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On the central configurations of the planar $1+3$ body problem

Montserrat Corbera (montserrat.corbera@uvic.cat)
Departament de Tecnologies Digitals i de la Informació, Universitat de Vic, Laura 13, 08500 Vic, Barcelona, Catalonia, Spain

Josep Maria Cors (cors@epsem.upc.edu)
Matemàtica Aplicada III, Universitat Politècnica de Catalunya, 08242 Manresa, Barcelona, Catalonia, Spain.

Jaume Llibre (jllibre@mat.uab.cat)
Departament de Matemàtiques, Universitat Autònoma de Barcelona, Bellaterra, 08193 Barcelona, Catalonia, Spain.

Abstract. We consider the Newtonian 4 body problem in the plane with a dominat mass M. We study the planar central configurations of this problem when the remaining masses are infinitesimal. We obtain two different classes of central configurations depending on the mutual distances between the infinitesimals masses. Both classes exhibit symmetric and non–symmetric configurations. And when two infinitesimal masses are equal, the number of central configurations varies from five to seven.

Keywords: $1+3$ body problem, central configurations, coorbital satellites

1. Introduction

We study configurations with one massive central mass and several infinitesimal coorbital satellites describing the same circular orbit around the central massive mass. Such configurations are called *relative equilibra*, because in a rotation frame the satellites remain fixed. Recently these configurations have attracted the attention of astronomers. Renner and Sicardy (2004), suggest that the presence of coorbital satellites might explain, at least partly, the confinement of Neptune's ring arcs.

When the configuration of the coorbital satellites changes its size, but keep the shape, the motion is called *homographic*. In that case each satellite describe a Keplerian orbit around the central mass.

A configuration that allows relative equilibria and homographic motions is called a *central configuration* and equals to configurations such that the total Newtonian acceleration of every mass is equal to a constant multiplied by the position vector of this mass with respect to the center mass of the configuration, see (Lee and Santoprete, 2009) and the references quoted there.

The central configurations in the case of one large mass and n infinitesimal arbitrary masses are called *central configurations of the* $1+n$ *body problem*.

Maxwell (Maxwell, 1859) studied the case of n equal masses orbiting Saturn at a common radius and uniformly distributed about a cercle of this radius. He concluded that, for large n, the ring is stable if a convenient inequality between the mass of the ring and the mass of Saturn is satisfied. More recently, Moeckel (Moeckel, 1994) studied the linear stability of the N–body problem when the motion is a rotation about the center of mass and under the condition that all the masses except one become vanishingly small. Notice that the $1+n$ body problem is a particular case. Moeckel shows that the $1+n$–gon (when the small equal masses are uniformly distributed) is stable if and only if $n \geq 7$. For $n \leq 6$, he gives some exemples of the stable configurations where the small equal bodies are not uniformly distributed. Additionally Roberts (Roberts, 2000) carries the analysis a step further showing that the large mass has to grow proportionally to n^3 to ensure the linear stability. Such a criteria agrees with those given by Scheeres and Vinh (1991).

For large values of n, Hall (1988) shows that if $n \geq e^{27,000}$, then there is a unique class of central configurations, the regular $1+n$–gon. In (Casasayas, Llibre and Nunes, 1994) the same result is proved under the assumption that $n \geq e^{73}$.

When n is small and the small masses are equal, in (Cors, Llibre and Ollé, 2004) the authors obtain numerically that the $1+n$–gon is the only configuration when $n \geq 9$. In the case $n=4$ they proved that there are only three symmetric central configurations. Recently Albouy and Fu (2009) proved that any central configuration of the $1+4$ body problem must be symmetric.

In (Renner and Sicardy, 2004) the authors removed the condition that the infinitesimal masses are identical and obtained results about the inverse problem, that is, given a configuration of the coorbital satellites, find the infinitesimal masses making it a central configuration. They also studied the linear stability.

In this paper we study the planar central configurations of the $1+3$ body problem without collision between two infinitesimal satellites when two infinitesimal masses are equal. The characterization of these central configurations is given in Theorem 5.1.

2. Definitions and equations

In this section we present the equations of the central configurations of the $1+n$ body problem. More details can be found in (Casasayas, Llibre and Nunes, 1994) and (Cors, Llibre and Ollé, 2004).

Consider N particles of masses m_1,...,m_N in $\mathbb{R}^2$ subject to their mutual Newtonian gravitational attraction. In an inertial reference frame with the origin at the center of mass of the N bodies and choosing suitable units, the equations of motion of the N body problem in $\mathbb{R}^2$ are

$$Mq'' = -V_q, \tag{1}$$

where M is the mass matrix $M = \text{diag}(m_1, m_1, ..., m_N, m_N)$, $q = (q_1, ..., q_N)$ is the position vector with $q_i \in \mathbb{R}^2$, V the potential vector

$$V(q_1, ..., q_N) = \sum_{1\leq i<j\leq N} \frac{m_i m_j}{\|q_i - q_j\|},$$

and $V_q = (\partial V/\partial q_1, ..., \partial V/\partial q_N)$. Note that the Newtonian gravitational constant has been taken equal to one choosing conveniently the unit of time. Taking into account the singularities of equation (1), the configuration space of the planar N body problem associated with the mass matrix M is

$$\mathcal{M}(m_1, ..., m_N) = \{q \in \mathbb{R}^{2N} : \sum_{i=1}^{N} q_i = 0,\ q_i \neq q_j, \quad \mathit{for} \quad i \neq j\}.$$

Given a matrix M, we say that $q \in \mathcal{M}$ represents a *central configuration* of the associated planar N body problem if there exists a positive constant λ^2 such that

$$M^{-1}V_q = \lambda^2 q,$$

i.e., if the acceleration vector of every particle is directed towards the center of mass and its modulus is proportional to the distance from the particle to the center of mass. We shall denote by $\mathcal{C}$ the set of planar central configurations associated with a given mass matrix M. Notice that $\mathcal{C}$ is invariant with respect to homothetic transformations and rotations of $\mathbb{R}^2$. We shall denote by $\tilde{\mathcal{C}}$ the set of planar central configurations modulus the group $SO(2)$ of plane rotations.

Now we concentrate our interest on the central configurations of the planar $1+n$ body problem with infinitesimal unequal masses. That is, we consider $N = 1+n$ and let $q(\varepsilon) = (q_0(\varepsilon), q_1(\varepsilon), ..., q_n(\varepsilon)) \in \tilde{\mathcal{C}}$ be a central configuration of the planar $1+n$ body problem with masses $m_0 = 1$, $m_i = \mu_i\varepsilon$, $i = 1, ..., n$, which depends continuously on ε. We say that $q = (q_0, q_1, ..., q_N)$ is a *central configuration of the planar* $1+n$ *body problem* if there exists $\lim_{\varepsilon\to 0} q(\varepsilon)$ and this limit is equal to q.

The following two results can be found in the unpublished paper of Hall (1988) and in (Casasayas, Llibre and Nunes, 1994).

PROPOSITION 2.1. *All the central configurations of the planar* $1+n$ *body problem lie on a circle centered at* $q_0 = 0$.

Let $q = (q_0, ..., q_n)$ be a central configuration of the planar $1+n$ body problem. We say that q is *non–collision* if $q_i \neq q_j$ for $i \neq j$. That is, we exclude the possibility that the distance between two or more masses tends to zero as $\varepsilon \to 0$.

PROPOSITION 2.2. *Let* $q = (q_0, ..., q_n)$ *be a non–collision central configuration of the planar* $1+n$ *body problem. Denoting by* ϑ_i *the angle defined by the position of* q_i *for* $i = 1, ..., n.$*, we have*

$$\sum_{j=1, j\neq i}^{n} \mu_j \sin(\vartheta_j - \vartheta_i) \left(1 - \frac{1}{2\sqrt{2}\sqrt{(1-\cos(\vartheta_j - \vartheta_i))^3}}\right) = 0, \qquad (2)$$

for $i = 1, ..., n$.

