This is a preprint of: “On the planar integrable differential systems”, Jaume Giné, Jaume Llibre, *Z. Angew. Math. Phys.*, vol. 62(4), 567–574, 2011.
DOI: [10.1007/s00033-011-0116-5]

ALL PLANAR INTEGRABLE DIFFERENTIAL SYSTEMS ARE ESSENTIALLY THE LINEAR DIFFERENTIAL SYSTEM $\dot{x} = x$, $\dot{y} = y$

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Abstract. Under very general assumptions we prove that the planar differential systems having a first integral are essentially the linear differential systems $\dot{u} = u$, $\dot{v} = v$. Additionally such systems always have a Lie symmetry. We improve these results for polynomial differential systems defined in $\mathbb{R}^2$ or $\mathbb{C}^2$.

1. Introduction and statement of the results

The nonlinear ordinary differential equations appear in many branches of applied mathematics, physics and, in general, in applied sciences. For a differential system or a vector field defined on an open subset U of the plane the existence of a first integral determines completely its phase portrait. Since for such vector fields the notion of integrability is based on the existence of a first integral the following natural question arises: *Given a differential system on an open subset U of the plane, how to recognize if this differential system has a first integral*?

The more easiest planar vector fields having a first integral are the Hamiltonian ones, i.e. the differential systems of the form

$$\dot{x} = -H_y, \qquad \dot{y} = H_x,$$

where $H = H(x, y)$ is a differentiable function. Of course, these systems have the function H as a first integral. As usual H_x denotes the partial derivative of the function H with respect to the variable x. A planar differential system having a first integral is called *integrable*. The integrable planar differential systems which are not Hamiltonian are in general, very difficult to detect, see for instance [1].

Here we deal with the *planar differential systems*

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where $P, Q : U \to \mathbb{R}$ are C^1 functions in the variables x and y, U is an open subset of $\mathbb{R}^2$, and the dot denotes derivative with respect to the independent variable t.

A C^1 non–constant function $H = H(x, y)$ defined in an open and dense subset V of U is called a *first integral* of system (1) if

$$P\, H_x + Q\, H_y = 0,$$

1991 *Mathematics Subject Classification.* Primary 34C35, 34D30.
Key words and phrases. planar integrability, Darboux theory of integrability, polynomial differential systems, linear differential systems.
The first author is partially supported by a MCYT/FEDER grant number MTM2008-00694 and by a CIRIT grant number 2005SGR 00550. The second author is partially supported by a MCYT/FEDER grant number MTM2008-03437 and by a CIRIT grant number 2005SGR 00550.

on V. In other words, the function H is constant on the solutions of system (1) contained on V.

A C^1 non–constant function $R = R(x, y)$ defined in an open and dense subset V of U is called an *integrating factor* of system (1) if the *divergence*

$$\mathrm{div}(RP, RQ) = (RP)_x + (RQ)_y = 0,$$

on V. Then the differential system

$$\dot{x} = RP, \qquad \dot{y} = RQ,$$

is Hamiltonian, i.e. there exists a function H such that

$$\dot{x} = RP = -H_y, \qquad \dot{y} = RQ = H_x.$$

We say that H is the *first integral associated to the integrating factor* R, and vice versa R is the *integrating factor associated to the first integral* H.

Our first result is the following one.

Theorem 1. *Consider a C^1 planar differential system* (1) *defined in an open subset U of $\mathbb{R}^2$ having a C^1 first integral H and a C^1 integrating factor R defined in open and dense subsets V_H and V_R of U, respectively. Assume that the Lebesgue measure of the set $\{R(R_xH_y - R_yH_x)(P_x + Q_y) = 0\}$ in $V_H \cap V_R$ is zero. Then, the change of variables $(x, y) \mapsto (u, v)$ defined by*

$$u = R(x, y), \qquad v = R(x, y)H(x, y), \tag{2}$$

in the open and dense subset

$$(V_H \cap V_R) \setminus \{R(R_xH_y - R_yH_x)(P_x + Q_y) = 0\} \tag{3}$$

of U, transforms system (1) *into the linear differential system $\dot{u} = u$ and $\dot{v} = v$.*

In the statement of Theorem 1 the integrating factor R may be (but it is not necessary) the one associated to the first integral H.

Theorem 1 is proved in section 2.

From Theorem 1 we can say that *the integrability of a planar differential system is essentially due to the integrability of the linear differential system $\dot{u} = u$, $\dot{v} = v$.*

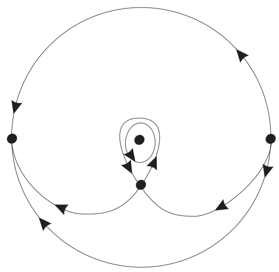

Figure 1. Phase portrait of system (4) in the Poincaré disc.

