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GENERALIZED RATIONAL FIRST INTEGRALS OF ANALYTIC DIFFERENTIAL SYSTEMS

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Abstract. In this paper we mainly study the necessary conditions for the existence of functionally independent generalized rational first integrals of ordinary differential systems via the resonances. The main results extend some of the previous related ones, for instance the classical Poincaré's one [16], the Furta's one [8], part of Chen *et al*'s ones [4], and the Shi's one [18]. The key point in the proof of our main results is that functionally independence of generalized rational functions implies the functionally independence of their lowest order rational homogeneous terms.

1. Introduction and statement of the main results

The rational first integrals in analytic differentiable systems and mainly in the particular case of polynomial differentiable systems has been studied intensively, specially inside the Darboux theory of integrability, see for instance [15], [9], [17], [19]. In this paper we want to study the generalized rational first integrals of the analytic differential systems.

Consider analytic differential systems in $(\mathbb{C}^n, 0)$

$$\dot{x} = f(x), \qquad x \in (\mathbb{C}^n, 0). \tag{1}$$

A function $F(x)$ of form $G(x)/H(x)$ with G and H analytic functions in $(\mathbb{C}^n, 0)$ is a *generalized rational first integral* if

$$\langle f(x), \partial_x F(x)\rangle \equiv 0, \qquad x \in (\mathbb{C}^n, 0),$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of two vectors in $\mathbb{C}^n$, and $\partial_x F = (\partial_{x_1} F, \ldots, \partial_{x_n} F)$ is the gradient of F and $\partial_{x_i} F = \partial F/\partial x_i$. As usually, if G and H are polynomial functions, then $F(x)$ is a *rational first integral*. If H is a non–zero constant, then $F(x)$ is an *analytic first integral*. So generalized rational first integrals include rational first integrals and analytic first integrals as particular cases.

If $f(0) \neq 0$, it is well–known from the Flow–Box Theorem that system (1) has an analytic first integral in a neighborhood of the origin. If $f(0) = 0$, i.e. $x = 0$ is a *singularity* of system (1), the existence of first integrals for system (1) in $(\mathbb{C}^n, 0)$ is usually much involved.

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Denote by $A = Df(0)$ the Jacobian matrix of $f(x)$ at $x = 0$. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the n–tuple of eigenvalues of A. We say that the eigenvalues λ satisfy a $\mathbb{Z}^+$–*resonant condition* if

$$\langle \lambda, \mathbf{k} \rangle = 0, \qquad \text{for some } \mathbf{k} \in \left(\mathbb{Z}^+\right)^n, \quad \mathbf{k} \neq 0,$$

where $\mathbb{Z}^+ = \mathbb{N} \cup \{0\}$ and $\mathbb{N}$ is the set of positive integers. The eigenvalues λ satisfy a $\mathbb{Z}$–*resonant condition* if

$$\langle \lambda, \mathbf{k} \rangle = 0, \qquad \text{for some } \mathbf{k} \in \mathbb{N}^n, \quad \mathbf{k} \neq 0,$$

where $\mathbb{Z}$ is the set of integers.

Poincaré [16] was the first one in studying the relation between the existence of analytic first integrals and resonance, he obtained the following classical result (for a proof, see for instance [8]).

Poincaré Theorem *If the eigenvalues λ of A do not satisfy any $\mathbb{Z}^+$–resonant conditions, then system* (1) *has no analytic first integrals in* $(\mathbb{C}^n, 0)$.

Recall that k ($k < n$) functions are *functionally independent* in an open subset U of $\mathbb{C}^n$ if their gradients have rank k in a full Lebesgue measure subset of U. Obviously an n–dimensional nontrivial autonomous system can have at most $n-1$ functionally independent first integrals, where nontrivial means that the associated vector field does not vanish identically.

In 2003 Li, Llibre and Zhang [14] extended the Poincaré's result to the case that λ admit one zero eigenvalue and the others are not $\mathbb{Z}^+$–resonant.

In 2008 Chen, Yi and Zhang [4] proved that the number of functionally independent analytic first integrals for system (1) does not exceed the maximal number of linearly independent elements of $\{\mathbf{k} \in (\mathbb{Z}^+)^n : \langle \mathbf{k}, \lambda \rangle = 0, \mathbf{k} \neq 0\}$.