We call any solution $(\vartheta_1, \cdots, \vartheta_n)$ of system (2) a *coorbital central configuration of the planar* $1+n$ *body problem.* Since we are interested in central configurations modulus rotations and homothetic transformations, without loss of generality we can assume that the circle has radius 1 and that $\vartheta_1 = 0$. Then the configuration space of the coorbital central configurations of the planar $1+n$ body problem is

$$\mathcal{A} = \{(\vartheta_1, \cdots, \vartheta_n) \in [0, 2\pi)^n \,:\, \vartheta_1 = 0, \, \vartheta_i \neq \vartheta_j \quad \text{for} \quad i \neq j\}.$$

The set $\mathcal{A}$ has $(n-1)!$ connected components, each of which corresponds to a given permutation of the n satellites on the circle. We shall consider one connected component of $\mathcal{A}$, i.e. a given counterclockwise ordering of the n satellites. Without loss of generality we can assume that $0 = \vartheta_1 < \vartheta_2 < \cdots < \vartheta_n < 2\pi$. In this case it is natural to take as coordinates the angles between two consecutive satellites

$$\theta_i = \vartheta_{i+1} - \vartheta_i, \qquad i = 1, ..., n-1.$$

Also it is convenient to work with an nth redundant coordinate angle

$$\theta_n = 2\pi - \sum_{i=1}^{n-1} \theta_i,$$

which measures the angular distance between particle n and particle 1. In this way the configuration space will be the simplex S given by the intersection of the hyperplane $\sum_{i=1}^{n}\theta_i = 2\pi$ and the space $\{(\theta_1, ..., \theta_n) : \theta_i > 0\}$.

In the coordinates $(\theta_1, \cdots, \theta_n)$ system (2) becomes

$$\begin{aligned} \mu_2 f(\theta_1) + \mu_3 f(\theta_1+\theta_2) + \cdots + \mu_n f(\theta_1 + ... + \theta_{n-1}) &= 0, \\ \mu_3 f(\theta_2) + \mu_4 f(\theta_2+\theta_3) + \cdots + \mu_1 f(\theta_2 + ... + \theta_n) &= 0, \\ \mu_4 f(\theta_3) + \mu_5 f(\theta_3+\theta_4) + \cdots + \mu_2 f(\theta_3 + ... + \theta_n + \theta_1) &= 0, \\ \cdots & \\ \mu_1 f(\theta_n) + \mu_2 f(\theta_n+\theta_1) + \cdots + \mu_{n-1} f(\theta_n + \theta_1 + \cdots + \theta_{n-2}) &= 0, \\ \theta_1 + \cdots + \theta_n &= 2\pi, \end{aligned} \tag{3}$$

where the function $f(\theta) = \sin\theta \left(1 - \dfrac{1}{2\sqrt{2}\sqrt{(1-\cos\theta)^3}}\right)$.

In the case of three satellites system (3) is

$$\begin{aligned} \mu_2 f(\theta_1) + \mu_3 f(\theta_1+\theta_2) &= 0, \\ \mu_3 f(\theta_2) + \mu_1 f(\theta_2+\theta_3) &= 0, \\ \mu_1 f(\theta_3) + \mu_2 f(\theta_3+\theta_1) &= 0, \\ \theta_1+\theta_2+\theta_3 &= 2\pi, \end{aligned}$$

or equivalently

$$\begin{aligned} \mu_2 f(\theta_1) - \mu_3 f(\theta_3) &= 0, \\ \mu_3 f(\theta_2) - \mu_1 f(\theta_1) &= 0, \\ \mu_1 f(\theta_3) - \mu_2 f(\theta_2) &= 0, \\ \theta_1+\theta_2+\theta_3 &= 2\pi, \end{aligned} \tag{4}$$

because $f(2\pi-\theta) = -f(\theta)$.

Notice that the first three equations in (4) are linearly dependent. So one equation is irrelevant. We also chose the unit of mass so that $\mu_3 = 1$. Under these assumptions equations (4) are equivalent to the system

$$\begin{aligned} \mu_1 f(\theta_1) - f(\theta_2) &= 0, \\ \mu_2 f(\theta_1) - f(\theta_3) &= 0, \\ \theta_1+\theta_2+\theta_3 &= 2\pi. \end{aligned} \tag{5}$$

Next results allows us to classify the coorbital central configurations of the $1+3$ body problem in two main classes, as we shall see latter on.

PROPOSITION 2.3. *Let* $(\theta_1, \theta_2, \theta_3)$ *be a coorbital central configuration solution of system (5).*

(i) If $f(\theta_i) = 0$ for some $i = 1, 2, 3$, then $\mu_j = 0$ for some $j = 1, 2$.

(ii) The sign of $f(\theta_i)$ must be positive or negative for all $i = 1, 2, 3$.

Proof. Supose that $f(\theta_1) = 0$, then from system (5) $f(\theta_2) = f(\theta_3) = 0$. The zeros of the function f are $\pi/3$, π and $5\pi/3$. Clearly the sum of any choice of these three values never equals to 2π.

Consider now $f(\theta_2) = 0$. Then either $\mu_1 = 0$ or $f(\theta_1) = 0$ (previous case). Finally if $f(\theta_3) = 0$ implies, using the same argument, that $\mu_2 = 0$. So statement (i) follows

On the other hand, supose that $f(\theta_1) < 0$, then from system (5) clearly $f(\theta_2) < 0$ and $f(\theta_3) < 0$, as well. The other cases run in a similar way. □

Proposition 2.3 says that any coorbital central configuration of the $1 + 3$ body problem cannot posses an angle θ_i equal to $\pi/3$, π or $5\pi/3$ for $i = 1, 2, 3$.

COROLLARY 2.4. *Let $(\theta_1, \theta_2, \theta_3)$ be a coorbital central configuration solution of system (5). Then either $\theta_i \in (\pi/3, \pi)$ for all $i = 1, 2, 3$, or two angles are less than $\pi/3$ and the third belongs to the interval $(4\pi/3, 5\pi/3)$.*

Proof. This follows from the Proposition 2.3 and the plot of the function f. □

In short, the $1 + 3$ body problem have two unconnected classes of coorbital central configurations. We call coorbital central configuration of class A the one having two angles less than $\pi/3$, and coorbital central configuration of class B the one having the three angles greater than $\pi/3$.

3. Symmetric central configurations

A coorbital central configuration $(\theta_1, \theta_2, \ldots, \theta_n)$ of the planar $1 + n$ body problem is *symmetric with respect to a straight line L containing the central body*, if modulus a cyclic permutation of the angles we have, when n is even either

$$\theta_1 = \theta_n, \quad \theta_2 = \theta_{n-1}, \quad \ldots, \quad \theta_{\frac{n}{2}} = \theta_{\frac{n+2}{2}},$$

(in this case the symmetry axis L contains two satellites), or

$$\theta_1 = \theta_{n-1}, \quad \theta_2 = \theta_{n-2}, \quad \ldots, \quad \theta_{\frac{n-2}{2}} = \theta_{\frac{n+2}{2}}, \quad \theta_{\frac{n}{2}}, \quad \theta_n,$$

(in this case the symmetry axis L contains no satellites) and when n is odd

$$\theta_1 = \theta_n, \quad \theta_2 = \theta_{n-1}, \quad \dots, \quad \theta_{\frac{n-1}{2}} = \theta_{\frac{n+3}{2}},$$

(in this case the symmetry axis L contains one satellite).

From the definition of symmetric coorbital central configuration in the case of three satellites the configuration is symmetric if and only if two of the angles $(\theta_1, \theta_2, \theta_3)$ are equal.