Example. *We consider the polynomial differential system*

$$\frac{dx}{dt} = -y - b(x^2 + y^2), \qquad \frac{dy}{dt} = x, \tag{4}$$

with $b \neq 0$. See its phase portrait in Figure 1. It is easy to check that this system has the first integral $H = e^{2by}(x^2+y^2)$, and the integrating factor $R = 2e^{2by}$. Then, by Theorem 1 the change of variables $(x, y, t) \mapsto (u, v, s)$ defined by

$$u = 2e^{2by}, \quad v = 2e^{4by}(x^2+y^2), \quad ds = 2\,b\,x\,dt,$$

defined for all $(x, y) \in \mathbb{R}^2 \setminus \{x = 0\}$ transforms the differential system (4) into the linear differential system

$$\frac{du}{ds} = u, \qquad \frac{dv}{ds} = v.$$

Lie's group theory is also applied for studying the integrability of ordinary differential equations, see for instance [3, 9, 12]. The fundamental observation of Lie was that the knowledge of a sufficient large group of symmetries of a system of ordinary differential equations allows to integrate the system by quadratures. See section 3 for the definitions and the results that we shall need on the Lie's group theory.

For planar differential systems a corollary of Theorem 1 in the context of the Lie's symmetries is the following one.

Theorem 2. *Assume that we are in the assumptions of Theorem 1. Then system (1) admits a Lie symmetry in the open and dense subset (3) of U.*

Theorem 2 is proved in section 3.

In section 5 we apply Theorem 2 to system (4) of the example.

Darboux [5] showed how we can constructed first integrals of planar polynomial differential systems in $\mathbb{R}^2$ or $\mathbb{C}^2$ possessing sufficient invariant algebraic curves. This theory is now known as the *Darboux theory of integrability*, see for more details the Chapter 8 of [6].

The more easy functions which can be first integrals of polynomial differential systems can be ordered as: polynomial, rational, Darboux, elementary and Liouvillian. Each one of these classes of functions is contained in the next class according the given order. The Darboux theory of integrability allows to compute all the first integrals belonging to one of these classes, see [6, 11, 13]. In the next result and for polynomial differential systems in $\mathbb{R}^2$ or $\mathbb{C}^2$ we summarize the explicit relationships between the functions defining the first integrals and their integrating factors inside the classes $\mathcal{C}$ of functions.

Theorem 3. *Consider the planar polynomial differential system (1).*

(a) *If system (1) has a Liouvillian first integral, then it has a Darboux integrating factor.*
(b) *If system (1) has an elementary first integral, then it has an integrating factor of the form a rational function to power $1/n$ for positive integer n.*
(c) *If system (1) has a Darboux first integral, then it has a rational integrating factor.*
(d) *If system (1) has a rational first integral, then it has a rational integrating factor.*
(e) *If system (1) has a polynomial first integral, then it has a polynomial integrating factor*

Theorem 3 is due to several authors, see section 4 for the definitions of the different kind of functions which appear in it, and for the references of its proof.

Let $\mathcal{C}$ be one of the following classes of functions, polynomial, rational, Darboux, elementary and Liouvillian. A *$\mathcal{C}$–curve* is a curve $f(x,y)=0$ where $f\in\mathcal{C}$. This curve is in $\mathbb{R}^2$ or $\mathbb{C}^2$ according with the polynomial differential system (1) is defined in $\mathbb{R}^2$ or $\mathbb{C}^2$, respectively. For polynomial differential systems in $\mathbb{R}^2$ or $\mathbb{C}^2$ we can improve slightly Theorem 1 as follows.

Theorem 4. *Consider a planar polynomial differential system* (1) *defined in* $\mathbb{R}^2$ *(respectively* $\mathbb{C}^2$*) having a first integral* H *belonging to the class* $\mathcal{C}$*. Let* R *be the integrating factor whose existence is given by Theorem 3, and let* V_H *and* V_R *be the domains of definition of the functions* H *and* R*, respectively. Assume that the function* $R(R_xH_y-R_yH_x)(P_x+Q_y)$ *is not identically zero. Then, the change of variables* $(x,y)\mapsto(u,v)$ *defined by* (2) *in the open and dense subset* (3) *of* $\mathbb{R}^2$ *(respectively* $\mathbb{C}^2$*), transforms system* (1) *into the linear differential system* $\dot u=u$ *and* $\dot v=v$*. Moreover the curve* $R(R_xH_y-R_yH_x)(P_x+Q_y)=0$ *is a* $\mathcal{C}$*–curve.*

Theorem 4 is proved in Section 4.