In 2007 the Poincaré's result was extended by Shi [18] to the $\mathbb{Z}$–resonant case. He proved that if system (1) has a rational first integral, then the eigenvalues λ of A satisfy a $\mathbb{Z}$–resonant condition. In other words, if λ do not satisfy any $\mathbb{Z}$–resonant condition, then system (1) has no rational first integrals in $(\mathbb{C}^n, 0)$.

The aim of this paper is to improve the above results by studying the existence of more than one functionally independent rational first integrals. Our first main result is the following.

Theorem 1. *Assume that the differential system* (1) *satisfies* $f(0) = 0$ *and let* $\lambda = (\lambda_1, \ldots, \lambda_n)$ *be the eigenvalues of* $Df(0)$. *Then the number of functionally independent generalized rational first integrals of system* (1) *in* $(\mathbb{C}^n, 0)$ *is at most the dimension of the minimal vector subspace of* $\mathbb{R}^n$ *containing the set* $\{\mathbf{k} \in \mathbb{Z}^n : \langle \mathbf{k}, \lambda \rangle = 0, \mathbf{k} \neq 0\}$.

We remark that Theorem 1 extends all the results mentioned above, i.e. the one of Poincaré [16], the Theorem 1.1 of Chen, Yi and Zhang [4], and the Theorem 1 of Shi [18]. We should mention that the methods of the above mentioned papers are not enough to study the existence of more than one functionally independent generalized rational first integrals. Here we will use a different approach to prove Theorem 1. Our key technique will be the Lemma 6 given in Section 2, which shows that the functionally independence of generalized rational functions implies the functionally independence of their lowest order rational homogeneous terms.

We should say that Theorem 1 has some relation with Propositions 3.5 and 5.4 of Goriely [10]. In the former the author established a relation between the weight degrees of independent algebraic first integrals of a weight homogeneous vector field and its Kowalevskaya exponents. And in the latter he provided a necessary condition for the existence of independent analytic first integrals of a weight homogeneous vector field.

We note that if the linear part $Df(0)$ of (1) has all its eigenvalues zero, then the result of Theorem 1 is trivial. For studying these cases we consider semi–quasi–homogeneous systems. Let $f = (f_1, \ldots, f_n)$ be vector functions. Then system (1) is *quasi–homogeneous of degree* $q \in \mathbb{N} \setminus \{1\}$ with exponents $s_1, \ldots, s_n \in \mathbb{Z} \setminus \{0\}$ if for all $\rho > 0$

$$f_i\left(\rho^{s_1} x_1, \ldots, \rho^{s_n} x_n\right) = \rho^{q+s_i-1} f_i(x_1, \ldots, x_n), \quad i = 1, \ldots, n.$$

The exponents $\mathbf{s} := (s_1, \ldots, s_n)$ are called *weight exponents*, and the number $q + s_i - 1$ is called the *weight degree* of f_i, i.e., f_i is a *quasi–homogeneous function of weight degree* $q+s_i-1$. A vector function f is *quasi–homogeneous of weight degree* q *with weight exponent*$\mathbf{s}$ if each component f_i is quasi–homogeneous of weight degree q, i.e. $f_i\left(\rho^{s_1} x_1, \ldots, \rho^{s_n} x_n\right) = \rho^{q} f_i(x_1, \ldots, x_n)$ for $i = 1, \ldots, n$.

System (1) is *semi–quasi–homogeneous* of degree q with the weight exponent $\mathbf{s}$ if

$$f(x) = f_q(x) + f_h(x), \tag{2}$$

where $\rho^{\mathbf{E}-\mathbf{S}} f_q$ is quasi–homogeneous of degree q with weight exponent $\mathbf{s}$ and $\rho^{\mathbf{E}-\mathbf{S}} f_h$ is the sum of quasi–homogeneous of degree either all larger than q or all less than q with weight exponent $\mathbf{s}$. The former (resp. latter) is called *positively* (resp. *negatively*) semi–quasi–homogeneous. Here we have used the notations: $\mathbf{E}$ is the $n \times n$ identity matrix, $\mathbf{S}$ is the $n \times n$ diagonal matrix $\mathrm{diag}(s_1, \ldots, s_n)$, and $\rho^{\mathbf{E}-\mathbf{S}} = \mathrm{diag}\left(\rho^{1-s_1}, \ldots, \rho^{1-s_n}\right)$.

We note that for any give exponent $\mathbf{s}$ an analytic differential system (1) can be written as a positively semi–quasi–homogeneous system, and also as a negatively one if it is a polynomial.