PROPOSITION 3.1. *If the coorbital central configuration of the* $1+3$ *body problem is symmetric, with respect to a straight line* L *containing the central body, then* $\mu_i = \mu_j$ *for some* $i, j = 1, 2, 3$ *with* $i \neq j$

Proof. Suppose that $\theta_2 = \theta_3$ then from system (5) we have that $\mu_1 f(\theta_1) = \mu_2 f(\theta_1)$ and so $\mu_1 = \mu_2$. Similar arguments can be used to prove cases $\theta_1 = \theta_2$ or $\theta_1 = \theta_3$. □

Without loss of generality, we assume that $\mu_1 = \mu_2 = \mu$. So system (5) is equivalent to system

$$\begin{aligned} \mu f(\theta_1) - f(\theta_2) &= 0, \\ \mu f(\theta_1) - f(\theta_3) &= 0, \\ \theta_1 + \theta_2 + \theta_3 &= 2\pi. \end{aligned} \tag{6}$$

By setting $\alpha = \theta/2$ we see that $f(\theta)$ can be written as

$$g(\alpha) = \frac{\cos\alpha}{4\sin^2\alpha}(8\sin^3\alpha - 1).$$

Let $\alpha_1 = \theta_1/2$ and $\alpha_2 = \theta_2/2$ and $\alpha_3 = \theta_3/2 = \pi - \alpha_1 - \alpha_2$. By writing $g(\alpha_3)$ in terms of α_1 and α_2 we get

$$g(\alpha_3) = \frac{\cos(\alpha_1 + \alpha_2)(1 - 8(\sin(\alpha_1 + \alpha_2))^3)}{4(\sin(\alpha_1 + \alpha_2))^2},$$

where $\alpha_1 \in (0, \pi)$ and $\alpha_2 \in (0, \pi)$. Since $\alpha_1 + \alpha_2 \neq 0$ and $\alpha_1 + \alpha_2 \neq \pi$ (notice that if $\alpha_1 + \alpha_2 = \pi$ then $\alpha_3 = 0$ which is not possible), the denominator of $g(\alpha_3)$ is different from zero. Let $e_1(\alpha_1, \alpha_2; \mu) = \mu g(\alpha_1) - g(\alpha_2)$ and $e_2(\alpha_1, \alpha_3; \mu) = \mu g(\alpha_1) - g(\alpha_3)$. The system (6) becomes

$$\begin{aligned} e_1(\alpha_1, \alpha_2; \mu) = 0, \\ e_2(\alpha_1, \alpha_3; \mu) = 0, \end{aligned} \tag{7}$$

where $\alpha_1, \alpha_2 \in (0, \pi)$ and $\alpha_3 = \pi - \alpha_1 - \alpha_2 \in (0, \pi)$.

Under the assumption $\mu_1 = \mu_2 = \mu$, the symmetric coorbital central configurations satisfy $\theta_2 = \theta_3$, or equivalently $\alpha_2 = \alpha_3$. Analyzing the solutions of (7) with $\alpha_2 = \alpha_3$ depending on the values of μ we obtain the following result.

PROPOSITION 3.2. *Let $u_1 = 0.89616\ldots$. The $1+3$ body problem has the following symmetric coorbital central configurations:*

(a) One symmetric coorbital central configuration A^s belonging to class A for $\mu \in (0, u_1)$.

(b) Two symmetric coorbital central configurations for $\mu = u_1$, one belonging to class A (A^s) and one belonging to class B ($B^s_{1,2}$).

(c) Three symmetric coorbital central configurations for $\mu > u_1$, one belonging to class A (A^s) and two belonging to class B (B^s_1, B^s_2). The two symmetric coorbital central configurations of class B coincide at $\mu = u_1$.

Proof. The proof is structured as follows. First, by using appropriate variables, we transform system (7) with $\alpha_2 = \alpha_3$ into a system of polynomial equations. Then by using resultant theory we obtain all the values of μ at which the number of solutions of that system changes, and we count the number of solutions of the system for the different values of μ. Finally we analyze the solution of the system in function of μ.

The assumption $\alpha_2 = \alpha_3$ implies that $\alpha_1 = \pi - 2\alpha_2$, so $\alpha_2 \in (0, \pi/2)$. Since $\cos\alpha_1 = \sin^2\alpha_2 - \cos^2\alpha_2$ and $\sin\alpha_1 = 2\sin\alpha_2\cos\alpha_2$, system (7) can be written in terms of $c = \cos\alpha_2$ and $s = \sin\alpha_2$. Then eliminating the denominators we get the following system equivalent to system (7)

$$-s^2\mathcal{E}_1 = 0, \qquad -s^2\left(c^2 + s^2\right)\mathcal{E}_2 = 0, \qquad F = 0, \tag{8}$$

where

$$\begin{aligned}
\mathcal{E}_1 &= 32s^3c^3 - 4c^3 + \left(64s^3c^5 - 64s^5c^3 - c^2 + s^2\right)\mu,\\
\mathcal{E}_2 &= 32s^3c^9 + 96s^5c^7 + 96s^7c^5 + 32s^9c^3 - 4c^3\\
&\quad + \left(64s^3c^7 - c^4 - 64s^7c^3 + s^4\right)\mu,\\
F &= c^2 + s^2 - 1,
\end{aligned}$$

and $0 < s \leqslant 1$ and $0 \leqslant c < 1$ because $\alpha_2 \in (0, \pi/2)$. Clearly $c^2 + s^2 \neq 0$ and $s \neq 0$. Therefore system (8) can be written as

$$\mathcal{E}_1 = 0, \qquad \mathcal{E}_2 = 0, \qquad F = 0. \tag{9}$$

We perform the following substitutions: $c^2 = 1 - s^2$, $c^3 = c(1 - s^2)$, $c^4 = (1 - s^2)^2$, $c^5 = c(1 - s^2)^2$, $c^7 = c(1 - s^2)^3$, and $c^9 = c(1 - s^2)^4$. After these substitutions $\mathcal{E}_1 = \mathcal{E}_2$ and system (9) becomes $E = 0$, $F = 0$ where

$$E = -32cs^5 + 32cs^3 + 4cs^2 - 4c + \left(128cs^7 - 192cs^5 + 64cs^3 + 2s^2 - 1\right)\mu.$$

From now on instead of writing the system $E = 0$, $F = 0$, we also shall write $E = F = 0$.

Using the resultant theory we can eliminate the variable c in the equations $E = F = 0$. The resultant of the polynomials E and F with respect to the variable c is the polynomial

$$\begin{aligned} R \;=\; & \Big(16384s^{16} - 65536s^{14} + 102400s^{12} - 77824s^{10} + 28672s^{8} \\ & -4096s^{6} + 4s^{4} - 4s^{2} + 1\Big)\mu^{2} + \Big(-8192s^{14} + 28672s^{12} \\ & +1024s^{11} - 36864s^{10} - 3584s^{9} + 20480s^{8} + 4608s^{7} - 4096s^{6} \\ & -2560s^{5} + 512s^{3}\Big)\mu + 1024s^{12} - 3072s^{10} - 256s^{9} + 3072s^{8} \\ & +768s^{7} - 1008s^{6} - 768s^{5} - 48s^{4} + 256s^{3} + 48s^{2} - 16, \end{aligned}$$

in the variables s and μ, and R satisfies the following property: if $(c, s) = (c^*(\mu), s^*(\mu))$ is a common root of the polynomials E and F, then $s^*(\mu)$ is a root of the polynomial R (see for instance (Lang, 1993) and (Olver, 1999) for more details on the resultant theory). Therefore the solutions of equation $R = 0$ give at least all the values $s(\mu)$ of the common solutions $(c(\mu), s(\mu))$ of $E = F = 0$. Notice that equation $R = 0$ could also give solutions $s(\mu)$ that do not provide common solutions of $E = F = 0$.

We are interested in the set of values of μ at which the number of real solutions $(c(\mu), s(\mu))$ of system $E = F = 0$ changes. Since equation $R = 0$ provides all the values $s(\mu)$ of the common solutions $(c(\mu), s(\mu))$ of $E = F = 0$, the set of the values of μ at which the number of real solutions of $R = 0$ changes contains at least all the values of μ at which the number of real solutions of $E = F = 0$ changes.

The number of real solutions of $R = 0$ can change only at the values of μ that provide solutions with multiplicity greater than 1, that is, at the values of μ where $R = dR/ds = 0$.

Again using resultant theory we can eliminate the variable s in the system $R = dR/ds = 0$. The resultant of the polynomials R and dR/ds with respect to s is the polynomial

$$H = k_0\mu^{34}(\mu - 4)(\mu + 4)H_0,$$

where $k_0 \neq 0$ is a large constant and H_0 is the following polynomial of degree 22

$$\begin{aligned} H_0 \;=\; & 5444002207736327208896\mu^{22} - 1149183543456203518771 2\mu^{20} \\ & +2898856085837444554752\mu^{19} + 4510139049847521493401 6\mu^{18} \\ & -5020717866207523913318 4\mu^{17} - 3597828247551090484224\mu^{16} \\ & +5444125268464780436275 2\mu^{15} - 9033026688160114998067 2\mu^{14} \end{aligned}$$

$$
\begin{aligned}
&+1050294675602269022008 32\mu^{13} - 870604829841626661611 52\mu^{12} \\
&+531926850612307293158 40\mu^{11} - 256080234645181810472 60\mu^{10} \\
&+924990674790352099161 6\mu^{9} - 177185206222332959386 8\mu^{8} \\
&-159500845521348320736\mu^{7} + 119990316536124348321\mu^{6} \\
&+12123699015371647968\mu^{5} - 11384652544065195120\mu^{4} \\
&+400296455761238280\mu^{3} + 345472406710968960\mu^{2} \\
&+2678966190976320\mu - 8438743501575408.
\end{aligned}
$$

By the properties of the resultant, the solutions of equation $H = 0$ provide at least all the values of μ of the common solutions (s, μ) of system $R = dR/ds = 0$, although they also can provide values of μ that do not correspond to common solutions of the system. Therefore the real solutions of $H = 0$ with $\mu > 0$ provide all the possible values of μ at which the number of real solutions of system $R = 0$ changes.

