(x)| < k < 1$ for all $x \in \mathbb{R}$ then $\mathrm{Id} + P$ is a $\mathcal{C}^k$ diffeomorphism of $\mathbb{R}$. Thus for each $a \in \mathbb{R}$ such that $|a| < 1$ we get that the function $h_a(x) = x + a\cos x$ is an analytic diffeomorphism of $\mathbb{R}$. Hence from the above Lemma we obtain that the map

$$F_a(x,y) = (y, y - h_a^{-1}(x + y - a\cos(y - x)))$$

is a global 4-periodic analytic recurrence.

3 On properties P1 and P2 and proof of Theorem 1

The next lemma shows that properties **P1** and **P2** are satisfied by linear recurrences. The first part of the lemma is already known, see [15, 18].

Lemma 10. *Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a periodic linear map of the form*

$$L(x_1, \ldots, x_n) = (x_2, \ldots, x_n, \ell(x_1, x_2, \ldots, x_n)).$$

Then the characteristic polynomial of L has no multiple roots. Moreover $(\sigma \circ L)^2 = \mathrm{Id}$, where $\sigma(x_1, x_2, \ldots, x_{n-1}, x_n) = (x_n, x_{n-1}, \ldots, x_2, x_1)$.

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of L. Since for any for any Jordan block and for any $m \in \mathbb{N}$,

$$
\begin{pmatrix}
\lambda & 1 & 0 & \ldots & 0 & 0 \\
0 & \lambda & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & \lambda & 1 \\
0 & 0 & 0 & \ldots & 0 & \lambda
\end{pmatrix}^m \neq \mathrm{Id},
$$

we obtain that L is diagonalizable over $\mathbb{C}$. So to prove the first part of statement it suffices to show that for any λ its space of eigenvectors has dimension one. Let $0 \neq v \in \mathbb{C}^n$ be such that $L(v) = \lambda\, v$. Then $v = (x_1, \lambda\, x_1, \ldots, \lambda^{n-1} x_1) = x_1(1, \lambda, \ldots, \lambda^{n-1})$.

Note also that if L is n-periodic all its eigenvalues must be n-th roots of the unity, so if λ is a eigenvalue of L then λ^{-1} must be also eigenvalue of L. Then L and L^{-1} have the same characteristic polynomial. Since $\ell(x_1,\ldots,x_n)=a_1x_1+\cdots+a_nx_n$,

$$L=\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0\\ 0 & 0 & 1 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & 0 & 1\\ a_1 & a_2 & a_3 & \ldots & a_{n-1} & a_n \end{pmatrix} \quad \text{and} \quad L^{-1}=\begin{pmatrix} b_1 & b_2 & b_3 & \ldots & b_{n-1} & b_n\\ 1 & 0 & 0 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ldots & 0 & 0\\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix},$$

it follows that $(a_1,\ldots,a_n)=(b_n,\ldots,b_1)$. Thus $\sigma\circ L\circ\sigma=L^{-1}$ and as a consequence $(\sigma\circ L)^2=\mathrm{Id}$, as we want to prove. □

As a corollary we obtain the following result:

Corollary 11. *Trivially linearizable recurrences satisfy properties* **P1** *and* **P2**.

Proof. It is easy to check that if recurrence (1) satisfies properties **P1** or **P2**, then the same happens for any recurrence of the form

$$y_{j+n}=\varphi^{-1}\left(f(\varphi(y_j),\varphi(y_{j+1}),\ldots,\varphi(y_{j+n-1}))\right),$$

where φ is an invertible map. Since trivially linearizable recurrences can be written as

$$y_{j+n}=\varphi^{-1}\left(\ell(\varphi(y_j),\varphi(y_{j+1}),\ldots,\varphi(y_{j+n-1}))\right),$$

for some linear map ℓ and some invertible map φ, the result follows from Lemma 10. □

By using the above corollary we have that both properties **P1** and **P2** hold for the difference equations appearing in [1, 4, 5, 6, 20], like

$$\begin{aligned} x_{j+n}&=\frac{C}{x_jx_{j+1}\cdots x_{j+n-1}}, \quad C>0, \quad (n+1)\text{-periodic},\\ x_{j+n}&=\frac{x_jx_{j+2}\cdots x_{j+n-1}}{x_{j+1}x_{j+3}\cdots x_{j+n-2}}, \quad n \text{ odd}, \quad (n+1)\text{-periodic},\\ x_{j+3}&=x_j\left(\frac{x_{j+2}}{x_{j+1}}\right)^{\phi}, \text{ where } \phi^2=\phi+1, \quad 5\text{-periodic}, \end{aligned}$$

which are trivially linearizable, as can be seen by using the function $\varphi(x)=\ln(x)$.