Assume that $\rho^{\mathbf{E}-\mathbf{S}} f_q$ is quasi–homogeneous of weight degree $q > 1$ with weight exponent $\mathbf{s}$. Set $\mathbf{W} = \mathbf{S}/(q-1)$. Any solution $c = (c_1, \ldots, c_n)$ of

$$f_q(c) + \mathbf{W} c = 0, \tag{3}$$

is called a *balance*. Denote by $\mathcal{B}$ the set of balances. For each balance c, the Jacobian matrix $K = Df_q(c) + \mathbf{W}$ is called the *Kowalevskaya matrix* at c and its eigenvalues are called the *Kowalevskaya exponents*, denoted by λ_c.

Let d_c be the dimension of the minimal vector subspace of $\mathbb{R}^n$ containing the set

$$\{\mathbf{k} \in \mathbb{Z}^n : \langle \mathbf{k}, \lambda_c \rangle = 0, \, \mathbf{k} \neq 0\}.$$

Theorem 2. *Assume that system* (1) *is semi–quasi–homogeneous of weight degree q with weight exponent* $\mathbf{s}$, *and $f(x)$ satisfies* (2) *with $f(0) = 0$. Then the number of functionally independent generalized rational first integrals of* (1) *is at most $d = \min_{c \in \mathcal{B}} d_c$.*

Theorem 2 is an extension of Theorem 1 of Furta [8], of Corollary 3.7 of [10] and of Theorem 2 of Shi [18]. In some sense it is also an extension of the main results of Yoshida [20, 21], where he proved that if a quasi–homogenous differential system is algebraically integrable, then every Kowalevkaya exponent should be a rational number.

Theorems 1 and 2 studied the existence of functionally independent generalized rational first integrals of system (1) in a neighborhood of a singularity. Now we turn to investigate the existence of generalized rational first integrals of system (1) in a neighborhood of a periodic orbit. The *multipliers* of a periodic orbit are the eigenvalues of the linear part of the Poincaré map at the fixed point corresponding to the periodic orbit. Recall that a Poincaré map associated to a periodic orbit is defined on a transversal section to the periodic orbit, and its linear part has the eigenvalue 1 along the direction tangent to the periodic orbit.

Theorem 3. *Assume that the analytic differential system* (1) *has a periodic orbit with multipliers $\mu = (\mu_1, \ldots, \mu_{n-1})$. Then the number of functionally independent generalized rational first integrals of system* (1) *in a neighborhood of the periodic orbit is at most the maximum number of linearly independent vectors in $\mathbb{R}^n$ of the set*

$$\{\mathbf{k} \in \mathbb{Z}^{n-1} : \mu^{\mathbf{k}} = 1, \, \mathbf{k} \neq 0\}.$$

Here and after, for vectors $x = (x_1, \ldots, x_n)$ and $\mathbf{k} = (k_1, \ldots, k_n)$ we use $x^{\mathbf{k}}$ to denote the product $x_1^{k_1} \cdot \ldots \cdot x_n^{k_n}$.

Finally we consider the periodic differential systems

$$\dot{x} = f(t, x), \qquad (t, x) \in \mathbb{S}^1 \times (\mathbb{C}^n, 0), \tag{4}$$

where $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{N})$, and $f(t, x)$ is analytic in its variables and periodic of period 2π in t. Assume that $x = 0$ is a constant solution of (4), i.e. $f(t, 0) = 0$.

A non–constant function $F(t, x)$ is a *generalized rational first integral* of system (4) if $F(t, x) = G(t, x)/H(t, x)$ with $G(t, x)$ and $H(t, x)$ analytic in their variables and 2π periodic in t, and it satisfies

$$\frac{\partial F(t, x)}{\partial t} + \langle \partial_x F(t, x), f(t, x) \rangle \equiv 0 \quad \text{in } \mathbb{S}^1 \times (\mathbb{C}^n, 0).$$

If $H(t,x)$is constant and non–zero, then $F(t,x)$ is an *analytic first integral.* If $G(t,x)$ and $H(t,x)$ are polynomials in x we say that $F(t,x)$ is a *rational first integral.* If $G(t,x)$ and $H(t,x)$ are both homogeneous polynomials in x of degrees l and m respectively, we say that G/H is a *rational homogeneous first integral of degree $l-m$.*

Since $f(t,0)=0$, we can write system (4) as

$$\dot{x} = A(t)x + g(t,x), \tag{5}$$

where $A(t)$ and $g(t,x) = O(x^2)$ are 2π periodic in t. Consider the linear equation

$$\dot{x} = A(t)x. \tag{6}$$

Let $x_0(t)$ be the solution of (6) satisfying the initial condition $x_0(0) = x_0$ with $x_0 \in (\mathbb{C}^n, 0)$. The *monodromy operator* associated to (6) is the map $\mathcal{P} : (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^{n-1}, 0)$ defined by $\mathcal{P}(x_0) = x_0(2\pi)$.