We compute all the solutions of the polynomial equation $H_0 = 0$ with the help of Mathematica and we find 22 different solutions of which only 3 are real solutions with $\mu > 0$

$$
\begin{aligned}
\mu &= u_1 = 0.8961616399532140\ldots \\
\mu &= u_2 = 1.2506375233635711\ldots \\
\mu &= u_4 = 4.0000142922001194\ldots
\end{aligned}
$$

On the other hand, equation $H = 0$ has the additional real solution with $\mu > 0$

$$\mu = u_3 = 4.$$

In short there are four possible values of μ at which the number of real solutions of $E = F = 0$ could change, $\mu = u_1, u_2, u_3, u_4$.

Since we only are interested in studying the solutions of $E = F = 0$ satisfying $0 < s \leqslant 1$ and $0 \leqslant c < 1$ we need to control not only the changes in the number of real solutions of the system $E = F = 0$ without constraints. We also need to control when a real solution $(s(\mu), c(\mu))$ starts to satisfy the conditions $0 < s \leqslant 1$ and $0 \leqslant c < 1$ for some values of μ.

Next we find values of μ where the number of real solutions of system $E = F = 0$ does not change but the number of real solutions satisfying $0 < s \leqslant 1$ and $0 \leqslant c < 1$ can change. For doing that, since all the solutions $s(\mu)$ of $E = F = 0$ are solutions of $R = 0$, we find the values of μ at which either $s = 0$ or $s = 1$ is a solution of $R = 0$. Evaluating $R = 0$ at $s = 0$ and $s = 1$ we get

$$
\begin{aligned}
R|_{s=0} &= \mu^2 - 16 = 0, \\
R|_{s=1} &= \mu^2.
\end{aligned}
$$

Therefore the unique values of μ at which the number of real solutions of system $E = F = 0$ satisfying $0 < s \leqslant 1$ and $0 \leqslant c < 1$ can change are $\mu = u_1, u_2, u_3, u_4$.

Finally we compute the number of real solutions of system $E = F = 0$ satisfying $0 < s \leqslant 1$ and $0 \leqslant c < 1$ at values $\mu = \{v_1, v_2, v_3, v_4, v_5\}$ with $0 < v_1 < u_1 < v_2 < u_2 < v_3 < u_3 < v_4 < u_4 < v_5$ in the following way. Let $\mathrm{Res}\,[P, Q, X]$ denote the resultant of the polynomials P and Q with respect to X. Fixed $\mu = v_i$ we compute the resultants $\mathrm{Res}\,[E, F, s]$ and $\mathrm{Res}\,[E, F, c]$, which are polynomials of the single variable c and s respectively. Using the properties of resultants, if (c_0, s_0) is a solution of system $E = F = 0$, then $\mathrm{Res}\,[E, F, s](c_0) = 0$ and $\mathrm{Res}\,[E, F, c](s_0) = 0$. We compute with the help of Mathematica all the solutions of the polynomial equations $\mathrm{Res}\,[E, F, s] = 0$ and $\mathrm{Res}\,[E, F, c] = 0$. Finally we check which pairs (c, s) formed by a real solution c of $\mathrm{Res}\,[E, F, s] = 0$ with $0 \leqslant c < 1$ and a real solution s of $\mathrm{Res}\,[E, F, c] = 0$ with $0 < s \leqslant 1$ provide a solution of system $E = F = 0$. Using this procedure for each v_i, $i = 1, \ldots, 5$ we get one solution for $\mu = v_1$ and three solutions for $\mu = \{v_2, v_3, v_4, v_5\}$, so the number of real solutions of system $E = F = 0$ with $0 < s \leqslant 1$ and $0 \leqslant c < 1$ changes at $\mu = u_1$.

Analyzing the solutions of system $E = F = 0$ as a function of μ we observe that the solution $(s(\mu), c(\mu))$ that we have obtained for $0 < \mu < u_1$ persists for all μ and it gives a family of symmetric coorbital central configurations of class A, denoted by A^s. At $\mu = u_1$ appears a new symmetric coorbital central configuration of class B, denoted by B^s which bifurcates in two families of symmetric coorbital central configurations, denoted by B_1^s and B_2^s for $\mu > u_1$. See Figures 1 and 2 where the angles $\theta_1(\mu)$ and $\theta_2(\mu)$ have been computed. Also see Figures 3 and 4 where qualitative pictures of coorbital central configurations have been plotted for different values of μ. □

4. Non–symmetric central configurations

In Section 3 we have analyzed the symmetric coorbital central configurations of the 1 + 3 body problem and we have found the value of μ at which the number of symmetric coorbital central configurations changes. Here we analyze the values μ where the number of coorbital central configurations change independently if such configurations are symmetric or not. In particular we prove the following result.

PROPOSITION 4.1. *The following statements hold.*

(a) At $\mu = 1.4238513\ldots$, the symmetric coorbital central configuration B_1^s given in Proposition 3.2, bifurcates to two non–symmetric

coorbital central configurations (B_1, B_2), which persist for all $\mu \in (0, 1.4238513\ldots)$.

(b) From the non–symmetric coorbital central configurations of the $1+3$ *body problem do not bifurcate any non–symmetric coorbital central configurations.*

Proof. System (7) can be written in terms of $c_1 = \cos\alpha_1$ and $s_1 = \sin\alpha_1$, $c_2 = \cos\alpha_2$ and $s_2 = \sin\alpha_2$, then by eliminating the denominators and by doing the substitutions

$$c_1^2 = 1 - s_1^2,\ c_1^3 = c_1(1-s_1^2),\ c_1^4 = (1-s_1^2)^2,$$
$$c_2^2 = 1 - s_2^2,\ c_2^3 = c_1(1-s_2^2),\ c_2^4 = (1-s_2^2)^2,$$

system (7) is equivalent to system $E_1 = E_2 = F_1 = F_2 = 0$ where

$$\begin{aligned}
E_1 &= -8c_2s_1^2s_2^3 + c_2s_1^2 + \left(8c_1s_1^3s_2^2 - c_1s_2^2\right)\mu,\\
E_2 &= 64c_2s_2^3s_1^6 - 32c_2s_2s_1^6 + 64c_1s_2^4s_1^5 - 64c_1s_2^2s_1^5 + 8c_1s_1^5 - 64c_2s_2^3s_1^4\\
&\quad +24c_2s_2s_1^4 - 32c_1s_2^4s_1^3 + 24c_1s_2^2s_1^3 + s_2s_1^3 + 8c_2s_2^3s_1^2 - c_1c_2s_1^2\\
&\quad + \Big(-16c_2s_2s_1^6 - 16c_1s_2^2s_1^5 + 8c_1s_1^5 + 16c_2s_2s_1^4 + 8c_1s_2^2s_1^3\\
&\quad +2c_2s_2s_1^3 + 2c_1s_2^2s_1^2 - c_1s_1^2 - 2c_2s_2s_1 - c_1s_2^2\Big)\mu,\\
F_1 &= c_1^2 + s_1^2 - 1,\\
F_2 &= c_2^2 + s_2^2 - 1.
\end{aligned}$$

We are interested in solutions of this system with $0 < s_1 \leqslant 1$, $0 < s_2 \leqslant 1$, $-1 < c_1 < 1$, $-1 < c_2 < 1$, and such that $\alpha_3 = \pi - \alpha_1 - \alpha_2 \in (0, \pi)$, here $\alpha_i = \arctan(s_i/c_i)$ when $s_i/c_i > 0$ and $\alpha_i = \arctan(s_i/c_i) + \pi$ when $s_i/c_i < 0$, $i = 1, 2$. Next we will find the values of μ at which the number of real solutions of system $E_1 = E_2 = F_1 = F_2 = 0$ satisfying those conditions changes. First we find the values of μ at which the number of real solutions of $E_1 = E_2 = F_1 = F_2 = 0$ changes.