A result of the same kind than the ones of Theorems 1 and 4 is done in [8]. More precisely there it is proved that all the known rational Abel differential equations having a first integral can be transformed through a rational change of variables to a Riccati or a linear differential equation.

2. Proof of Theorem 1

Under the assumptions of Theorem 1 we have a C^1 planar differential system (1) defined in an open subset U of $\mathbb{R}^2$ having a C^1 first integral H and a C^1 integrating factor R defined in open and dense subsets V_H and V_R of U, respectively. Moreover the Lebesgue measure of the set $\{R(R_xH_y-R_yH_x)(P_x+Q_y)=0\}$ in $V_H\cap V_R$ is zero.

Now we consider the transformation $(x,y)\mapsto(u,v)$ given by (2). This transformation is a change of variables in $(V_H\cap V_R)\setminus\{g=0\}$, where

$$g=g(x,y)=\det\begin{pmatrix} R_x & R_y \\ (RH)_x & (RH)_y\end{pmatrix}=R\,\det\begin{pmatrix} R_x & R_y \\ H_x & H_y\end{pmatrix}=0.$$

After the change of variables in $(V_H\cap V_R)\setminus\{g=0\}$ system (1) becomes

$$\begin{aligned}\dot u&=\dot R=R_xP+R_yQ=-\mathrm{div}(P,Q)R=-\mathrm{div}(P,Q)u,\\ \dot v&=\dot{RH}=(R_xP+R_yQ)H+R(H_xP+H_yQ)=-\mathrm{div}(P,Q)RH=-\mathrm{div}(P,Q)v.\end{aligned}$$

Now changing the independent variable from t to s doing the rescaling

$$ds=-\mathrm{div}(P,Q)\,dt \tag{5}$$

in $(V_H\cap V_R)\setminus\{g\,\mathrm{div}(P,Q)=0\}$, (i.e. in (3)), we get the linear differential system $u'=u$, $v'=v$, where the prime denotes derivative with respect to the variable s. Hence Theorem 1 is proved.

3. Lie's symmetries and planar differential systems

Let $\mathcal{X}$ be the planar vector field associated to system (1), that is

$$\mathcal{X}=P(x,y)\frac{\partial}{\partial x}+Q(x,y)\frac{\partial}{\partial y}.$$

Let G be the *one–parameter Lie group* of transformations

$$x^*(x,y;\varepsilon)=x+\varepsilon\,\xi(x,y)+O(\varepsilon^2),\quad y^*(x,y;\varepsilon)=y+\varepsilon\,\eta(x,y)+O(\varepsilon^2), \tag{6}$$

acting on the open subset U of $\mathbb{R}^2$ with associated *infinitesimal generator* $\mathcal{Y}$ defined like

$$\mathcal{Y} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y},$$

with $\xi, \eta \in C^1(U)$. A *symmetry* of differential system (1) is defined to be a group of transformations (6) such that under the action of this group, a solution curve of system (1) is mapped into another solution curve of (1).

We define the *Lie bracket* of the C^1–vector fields $\mathcal{X}$ and $\mathcal{Y}$ as $[\mathcal{X},\mathcal{Y}] := \mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X}$. In other words we have

$$[\mathcal{X},\mathcal{Y}] = \left(P\frac{\partial \xi}{\partial x} - \xi\frac{\partial P}{\partial x} + Q\frac{\partial \xi}{\partial y} - \eta\frac{\partial P}{\partial y}\right)\frac{\partial}{\partial x} + \left(P\frac{\partial \eta}{\partial x} - \xi\frac{\partial Q}{\partial x} + Q\frac{\partial \eta}{\partial y} - \eta\frac{\partial Q}{\partial y}\right)\frac{\partial}{\partial y}.$$

The following proposition is well known, see for instance [13].