We say that functions $F_1(t,x), \ldots, F_m(t,x)$ are *functionally independent* in $\mathbb{S}^1 \times (\mathbb{C}^n, 0)$ if $\partial_x F_1(t,x), \ldots, \partial_x F_m(t,x)$ have the rank m in a full Lebesgue measure subset of $\mathbb{S}^1 \times (\mathbb{C}^n, 0)$.

Theorem 4. *Let $\mu = (\mu_1, \ldots, \mu_n)$ be the eigenvalues of the monodromy operator* (*i.e. the characteristic multipliers of* (6))*. Then the number of functionally independent generalized rational first integrals of system* (4) *is at most the maximum number of linearly independent vectors in $\mathbb{R}^n$ of the set*

$$\Xi := \left\{ \mathbf{k} \in \mathbb{Z}^n : \mu^{\mathbf{k}} = 1, \mathbf{k} \neq 0 \right\} \subset \mathbb{Z}^n.$$

We remark that Theorem 4 is an improvement of Theorem 5 of [14] in two ways: one is from analytic (formal) first integrals to generalized rational first integrals, and second our result is for more than one first integral.

If there exits a non–zero $\mathbf{k} \in \mathbb{Z}^n$ such that $\mu^{\mathbf{k}} = 1$, we say that μ is *resonant.* The set Ξ in Theorem 4 is called the *resonant lattice.* For a $\mathbf{k} \in \Xi$, we say that $y^{\mathbf{k}}$ is a *resonant monomial.*

The rest of this paper is dedicated to prove our main results. The proof of Theorems 1, 2, 3 and 4 is presented in sections 2, 3, 4 and 5, respectively.

2. Proof of Theorem 1

Before proving Theorem 1 we need some preliminaries, which are the heart of the proof of Theorem 1.

Let $\mathbb{C}(x)$ be the field of rational functions in the variables of x, and $\mathbb{C}[x]$ be the ring of polynomials in x. We say that the functions $F_1(x), \ldots, F_k(x) \in \mathbb{C}(x)$ are *algebraically dependent* if there exists a complex polynomial P of k variables such that $P(F_1(x), \ldots, F_k(x)) \equiv 0$. For general definition on algebraical dependence in a more general field, see for instance [6, p.152]).

The first result provides an equivalent condition on functional independence. The main idea of the proof follows from that of Ito [12, Lemma 9.1].

Lemma 5. *The functions $F_1(x),\ldots,F_k(x)\in\mathbb{C}(x)$ are algebraically independent if and only if they are functionally independent.*

Proof. Sufficiency. By contradiction, if $F_1(x),\ldots,F_k(x)$ are algebraically dependent, then there exists a complex polynomial $P(z_1,\ldots,z_k)$ of minimal degree such that

$$P(F_1(x),\ldots,F_k(x))\equiv 0. \tag{7}$$

Here minimal degree means that for any polynomial $Q(z_1,\ldots,z_k)$ of degree less than deg$P$ we have that $Q(F_1(x),\ldots,F_k(x))\not\equiv 0$.

From (7) it follows that $\partial_{x_j}P(F_1(x),\ldots,F_k(x))\equiv 0$ for $j=1,\ldots,n$. These are equivalent to

$$\begin{pmatrix} \dfrac{\partial F_1(x)}{\partial x_1} & \cdots & \dfrac{\partial F_k(x)}{\partial x_1}\\ \vdots & \ddots & \vdots\\ \dfrac{\partial F_1(x)}{\partial x_n} & \cdots & \dfrac{\partial F_k(x)}{\partial x_n}\end{pmatrix}\begin{pmatrix} \dfrac{\partial P}{\partial z_1}(F_1(x),\ldots,F_k(x))\\ \vdots\\ \dfrac{\partial P}{\partial z_k}(F_1(x),\ldots,F_k(x))\end{pmatrix}\equiv 0. \tag{8}$$