We eliminate the variable c_1 by doing the resultants

$$R_1 = \mathrm{Res}\,[E_1, E_2, c_1],\ R_2 = \mathrm{Res}\,[E_1, F_1, c_1],\ R_3 = \mathrm{Res}\,[E_1, F_2, c_1].$$

Next we eliminate the variable c_2 by doing the resultants

$$S_1 = \mathrm{Res}\,[R_3, R_1, c_2],\ S_2 = \mathrm{Res}\,[R_3, R_2, c_2].$$

The factorization of the polynomials S_1 and S_2 contains factors with multiplicity greater than 1. We define two new polynomials $\overline{S}_1$ and $\overline{S}_2$ as the polynomials that contain exactly the same factors than S_1 and

S_2 but all of them with multiplicity one. Then we eliminate the variable s_2 by doing the resultant

$$T = \text{Res}\,[\overline{S}_1, \overline{S}_2, s_2].$$

The factorization of T is formed by a large constant, a power of μ and five factors in the variables s_1 and μ. We eliminate the constant, the power of μ (we are only interested in solutions with $\mu > 0$) and the multiplicities of the factors and we obtain a new polynomial

$$\overline{T} = \overline{T}_1 \cdot \overline{T}_2 \cdot \overline{T}_3 \cdot \overline{T}_4 \cdot \overline{T}_5,$$

where

$$\begin{aligned}
\overline{T}_1 &= s_1, \\
\overline{T}_2 &= 2s_1 - 1, \\
\overline{T}_3 &= 4s_1^2 + 2s_1 + 1, \\
\overline{T}_4 &= \Big(4096s_1^{16} - 8192s_1^{14} - 2048s_1^{13} + 4096s_1^{12} + 4096s_1^{11} + 384s_1^{10} \\
&\quad -2048s_1^9 - 768s_1^8 - 32s_1^7 + 384s_1^6 + 64s_1^5 + s_1^4 - 32s_1^3 - 2s_1^2 \\
&\quad +1\Big)\,\mu^4 + \Big(2048s_1^{14} - 2048s_1^{12} - 512s_1^{11} - 768s_1^{10} + 512s_1^9 \\
&\quad +1824s_1^8 + 192s_1^7 - 1056s_1^6 - 448s_1^5 - 12s_1^4 + 256s_1^3 + 28s_1^2 \\
&\quad -16\Big)\,\mu^2 + \Big(256s_1^{10} - 1280s_1^8 - 32s_1^7 + 1024s_1^6 + 160s_1^5 \\
&\quad -128s_1^3\Big)\,\mu + 256s_1^{12} + 192s_1^8 - 252s_1^6,
\end{aligned}$$

and $\overline{T}_5$ is a polynomial depending on s_1 and μ of degree 44 in the variable s_1.

By the properties of resultants, the new equation $\overline{T} = 0$ contains all the values $s_1(\mu)$ of the solutions $(c_1(\mu), s_1(\mu), c_2(\mu), s_2(\mu))$ of the system $E_1 = E_2 = F_1 = F_2 = 0$ and probably values $s_1(\mu)$ that do not provide solutions of the initial system. Therefore the values of μ at which the number of real solutions of $E_1 = E_2 = F_1 = F_2 = 0$ changes are also values of μ at which the number of real solutions of $T = 0$ changes.

The number of real solutions of $\overline{T} = 0$ can change either at the values of μ at which a real root of a factor $\overline{T}_i$ has multiplicity greater than one, or at the values of μ at which a real root of a factor $\overline{T}_i$ is also a root of a factor $\overline{T}_j$ with $i \neq j$.

Clearly the root of the factor $\overline{T}_1$, $s_1 = 0$ is not a root of the factors $\overline{T}_2$ and $\overline{T}_3$. Moreover

$$\overline{T}_4|_{s_1=0} = \mu^4 - 16\mu^2, \qquad \overline{T}_5|_{s_1=0} = -\mu^{10} - 4\mu^8.$$

So $\mu = 4$ is the unique value of $\mu > 0$ at which the root of the factor $\overline{T}_1$ is a root of some of the other factors. Proceeding in a similar way for the root of the factor $\overline{T}_2$, $s_1 = 1/2$, we get

$$\overline{T}_3|_{s_1=1/2} = 3, \qquad \overline{T}_4|_{s_1=1/2} = -25/8, \qquad \overline{T}_5|_{s_1=1/2} = 2197/4096,$$

so $s_1 = 1/2$ is only root of the factor $\overline{T}_2$. The roots of factor $\overline{T}_3$ are non–real, therefore we do not consider them.

Next we compute at which values of $\mu > 0$ there are real roots of $\overline{T}_4$ that are roots of $\overline{T}_5$ by solving the polynomial equation $\text{Res}\,[\overline{T}_4, \overline{T}_5, s_1] = 0$ and we get

$$\begin{aligned}
\mu &= \widetilde{u}_1 = 0.1325450004033988\ldots\\
\mu &= \widetilde{u}_3 = 0.1541643785502070\ldots\\
\mu &= \widetilde{u}_6 = 0.5250057176263389\ldots\\
\mu &= \widetilde{u}_8 = 0.6784610926616778\ldots\\
\mu &= \widetilde{u}_{13} = 1.0011322554499964\ldots\\
\mu &= \widetilde{u}_{15} = 1.0076481112559950\ldots\\
\mu &= \widetilde{u}_{16} = 1.0606474258429618\ldots\\
\mu &= \widetilde{u}_{19} = 1.4238513421761745\ldots\\
\mu &= \widetilde{u}_{20} = 2.5123408303563733\ldots\\
\mu &= \widetilde{u}_{21} = 3.2950663210056425\ldots
\end{aligned}$$

Now we find the values of $\mu > 0$ at which the factor $\overline{T}_4$ has real roots with multiplicity greater than ones by solving the polynomial equation $\text{Res}\,[\overline{T}_4, \partial\overline{T}_4/\partial s_1, s_1] = 0$ and we get

$$\begin{aligned}
\mu &= \widetilde{u}_5 = 0.4816163432110476\ldots\\
\mu &= \widetilde{u}_7 = 0.6154472608017034\ldots\\
\mu &= u_1 = \widetilde{u}_{10} = 0.8961616399532140\ldots\\
\mu &= \widetilde{u}_{11} = 0.8977353306077976\ldots\\
\mu &= \widetilde{u}_{17} = 1.2506375233635711\ldots\\
\mu &= \widetilde{u}_{18} = 1.2615567635193134\ldots\\
\mu &= \widetilde{u}_{22} = 4\\
\mu &= \widetilde{u}_{23} = 4.0000142922001194\ldots
\end{aligned}$$

Finally we find the values of $\mu > 0$ at which the factor $\overline{T}_5$ has real roots with multiplicity greater than ones by solving the polynomial equation $\text{Res}\,[\overline{T}_5, \partial\overline{T}_5/\partial s_1, s_1] = 0$ and we get

$$\mu = \widetilde{u}_2 = 0.1424427964946848\ldots$$

$$
\begin{aligned}
\mu &= \widetilde{u}_4 = 0.2055692659990572\ldots \\
\mu &= \widetilde{u}_9 = 0.6791080539220485\ldots \\
\mu &= \widetilde{u}_{12} = 0.9939202640321302\ldots \\
\mu &= \widetilde{u}_{14} = 1.0072825044062513\ldots
\end{aligned}
$$

In short, the possible values of $\mu > 0$ at which the number of real solutions of $\overline{T} = 0$ can change are $\mu = \widetilde{u}_i$ for $i = 1, \ldots, 23$.