Proposition 5. *Let G be the one–parameter Lie group of transformations* (6). *Then G is a symmetry of system* (1) *if and only if the commutation relation $[\mathcal{X},\mathcal{Y}] = \mu(x,y)\ \mathcal{X}$ is satisfied for some smooth scalar function $\mu(x,y)$.*

Proof of Theorem 2. Let ϕ be the change of variables $(x,y) \mapsto (u,v)$ defined by (2) in (3), and let $\mathcal{X}$ be the planar vector field associated to system (1). This means that $\bar{\mathcal{X}} = \phi_* \mathcal{X}$ with $\bar{\mathcal{X}} = u\, h(u,v)\,\partial/\partial u + v\, h(u,v)\,\partial/\partial v$ and where ϕ_* and ϕ^* is the push–forward and the pull–back defined by the diffeomorphism ϕ in (3). The proof follows easily taking into account that the vector field $\bar{\mathcal{X}} = u\, h(u,v)\,\partial/\partial u + v\, h(u,v)\,\partial/\partial v$ admits $\bar{\mathcal{Y}} = -v\,\partial/\partial x + u\,\partial/\partial y$ as an infinitesimal generator of a Lie symmetry in the sense of Proposition 5. This is true because $\tilde{\mathcal{X}} = u\,\partial/\partial u + v\,\partial/\partial v$ commutes with $\bar{\mathcal{Y}}$ i.e. $[\tilde{\mathcal{X}}, \bar{\mathcal{Y}}] = 0$, and therefore

$$[\bar{\mathcal{X}}, \bar{\mathcal{Y}}] = [h\tilde{\mathcal{X}}, \bar{\mathcal{Y}}] = h[\tilde{\mathcal{X}}, \bar{\mathcal{Y}}] - \bar{\mathcal{Y}}(h)\tilde{\mathcal{X}} = -\bar{\mathcal{Y}}(h)/h\ \bar{\mathcal{X}} = \bar{\mu}\ \bar{\mathcal{X}}, \tag{7}$$

where $\bar{\mu} = -\bar{\mathcal{Y}}(h)/h$. Hence, in the image under ϕ of (3), the inverse of the change of variables (2) applied to system $\dot{u} = -v$ and $\dot{v} = u$; i.e. $\mathcal{Y} = \phi^* \bar{\mathcal{Y}}$ gives a Lie symmetry of system (1), because the Lie bracket is a coordinate free geometrical object, i.e. $\phi^*[\bar{\mathcal{X}}, \bar{\mathcal{Y}}] = [\phi^*\bar{\mathcal{X}}, \phi^*\bar{\mathcal{Y}}]$. Therefore the transformed equation (7) is $[\mathcal{X},\mathcal{Y}] = \mu\ \mathcal{X}$ where $\mu = \phi^*\bar{\mu}$. □

4. On the Darboux theory of integrability

In this section we summarize the effectiveness of the Darboux theory of integrability, i.e. what sort of integrals does it capture. For additional details on this section see [6, 11, 12].

The idea of calculating what sort of functions can arise as the result of evaluating an indefinite integral or solving a differential equation goes back to Liouville. The modern formulation of these ideas is usually done through differential algebra. Some of the advantages over an analytic approach are first that the messy details of branch points etc, is hidden completely, and second the Darboux theory of integrability can be studied using symbolic computation.

We assume that the set of functions we are interested form a field together with a number of *derivations*. We call such an object a *differential field*. The process of adding more functions to a given set of functions is described by a tower of such fields:

$$F_0 \subset F_1 \subset \cdots \subset F_n.$$

Of course, we must also specify how the derivations of F_0 are extended to derivations on each F_i.

The fields we are interested in arise by adding exponentials, logarithms or the solutions of algebraic equations based on the previous set of functions. That is we take

$$F_i = F_0(\theta_1, \dots \theta_i),$$

where one of the following holds:

(i) $\delta\theta_i = \theta_i \delta g$, for some $g \in F_{i-1}$ and for each derivation δ;
(ii) $\delta\theta_i = g^{-1}\delta g$, for some $g \in F_{i-1}$ and for each derivation δ;
(iii) θ_i is algebraic over F_{i-1}. If we have such a tower of fields, F_n is called an *elementary* extension of F_0.

This is essentially what we mean by a function being expressible in closed form. We call the set of all elements of a differential field which are annihilated by all the derivations of the field the *field of constants*. We shall always assume that the field of constants is algebraically closed.

We say that our system (1) has an *elementary* first integral if there is an element u in an elementary extension field of the field of rational functions $\mathbb{C}(x, y)$ with the same field of constants such that $Du = 0$. The derivations on $\mathbb{C}(x, y)$ are of course d/dx and d/dy.

Another class of integrals we are interested in are the *Liouvillian* ones. Here we say that an extension F_n is a *Liouvillian* extension of F_0 if there is a tower of differential fields as above which satisfies conditions (i), (iii) or

$(ii)'$ $\delta_\alpha\theta_i = h_\alpha$ for some elements $h_\alpha \in F_{i-1}$ such that $\delta_\alpha h_\beta = \delta_\beta h_\alpha$.