Since $P(z_1,\ldots,z_n)$ has the minimal degree and some of the derivatives $\dfrac{\partial P}{\partial z_l}(z_1,\ldots,z_n)\neq 0$, it follows that

$$\frac{\partial P}{\partial z_1}(F_1(x),\ldots,F_k(x))\equiv 0,\ \ldots,\ \frac{\partial P}{\partial z_k}(F_1(x),\ldots,F_k(x))\equiv 0,$$

cannot simultaneously hold. This shows that the rank of the $n\times k$ matrix in (8) is less than k. Consequently $F_1(x),\ldots,F_k(x)$ are functionally dependent, this is in contradiction with the assumption. So we have proved that if $F_1(x),\ldots,F_n(x)$ are functionally independent, then they are algebraically independent.

Necessity. For proving this part, we will use the theory of field extension. For any $v_1,\ldots,v_r$ which are elements of some finitely generated field extension of $\mathbb{C}$, we denote by $\mathbb{C}(v_1,\ldots,v_r)$ the minimal field containing $v_1,\ldots,v_r$ (for more information on finitely generated field extensions, see for instance [6, Chapter one, § 8]).

Recall that for a finitely generated field extension K of the field $\mathbb{C}$, denoted by $K/\mathbb{C}$, the *transcendence degree* of K over $\mathbb{C}$ is defined to be the smallest integer m such that for some $y_1,\ldots,y_m\in K$, K is algebraic over $\mathbb{C}(y_1,\ldots,y_m)$, the field of complex coefficient rational functions in $y_1,\ldots,y_m$ (see for instance [6, p.149]). We say that K is *algebraic* over $\mathbb{C}(y_1,\ldots,y_n)$ if every element, saying k, of K is algebraic over $\mathbb{C}(y_1,\ldots,y_m)$, i.e. there exists a monic polynomial $p(X)=X^l+p_1X^{l-1}+\ldots+p_l\in\mathbb{C}(y_1,\ldots,y_m)[X]$ such that $p(k)=0$, where by definition $p_j\in\mathbb{C}(y_1,\ldots,y_m)$ for $j=1,\ldots,l$

(see for instance [6, p.29]). The elements $\{y_1, \ldots, y_m\}$ is called a *transcendence base* of K over $\mathbb{C}$ (see for instance [6, p.152]). A finitely generated field extension K of $\mathbb{C}$ is *separably generated* if there is a transcendence base $\{z_1, \ldots, z_m\}$ of K over $\mathbb{C}$ such that K is a separable algebraic extension of $\mathbb{C}(z_1, \ldots, z_m)$ (see for instance [11, p.27]). Let K be a field extension of a field L. The field extension K/L is called *algebraic* if K is algebraic over L. An algebraic extension K/L is *separable* if for every $\alpha \in K$, the minimal polynomial of α over L is separable. A polynomial is *separable* over a field L if all of its irreducible factors have distinct roots in an algebraic closure of L.

Since $F_1, \ldots, F_k$ are algebraically independent, $\mathbb{C}(F_1, \ldots, F_k)$ is a separably generated and finitely generated field extension of $\mathbb{C}$ of transcendence degree k. Here separabililty follows from the fact that $\mathbb{C}$ is of characteristic 0, and so is $\mathbb{C}(F_1, \ldots, F_k)$. This last claim follows from [11, Theorem 4.8A], which states that if k is an algebraically closed field, then any finitely generated field extension K of k is separably generated.

From the theory of derivations over a field (see for instance [13, Chapter X, Theorem 10]), there exist k derivations $D_r (r = 1, \ldots, k)$ on $\mathbb{C}(F_1, \ldots, F_k)$ satisfying

$$D_r F_s = \delta_{rs}, \tag{9}$$

where $\delta_{rs} = 0$ if $r \neq s$, or $\delta_{rs} = 1$ if $r = s$.