We are interested in the values of μ at which the number of real solutions of $E_1 = E_2 = F_1 = F_2 = 0$ with $0 < s_1 \leqslant 1$, $0 < s_2 \leqslant 1$, $-1 < c_1 < 1$, $-1 < c_2 < 1$, and $\alpha_3 = \pi - \alpha_1 - \alpha_2 \in (0, \pi)$. Using the same argument as in Section 3 we need to control not only the changes in the number of real solutions of the system $E_1 = E_2 = F_1 = F_2 = 0$ without constraints. We also need to control when a real solution starts to satisfy the conditions $0 < s_j \leqslant 1$ and $-1 < c_j < 1$, $j = 1, 2$ for some values of μ. Since the set of real solutions $s_1(\mu)$ of system $E_1 = E_2 = F_1 = F_2 = 0$ is contained in the set of real solutions of $\overline{T} = 0$, in order to find these last possible values of μ it is sufficient to find the values μ at which a solution $s_1(\mu)$ of $\overline{T} = 0$ satisfies that $s_1(\mu) = 0$ or $s_1(\mu) = 1$. Notice that $s_1 = 0$ is a solution of $\overline{T} = 0$ but this solution does not depend on μ so we do not consider it. Evaluating $\overline{T}/s_1$ at $s = 0$ and $s = 1$ we get

$$
\begin{aligned}
(\overline{T}/s_1)|_{s_1=0} &= -\left(\mu^4 - 16\mu^2\right)\left(-\mu^{10} - 4\mu^8\right) \\
(\overline{T}/s_1)|_{s_1=1} &= 92747200.
\end{aligned}
$$

So we do not obtain new possible values of μ at which the number of solutions real solutions of the system $E_1 = E_2 = F_1 = F_2 = 0$ with satisfying the desired conditions changes.

Finally we compute the number of real solutions of the system $E_1 = E_2 = F_1 = F_2 = 0$ with $0 < s_1 \leqslant 1$, $0 < s_2 \leqslant 1$, $-1 < c_1 < 1$, $-1 < c_2 < 1$, and $\alpha_3 = \pi - \alpha_1 - \alpha_2 \in (0, \pi)$ at values $\mu = v_i$ with $v_1 \in (0, \widetilde{u}_1)$, $v_{i+1} \in (\widetilde{u}_i, \widetilde{u}_{i+1})$ for $i = 1, \ldots, 23$ and $v_{24} > \widetilde{u}_{23}$ by proceeding of the following way. System (7) can be written in terms of $c_1 = \cos\alpha_1$ and $s_1 = \sin\alpha_1$, $c_2 = \cos\alpha_2$ and $s_2 = \sin\alpha_2$. In order to simplify our computations we introduce two new variables $z_1 = e^{i\alpha_1}$ and $z_2 = e^{i\alpha_2}$ where $i = \sqrt{-1}$. Using these variables, $\cos\alpha_j = (z_j^2 + 1)/(2\,z_j)$ and $\sin\alpha_j = (z_j^2 - 1)/(2\,i\,z_j)$ for $j = 1, 2$. Then by doing the previous substitutions and by eliminating the denominators system (7) is equivalent to system $\overline{e}_1 = \overline{e}_2 = 0$ where

$$
\begin{aligned}
\overline{e}_1 = {} & iz_1^6z_2^8 - 2iz_1^4z_2^8 + iz_1^2z_2^8 - 2iz_1^6z_2^6 + 4iz_1^4z_2^6 - 2iz_1^2z_2^6 - z_1^6z_2^5 + 2z_1^4z_2^5 \\
& -z_1^2z_2^5 - z_1^6z_2^3 + 2z_1^4z_2^3 - z_1^2z_2^3 + 2iz_1^6z_2^2 - 4iz_1^4z_2^2 + 2iz_1^2z_2^2 - iz_1^6
\end{aligned}
$$

$$
\begin{aligned}
&+2iz_1^4 - iz_1^2 + \Big(-iz_2^6 z_1^8 + 2iz_2^4 z_1^8 - iz_2^2 z_1^8 + 2iz_2^6 z_1^6 - 4iz_2^4 z_1^6 \\
&+2iz_2^2 z_1^6 + z_2^6 z_1^5 - 2z_2^4 z_1^5 + z_2^2 z_1^5 + z_2^6 z_1^3 - 2z_2^4 z_1^3 + z_2^2 z_1^3 - 2iz_2^6 z_1^2 \\
&+4iz_2^4 z_1^2 - 2iz_2^2 z_1^2 + iz_2^6 - 2iz_2^4 + iz_2^2 \Big) \mu, \\
\overline{e}_2 \; = \; & i - iz_2^8 z_1^{12} + 2iz_2^8 z_1^{10} + 2iz_2^6 z_1^{10} + z_2^5 z_1^9 - iz_2^8 z_1^8 - 4iz_2^6 z_1^8 - 2z_2^5 z_1^7 \\
&+z_2^3 z_1^7 + 2iz_2^6 z_1^6 - 2iz_2^2 z_1^6 + z_2^5 z_1^5 - 2z_2^3 z_1^5 + 4iz_2^2 z_1^4 + iz_1^4 + z_2^3 z_1^3 \\
&-2iz_2^2 z_1^2 - 2iz_1^2 + \Big(-iz_2^6 z_1^{12} + 2iz_2^6 z_1^{10} + 2iz_2^4 z_1^{10} + z_2^6 z_1^9 \\
&-4iz_2^4 z_1^8 - iz_2^2 z_1^8 + z_2^6 z_1^7 - 2z_2^4 z_1^7 - 2iz_2^6 z_1^6 + 2iz_2^2 z_1^6 - 2z_2^4 z_1^5 \\
&+z_2^2 z_1^5 + iz_2^6 z_1^4 + 4iz_2^4 z_1^4 + z_2^2 z_1^3 - 2iz_2^4 z_1^2 - 2iz_2^2 z_1^2 + iz_2^2 \Big) \mu,
\end{aligned}
$$

and $|z_1| = |z_2| = 1$. Since we are interested in solutions of (7) with $\alpha_1, \alpha_2 \in (0, \pi)$ and $\alpha_3 = \pi - \alpha_1 - \alpha_2 \in (0, \pi)$, we only consider solutions of $\overline{e}_1 = \overline{e}_2 = 0$ with $|z_1| = |z_2| = 1$ such that $\mathrm{Im}(z_1), \mathrm{Im}(z_2) > 0$ and $\alpha_3 = \pi - \alpha_1 - \alpha_2 \in (0, \pi)$ where $\alpha_i = \arctan(\mathrm{Im}(z_i)/\mathrm{Re}(z_i))$ when $\mathrm{Im}(z_i)/\mathrm{Re}(z_i) > 0$ and $\alpha_i = \arctan(\mathrm{Im}(z_i)/\mathrm{Re}(z_i)) + \pi$ when $\mathrm{Im}(z_i)/\mathrm{Re}(z_i) < 0$, $i = 1, 2$.

Fixed $\mu = v_i$ for $i = 1, \ldots, 24$ we solve the polynomial system $\overline{e}_1 = \overline{e}_2 = 0$ by means of resultants. With the help of Mathematica, we compute the resultants $\mathrm{Res}\,[\overline{e}_1, \overline{e}_2, z_1]$ and $\mathrm{Res}\,[\overline{e}_1, \overline{e}_2, z_2]$, which are polynomials of the single variable z_2 and z_1 respectively. Then we compute all the solutions of the polynomial equations $\mathrm{Res}\,[\overline{e}_1, \overline{e}_2, z_1] = 0$ and $\mathrm{Res}\,[\overline{e}_1, \overline{e}_2, z_2] = 0$. We check which pairs (z_1, z_2) formed by a solution z_2 of $\mathrm{Res}\,[\overline{e}_1, \overline{e}_2, z_1] = 0$ and a solution z_1 of $\mathrm{Res}\,[\overline{e}_1, \overline{e}_2, z_2] = 0$ provide a solution of system $\overline{e}_1 = \overline{e}_2 = 0$. Finally we chose the solutions (z_1, z_2) of system $\overline{e}_1 = \overline{e}_2 = 0$ satisfying the conditions $|z_1| = |z_2| = 1$, $\mathrm{Im}(z_1), \mathrm{Im}(z_2) > 0$ and $\alpha_3 = \pi - \alpha_1 - \alpha_2 \in (0, \pi)$.

After all these computations we get the following: system $\overline{e}_1 = \overline{e}_2 = 0$ has 5 solutions for $\mu = v_1, \ldots, v_{10}$, four corresponding to non–symmetric coorbital central configuration and one corresponding to a symmetric one; it has 7 solutions for $\mu = v_{11}, \ldots, v_{19}$, four corresponding to non–symmetric coorbital central configurations and three corresponding to symmetric ones; it has 5 solutions for $\mu = v_{20}, \ldots, v_{24}$, two corresponding to non–symmetric coorbital central configurations and three corresponding to symmetric ones. So the number of solutions of system $\overline{e}_1 = \overline{e}_2 = 0$ satisfying the desired conditions changes at $\mu = \widetilde{u}_{10} = u_1$ and at $\mu = \widetilde{u}_{19}$.