This last condition, mimics the introduction of line integrals into the class of functions. Clearly (ii) is included in $(ii)'$.

This class of functions represents those functions which are obtainable "by quadratures". An element u of a Liouvillian extension field of $\mathbb{C}(x, y)$ with the same field of constants is said to be a *Liouvillian* first integral.

A function of the form $e^{w_0+\sum c_i \ln(w_i)}$, where c_i are constants and w_i are rational functions is called *Darboux* function.

We note that the Darboux theory of integrability finds all Liouvillian first integrals of the planar polynomial vector fields.

Proof of Theorem 3. Statement (a) is due to Singer [12], see also Christopher [4] and Pereira [10]. Statement (b) was proved by Prelle and Singer [11]. Statement (c) was shown in [2]. The proof of statement (d) follows easily. Finally the proof of statement (e) follows from [2] and [7]. □

Proof of Theorem 4. Under the assumptions of Theorem 4 we have a planar polynomial differential system (1) defined in $\mathbb{R}^2$ (respectively $\mathbb{C}^2$) having a first integral H belonging to the class $\mathcal{C}$. Moreover we also have the integrating factor R whose existence is given by Theorem 3, and let V_H and V_R be the domains of definition of the functions H and R, respectively. We also know that the function $R(R_xH_y - R_yH_x)(P_x + Q_y)$ is not identically zero.

Due to Theorem 3 the functions $g = R(R_xH_y - R_yH_x)$ and H belong to the same class $\mathcal{C}$. Since the function $\text{div}(P, Q) = P_x + Q_y$ is polynomial, the $\mathcal{C}$ type of the curves $g\,\mathrm{div}(P,Q)=0$ and $g=0$ coincide. Hence Theorem 4 follows easily from Theorem 1. □

5. An application

Here we apply Theorem 2 to the system (4). Let ϕ be the change of variables $(x,y)\mapsto(u,v)=(2e^{2by},2e^{4by}(x^2+y^2))$ in $\mathbb{R}^2\setminus\{x=0\}$ and $\mathcal{X}$ the planar vector field associated to system (4). We have that

$$\bar{\mathcal{X}}=\phi_*\mathcal{X}=b\sqrt{8v-\frac{u^2}{b^2}\log\left[\frac{u}{2}\right]^2}\,\frac{\partial}{\partial u}\;+\;\frac{b\,v}{u}\sqrt{8v-\frac{u^2}{b^2}\log\left[\frac{u}{2}\right]^2}\,\frac{\partial}{\partial v}.$$

Applying Theorem 2 the associated vector field $\bar{\mathcal{X}}$ has the infinitesimal generator of a Lie symmetry $\bar{\mathcal{Y}}=-v\,\partial/\partial x+u\,\partial/\partial y$, i.e. $[\bar{\mathcal{X}},\bar{\mathcal{Y}}]=\bar{\mu}\;\bar{\mathcal{X}}$ where

$$\bar{\mu}=-\frac{4b^2(u^2+2v^2)+u^2v\log\left[\frac{u}{2}\right]}{u(8b^2v-u^2\log\left[\frac{u}{2}\right]^2)}.$$

Therefore the inverse of the change of variables ϕ (defined in $\phi(\mathbb{R}^2\setminus\{x=0\})$) applied to $\bar{\mathcal{Y}}$, that is, $\mathcal{Y}=\phi^*\bar{\mathcal{Y}}$ gives a Lie symmetry of system (4). Hence the associated vector field $\mathcal{X}$ to system (4) has the following infinitesimal generators of a Lie symmetry $\mathcal{Y}=\mathcal{Y}_1\,\partial/\partial x+\mathcal{Y}_2\,\partial/\partial y$ given by

$$\mathcal{Y}_1=\frac{e^{-2by}}{2b\,x}(b+2be^{4by}x^4+e^{4by}x^2y+4be^{4by}x^2y^2+e^{4by}y^3+2be^{4by}y^4),$$

and

$$\mathcal{Y}_2=-\frac{e^{-2by}}{2b}(x^2+y^2),$$

which verifies that $[\mathcal{X},\mathcal{Y}]=\mu(x,y)\;\mathcal{X}$ where the scalar function $\mu(x,y)$ is

$$\mu(x,y)=\frac{e^{-2by}}{2b\,x^2}(b+2be^{4by}x^4+e^{4by}x^2y+4be^{4by}x^2y^2+e^{4by}y^3+2be^{4by}y^4).$$

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