Since $F_1, \ldots, F_k$ are algebraically independent, it follows that $\mathbb{C}(x)$ is a finitely generated field extension of $\mathbb{C}(F_1, \ldots, F_k)$ of transcendence degree $n-k$. Hence there exist n derivations $\widetilde{D}_1, \ldots, \widetilde{D}_n$ on $\mathbb{C}(x)$ satisfying $\widetilde{D}_j = D_j$ on $\mathbb{C}(F_1, \ldots, F_k)$ for $j = 1, \ldots, k$. In addition, all derivations on $\mathbb{C}(x)$ form an n–dimensional vector space over $\mathbb{C}(x)$ with base $\{\frac{\partial}{\partial x_j} : j = 1, \ldots, n\}$. So we have

$$\widetilde{D}_s = \sum_{j=1}^{n} d_{sj} \frac{\partial}{\partial x_j},$$

where $d_{sj} \in \mathbb{C}(x)$. The derivations $\widetilde{D}_s$ acting on $\mathbb{C}(F_1, \ldots, F_k)$ satisfy

$$\delta_{sr} = D_s F_r = \widetilde{D}_s F_r = \sum_{j=1}^{n} d_{sj} \frac{\partial F_r}{\partial x_j}, \qquad r, s \in \{1, \ldots, k\}.$$

This shows that the gradients $\nabla_x F_1, \ldots, \nabla_x F_k$ have the rank k, and consequently $F_1, \ldots, F_k$ are functionally independent. $\square$

We remark that Lemma 5 has a relation in some sense with the result of Bruns in 1887 (see [7]), which stated that if a polynomial differential system of dimension n has $l \, (1 \leq l \leq n-1)$ independent algebraic first integrals, then it has l independent rational first integrals. For a short proof of this result, see Lemma 2.4 of Goriely [10].

For an analytic or a polynomial function $F(x)$ in $(\mathbb{C}^n, 0)$, in what follows we denote by $F^0(x)$ its lowest degree homogeneous term. For a rational or a generalized rational function $F(x) = G(x)/H(x)$ in $(\mathbb{C}^n, 0)$, we denote by $F^0(x)$ the rational function $G^0(x)/H^0(x)$. We expand the analytic functions $G(x)$ and $H(x)$ as

$$G^0(x) + \sum_{i=1}^{\infty} G^i(x) \quad \text{and} \quad H^0(x) + \sum_{i=1}^{\infty} H^i(x),$$

where $G^i(x)$ and $H^i(x)$ are homogeneous polynomials of degrees $\deg G^0(x) + i$ and $\deg H^0(x) + i$, respectively. Then we have

$$\begin{aligned} F(x) = \frac{G(x)}{H(x)} &= \left(\frac{G^0(x)}{H^0(x)} + \sum_{i=1}^{\infty} \frac{G^i(x)}{H^0(x)} \right) \left(1 + \sum_{i=1}^{\infty} \frac{H^i(x)}{H^0(x)} \right)^{-1} \\ &= \frac{G^0(x)}{H^0(x)} + \sum_{i=1}^{\infty} \frac{A^i(x)}{B^i(x)}, \end{aligned} \tag{10}$$

where $A^i(x)$ and $B^i(x)$ are homogeneous polynomials. Clearly

$$\deg G^0(x) - \deg H^0(x) < \deg A^i(x) - \deg B^i(x) \qquad \text{for all } i \geq 1.$$

In what follows we will say that $\deg A^i(x) - \deg B^i(x)$ is the degree of $A^i(x)/B^i(x)$, and $G^0(x)/H^0(x)$ is the lowest degree term of $F(x)$ in the expansion (10). For simplicity we denote

$$d(G) = \deg G^0(x), \qquad d(F) = d(G) - d(H) = \deg G^0(x) - \deg H^0(x),$$

and call $d(F)$ the *lowest degree* of F.

Lemma 6. *Let*

$$F_1(x) = \frac{G_1(x)}{H_1(x)}, \ldots, F_m(x) = \frac{G_m(x)}{H_m(x)},$$

be functionally independent generalized rational functions in $(\mathbb{C}^n, 0)$*. Then there exist polynomials* $P_i(z_1, \ldots, z_m)$ *for* $i = 2, \ldots, m$ *such that* $F_1(x), \widetilde{F}_2(x) = P_2(F_1(x), \ldots, F_m(x)), \ldots, \widetilde{F}_m(x) = P_m(F_1(x), \ldots, F_m(x))$ *are functionally independent generalized rational functions, and that* $F_1^0(x), \widetilde{F}_2^0(x), \ldots, \widetilde{F}_m^0(x)$ *are functionally independent rational functions.*

Proof. This result was first proved by Ziglin [22] in 1983, and then proved by Baider *et al* [1] in 1996. In order that this paper is self–contained, we provide a proof here (see also the idea of the proof of Lemma 2.1 of [12]).