We remark that we have studied all the bifurcations values using the changes of solutions in the variable s_1, we have a similar study using the changes of solutions in the variable s_2, obtaining exactly the same results.

Notice that $\mu = u_1$ is the bifurcation value that we have obtained in Section 3. Analyzing the solutions of system $\overline{e}_1 = \overline{e}_2 = 0$ as a function of μ in a neighbourhood of $\mu = u_1$ and $\mu = \widetilde{u}_{19}$ we observe the following. In a neighbourhood of $\mu = u_1$ the four solutions corresponding to non–symmetric coorbital central configurations that we have found for $\mu < u_1$ persist when $\mu \geqslant u_1$, and at $\mu = u_1$ appears a new solution that bifurcates into two ones for $\mu > u_1$ providing symmetric coorbital central configurations. Two of the four solutions corresponding to non–symmetric coorbital central configurations that we have found for $\mu < \widetilde{u}_{19}$ match to one of the solutions providing a symmetric coorbital central configuration at $\mu = \widetilde{u}_{19}$ and these last solution persist for $\mu > \widetilde{u}_{19}$. In short, at $\mu = \widetilde{u}_{19}$ a symmetric coorbital central configuration bifurcates to two non–symmetric ones for $\mu < \widetilde{u}_{19}$. This proves statements (a), and since there is no other bifurcation value (except $\mu = u_1$) it follows statement (b). □

Analyzing the solutions of system $E_1 = E_2 = F_1 = F_2 = 0$ (or equivalently of system $\overline{e}_1 = \overline{e}_2 = 0$) as a function of μ we observe that we have five families of coorbital central configurations for $\mu < u_1$, the symmetric family A^s found in Section 3, two non–symmetric families of class A denoted by A_1 and A_2 and two non–symmetric families of class B denoted by B_1 and B_2; seven families of coorbital central configurations for $u_1 < \mu < \widetilde{u}_{19}$, the symmetric family A^s, the two symmetric families B_1^s and B_2^s found in Section 3, which match at $\mu = u_1$, plus the four non–symmetric families A_1, A_2, B_1 and B_2; and finally five families of coorbital central configurations for $\mu > \widetilde{u}_{19}$, the three symmetric families A^s, B_1^s and B_2^s plus the two non–symmetric families A_1, A_2. We note that the two non–symmetric families B_1 and B_2 match at $\mu = \widetilde{u}_{19}$ to one of the symmetric families of class B, exactly B_1^s. Again see Figures 1, 2, 3 and 4.

5. Conclusions

We have derived results concerning to the number of coorbital central configurations of the planar $1+3$ body problem when two infinitesimal masses are equal, that is, $m_1 = m_2 = \mu$ and $m_3 = 1$. The problem exhibits two unconnected classes of central configurations denoted A, when two angles are less than $\pi/3$ and B, when the three angles are greater than $\pi/3$. Each class have symmetric (denoted by A^s, B_1^s and B_2^s) and non–symmetric configurations.

The results obtained in Sections 3 and 4 can be summarized as follows.

THEOREM 5.1. *The coorbital central configurations of the* $1+3$ *body problem with two equal infinitesimal masses are the following ones.*

(a) For $\mu \in (0, 0.89616...)$ *there are* 5 *coorbital central configurations,* 3 *of class* A *(*A^s*,* A_1 *and* A_2*) and* 2 *of class* B *(*B_1 *and* B_2*).*

(b) For $\mu = 0.89616...$ *there are* 6 *coorbital central configurations,* 3 *of class* A *(*A^s*,* A_1 *and* A_2*) and* 3 *of class* B *(*B_1*,* B_2 *and* $B^s_{1,2}$*).*

(c) For $\mu \in (0.89616..., 1.42385...)$ *there are* 7 *coorbital central configurations,* 3 *of class* A *(*A^s*,* A_1 *and* A_2*) and* 4 *of class* B *(*B_1*,* B_2*,* B^s_1 *and* B^s_2*).*

(d) For $\mu = 1.42385...$ *there are* 5 *coorbital central configurations,* 3 *of class* A *(*A^s*,* A_1 *and* A_2*) and* 2 *of class* B *(*B^s_1 *and* B^s_2*).*

(e) For $\mu > 1.42385...$ *there are* 5 *coorbital central configurations,* 3 *of class* A *(*A^s*,* A_1 *and* A_2*) and two of class* B *(*B^s_1 *and* B^s_2*).*

Notice that the proof that we have done of Theorem 5.1 is analytical with the exception that we have computed numerically the real roots of some polynomials of one variable. We also remark that for a polynomial of one variable we can compute all the roots with the precision as we want.

In order to see how the classes A and B of the coorbital central configurations change in function of the parameter μ, we have computed numerically the values of the angles $\theta_1(\mu)$ and $\theta_2(\mu)$, for each family of coorbital central configurations, see Figures 1 and 2.

In Figures 3 and 4 qualitative pictures of the classes A and B of the coorbital central configurations are showed for different values of μ.

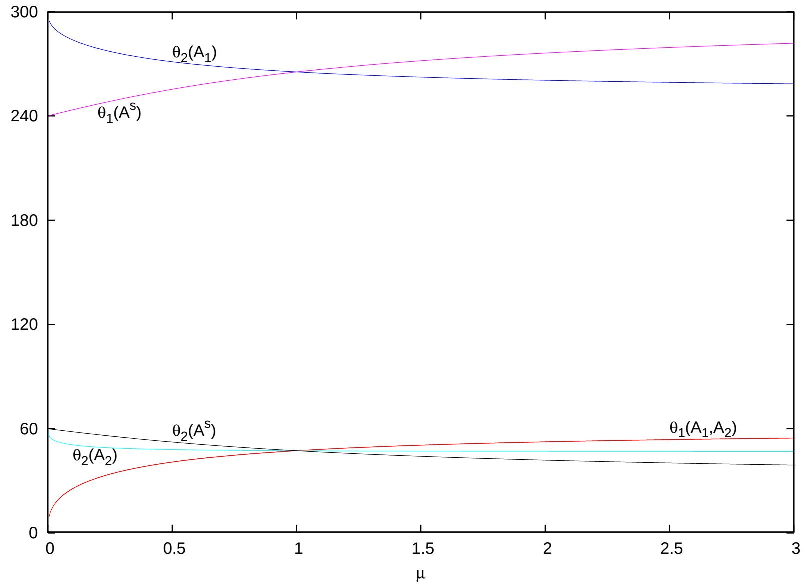

Figure 1. The angles $\theta_1(\mu)$ and $\theta_2(\mu)$ along the class A.

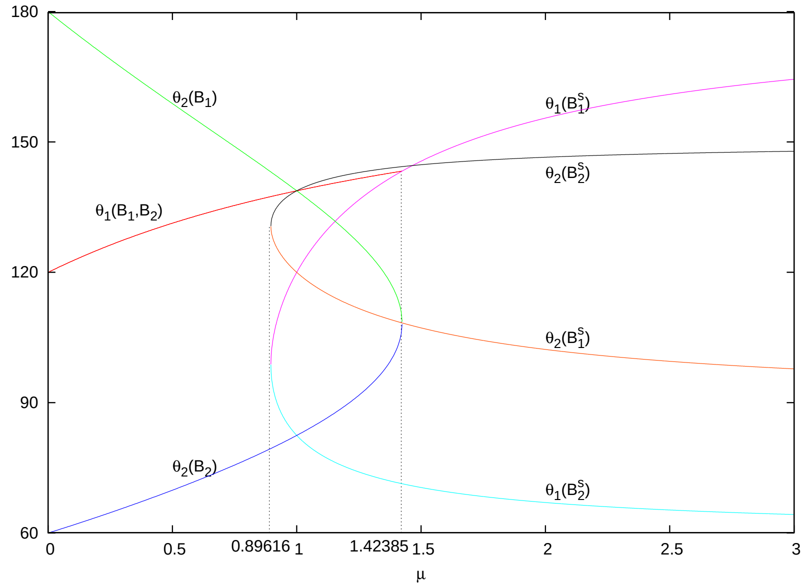

Figure 2. The angles $\theta_1(\mu)$ and $\theta_2(\mu)$ along the class B.