Moreover it is easy to prove that many of the well-known periodic recurrences like the Lyness or the max-type ones, see [3, 4, 9, 10] also satisfy properties **P1** and **P2**, although they are not trivially linearizable, see again [10]. The same happens with the Coxeter periodic recurrences, see [11, 14]. Nevertheless we give a counterexample in the proof of Theorem 1.

Proof of Theorem 1. To show that **P1** is not a necessary condition of global periodicity we will give a second order, 3-periodic recurrence. Notice that since for order 2 recurrences both properties are equivalent, this example will be also a counterexample for property **P2**.

Consider $F = \Phi \circ L \circ \Phi^{-1}$, where

$$\Phi(x, y) = (x + g(y), y + g(-x - y))$$

with $g(z) = -z - z^2 - z^3$ and $L(x, y) = (y, -x - y)$. That F is a second order, 3-periodic recurrence follows from Lemmas 7 and 8 applied to $d(x) = x$, $h(y) = -y - y^2 - y^3$ and $\alpha = (-1 + i\sqrt{3})/2$. So we can put $F(x, y) = (y, f(x, y))$. Since F is 3-periodic we get that $F^{-1} = (f(x, y), x)$. On the other hand we have that $(\sigma \circ F \circ \sigma)(x, y) = (f(y, x), x)$. Then the equality $\sigma \circ F \circ \sigma = F^{-1}$ is equivalent to $(\sigma \circ F)^2 = \mathrm{Id}$ and also to $f(x, y) = f(y, x)$. Computing the Taylor series of f we obtain

$$\begin{aligned} f(x, y) = \;& -y - x - \tfrac{2}{3}x^2 - \tfrac{2}{3}xy - \tfrac{2}{3}y^2 - \tfrac{4}{9}x^3 - \tfrac{5}{9}x^2y - \tfrac{5}{9}xy^2 - \tfrac{4}{9}y^3 \\ & -\tfrac{1}{9}x^4 - \tfrac{2}{27}x^3y + \tfrac{1}{27}x^2y^2 - \tfrac{2}{27}xy^3 - \tfrac{1}{9}y^4 \\ & +\tfrac{8}{243}x^5 + \tfrac{29}{243}x^4y + \tfrac{161}{243}x^3y^2 + \tfrac{229}{243}x^2y^3 + \tfrac{199}{243}xy^4 + \tfrac{76}{243}y^5 + O(6). \end{aligned}$$

Since the degree 5 homogeneous part of the Taylor series of f is non symmetric it follows that $f(x, y) \neq f(y, x)$ and so that $\sigma \circ F$ is not an involution, as we wanted to prove. $\square$

4 Proof of Theorem 2

Our proof of Theorem 2 is based on the following result:

Theorem 12 (See [12, 23])**.** *The origin of*

$$\dot{x} = -y, \quad \dot{y} = V'(x),$$

is a global analytic isochronous center if and only if there exists an analytic involution ψ, different from the identity, and satisfying $\psi(0) = 0$, such that

$$V(x) = K(x - \psi(x))^2,$$

for some positive real number K.

Proof of Theorem 2. It is well known that $\mathcal{C}^k$-periodic maps in $\mathbb{R}$ are either the identity or globally $\mathcal{C}^k$-conjugated to $-\mathrm{Id}$, see [17]. Then the way to obtain explicit examples of $\mathcal{C}^\omega$-involutions in $\mathbb{R}$ is to consider maps of the form $\psi(x) = \varphi(-(\varphi^{-1}(x))$, where $\varphi \in \mathcal{E}_\omega$. However note that if φ is odd then $\varphi(-(\varphi^{-1}(x)) = -x$. So to obtain non-trivial global involutions we must choose non odd elements of $\mathcal{E}_\omega$. Consider the map $\varphi(x) = x^3 + rx^2 + sx$, with $\Delta = 4(3s - r^2) > 0$. Since $\varphi'(x) > 0$ for all $x \in \mathbb{R}$ we know that it is an analytic diffeomorphism. By using suitably Cardano's formula (7) to compute φ^{-1} we obtain an explicit expression for this function. After some more computations we get that $\psi(x) = \varphi^{-1}(-\varphi(x))$ is the function (6) given in the statement of the theorem. Finally observe that in the expression $\sqrt[3]{A(x) + \sqrt{A^2(x) + \Delta^3}}$, neither $A(x) + \sqrt{A^2(x) + \Delta^3}$ nor $A^2(x) + \Delta^3$ vanish. So ψ is a global analytic involution. Thus the result follows by Theorem 12. $\square$

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