If $F_1^0(x), \ldots, F_m^0(x)$ are functionally independent, the proof is done.

Without loss of generality we assume that $F_1^0(x), \ldots, F_k^0(x)$ $(1 \leq k < m)$ are functionally independent, and $F_1^0(x), \ldots, F_{k+1}^0(x)$ are functionally dependent. By Lemma 5 it follows that $F_1^0(x), \ldots, F_{k+1}^0(x)$ are algebraically dependent. So there exists a polynomial $P(z)$ of minimal degree with $z = (z_1, \dots, z_{k+1})$ such that

$$P(F_1^0(x), \dots, F_{k+1}^0(x)) \equiv 0.$$

The fact that $F_1^0(x), \dots, F_k^0(x)$ are algebraically independent implies

$$\frac{\partial P}{\partial z_{k+1}}(z) \not\equiv 0.$$

Since $F_1(x), \dots, F_m(x)$ are functionally independent, and so also $F_1(x), \dots, F_{k+1}(x)$ are functionally independent. Hence there exists a $(k+1) \times (k+1)$ minor $M = \frac{\partial(F_1(x),\dots,F_{k+1}(x))}{\partial(x_{i_1},\dots,x_{i_{k+1}})}$ of the matrix $\frac{\partial(F_1(x),\dots,F_{k+1}(x))}{\partial(x_1,\dots,x_n)}$ such that its determinant does not vanish. We denote by $\mathcal{D}$ this last determinant, and denote by $\mathcal{D}^0$ the determinant of the square matrix $M^0 := \frac{\partial(F_1^0(x),\dots,F_{k+1}^0(x))}{\partial(x_{i_1},\dots,x_{i_{k+1}})}$. We note that $\mathcal{D}^0$ is the lowest degree rational homogeneous term of $\mathcal{D}$.

Define

$$\mu(F_1, \dots, F_{k+1}) = d(\mathcal{D}) + k + 1 - \sum_{j=1}^{k+1} d(F_j),$$

where $d(\mathcal{D})$ is well–defined, because $\mathcal{D}$ is also a generalized rational function. Since $\partial F_i/\partial x_j = \left(H_i \frac{\partial G_i}{\partial x_j} - G_i \frac{\partial H_i}{\partial x_j} \right) / (H_i)^2$ has the lowest degree larger than or equal to $d(F_i) - 1$ (the former happens if $H_i^0 \frac{\partial G_i^0}{\partial x_j} - G_i^0 \frac{\partial H_i^0}{\partial x_j} \equiv 0$), it follows from the definition of $\mathcal{D}$ that

$$\mu(F_1, \dots, F_{k+1}) \geq 0.$$

Furthermore $\mu(F_1, \dots, F_{k+1}) = 0$ if and only if $d(\mathcal{D}^0) = d(\mathcal{D})$, because $\mathcal{D}^0$ is the lowest degree rational homogeneous part of $\mathcal{D}$. We note that $d(\mathcal{D}^0) = d(\mathcal{D})$ if and only if $\det(\mathcal{D}^0) \neq 0$, and this is equivalent to the functional independence of $F_1^0(x), \dots, F_{k+1}^0(x)$. So by the assumption we have $\mu(F_1, \dots, F_{k+1}) > 0$. Set

$$\widehat{F}_{k+1}(x) = P(F_1(x), \dots, F_{k+1}(x)).$$

First we claim that the functions $F_1(x), \dots, F_k(x), \widehat{F}_{k+1}(x)$ are functionally independent. Indeed, define

$$\widehat{\mathcal{D}} := \det(\partial(F_1(x), \dots, F_k(x), \widehat{F}_{k+1}(x))/\partial(x_{i_1}, \dots, x_{i_k}, x_{i_{k+1}}).$$

Then it follows from the functional independence of $F_1(x), \dots F_{k+1}(x)$ that

$$\begin{aligned} \widehat{\mathcal{D}} &= \det\left(\frac{\partial(F_1(x), \dots, F_k(x), \widehat{F}_{k+1}(x))}{\partial(F_1, \dots, F_k, F_{k+1})} \, \frac{\partial(F_1, \dots, F_k, F_{k+1})}{\partial(x_{i_1}, \dots, x_{i_k}, x_{i_{k+1}})} \right) \\ &= \mathcal{D} \frac{\partial P}{\partial z_{k+1}}(F_1, \dots, F_{k+1}). \qquad (11) \end{aligned}$$