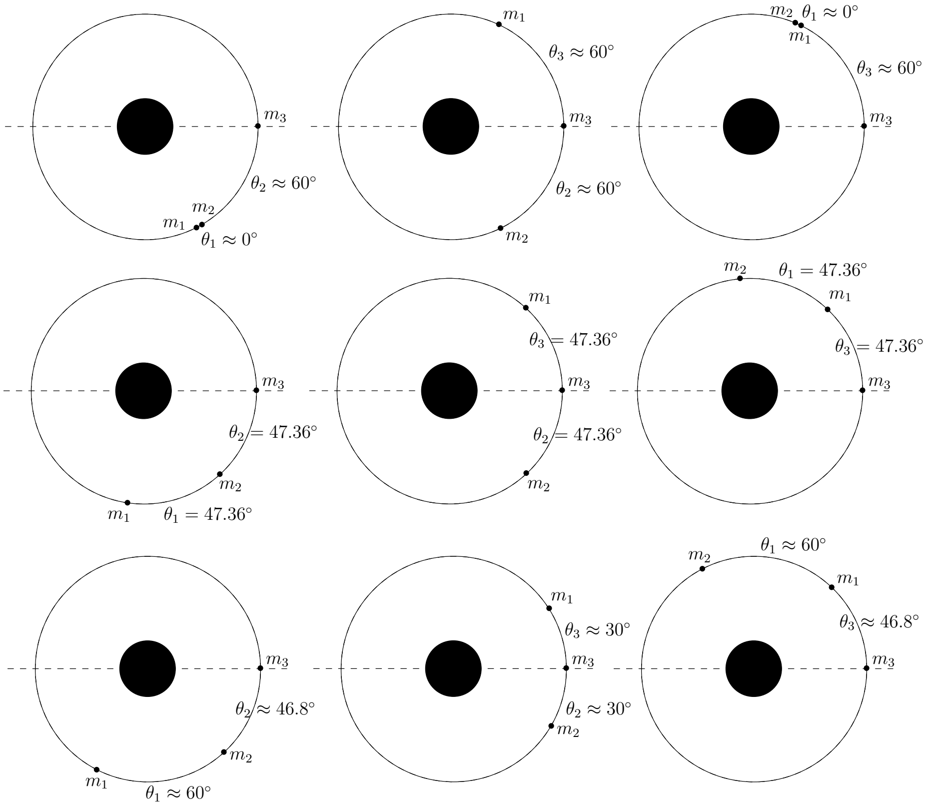

Figure 3. Coorbital central configurations of class A for different values of μ. Top $\mu \approx 0$, middle $\mu = 1$ and bottom $\mu \gg 1$. Left A_1, center A^s and right A_2.

6. Acknowledgments

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References

A. Albouy and Y. Fu, (2009), Relative equilibria of four identical satellites, *Proc. R. Soc. A*, **465**, 2633–2645.

J. Casasayas, J. Llibre and A. Nunes, (1994), Central configurations of the planar $1+n$ body problem, *Cel. Mech. and Dynam. Astr.*, **60**, 273–288.

J.M. Cors, J. Llibre and M. Ollé, (2004), Central configurations of the planar coorbital satellite problem, *Cel. Mech. and Dynam. Astr.*, **89**, 319–342.

G.R. Hall, (1988), Central configurations in the planar $1+n$ body problem, Preprint.

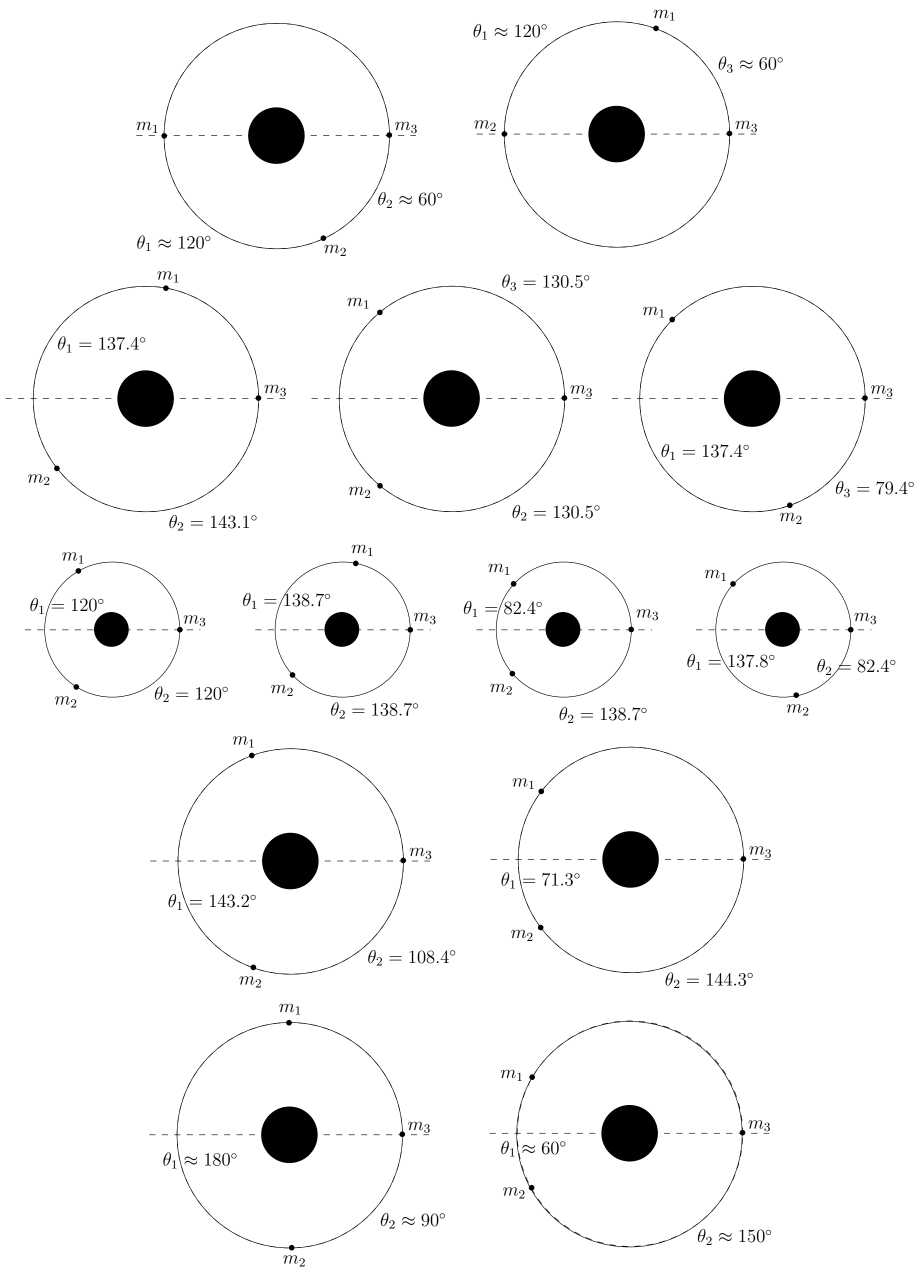

Figure 4. Coorbital central configurations of class B for different values of μ. From the top to the bottom $\mu \approx 0$, $\mu = 0.89...$, $\mu = 1$, $\mu = 1.42...$ and $\mu \gg 1$

S. Lang, (1993), Algebra, 3rd. Edition, Addison–Wesley.

T-L. Lee and M. Santoprete, (2009), Central configurations of the five-body problem with equal masses, *Cel. Mech. and Dynam. Astr.*, **104**, 369–381.

J.C. Maxwell, (1859) On the stability of the motion of Saturn's rings", *Maxwell on Saturn's Rings*, MIT Press, pp 69–158.

R. Moeckel, (1994) Linear stability of relative equilibria with a dominant mass, *J. of Dyn. and Diff. Equat.*, **6**, 37–51.

P. Olver, (1999) Classical invariant theory, London Math. Soc. Student Texts, Vol. 44, Cambbridge Univ. Press, New York.

S. Renner and B. Sicardy, (2004) Stationary configurations for co–orbital satellites with small arbitary masses, *Cel. Mech. and Dynam. Astr.*, **88**, 397–414.

D.J. Scheeres and N.X. Vinh, (1991), Linear stability of a self–graviting rings, *Cel. Mech. and Dynam. Astr.*, **51**, 83–103.

G. Roberts, (2000), Linear stability of the $n+1$–gon relative equilibrium, Hamiltonian Syst. Celest. Mech., World Scient. Monogr. Ser. Math. **6**, 303–330.