The first equality is obtained by using the derivative of composite functions, and the second equality follows from the fact that

$$\det\left(\frac{\partial(F_1(x),\dots,F_k(x),\widehat{F}_{k+1}(x))}{\partial(F_1,\dots,F_k,F_{k+1})}\right)=\frac{\partial P}{\partial z_{k+1}}(F_1,\dots,F_{k+1}).$$

Since $(\partial P/\partial z_{k+1})(z_1,\dots,z_{k+1})\not\equiv 0$ and P has the minimal degree such that $P(F_1^0(x),\dots,F_{k+1}^0(x))\equiv 0$, we have $(\partial P/\partial z_{k+1})(F_1^0(x),\dots,F_{k+1}^0(x))\not\equiv 0$, and so $\widehat{\mathcal{D}}\not\equiv 0$. This proves that $F_1(x),\dots,F_k(x),\widehat{F}_{k+1}(x)$ are functionally independent. The claim follows.

Second we claim that

$$\mu(F_1,\dots,F_k,\widehat{F}_{k+1})<\mu(F_1,\dots,F_{k+1}).$$

Indeed, writing the polynomial P as the summation

$$P(z)=\sum_\alpha p_\alpha z^\alpha,\quad z^\alpha=z_1^{\alpha_1}\cdot\ldots\cdot z_{k+1}^{\alpha_{k+1}},\ \ \alpha=(\alpha_1,\dots,\alpha_{k+1})\in(\mathbb{Z}^+)^{k+1}.$$

Define

$$\nu=\min\{\langle\alpha,\,(d(F_1),\dots,d(F_{k+1})\rangle\,:\,p_\alpha\neq 0,\ \alpha_{k+1}\neq 0\}.$$

We have $\nu<d(\widehat{F}_{k+1})$, because $\widehat{F}_{k+1}(x)=P(F_1(x),\dots,F_{k+1}(x))$ contains F_{k+1} and $P(F_1^0(x),\dots,F_{k+1}^0(x))\equiv 0$, so the lowest degree part in the expansion of $P(z)$ must contain $F_{k+1}(x)$. This implies the lowest degree of $\widehat{F}(x)$ is larger than that of $P(z)$. Moreover we get from (11) and the definition of ν that

$$d(\widehat{\mathcal{D}})=d(\mathcal{D})+d\left(\frac{\partial P}{\partial z_{k+1}}(F_1,\dots,F_{k+1})\right)=d(\mathcal{D})+\nu-d(F_{k+1}),$$

where the last equality holds because the partial derivative of P with respect to z_{k+1} is such that P loses one F_{k+1}, and so the total degree loses the degree of F_{k+1}. Hence from the definition of the quantity μ it follows that

$$\begin{aligned}\mu(F_1,\dots,F_k,\widehat{F}_{k+1}) &= d(\widehat{\mathcal{D}})+k+1-\sum_{j=1}^k d(F_j)-d(\widehat{F}_{k+1})\\ &= \mu(F_1,\dots,F_{k+1})+\nu-d(\widehat{F}_{k+1})\\ &< \mu(F_1,\dots,F_{k+1}).\end{aligned}$$

This proves the claim.

By the two claims, from the functionally independent generalized rational functions $F_1(x),\dots,F_k(x),F_{k+1}(x)$ with $F_1^0(x),\dots,F_k^0(x),F_{k+1}^0(x)$ being functionally dependent, we get functionally independent generalized rational functions $F_1(x),\dots,F_k(x),\widehat{F}_{k+1}(x)$, which satisfy $\mu(F_1,\dots,F_k,\widehat{F}_{k+1})<\mu(F_1,\dots,F_{k+1})$.

If $\mu(F_1,\dots,F_k,\widehat{F}_{k+1})=0$, then $F_1^0(x),\dots,F_k^0(x),\widehat{F}_{k+1}^0(x)$ are functionally independent. The proof is done.

If $\mu(F_1,\ldots,F_k,\widehat{F}_{k+1})>0$, then $F_1^0(x),\ldots,F_k^0(x),\widehat{F}_{k+1}^0$ are also functionally dependent. Continuing the above the procedure, finally we can get a polynomial $\widetilde{P}(z)$ with $z=(z_1,\ldots,z_{k+1})$ such that

$$F_1(x),\ldots,F_k(x),\widetilde{F}_{k+1}(x)=\widetilde{P}(F_1(x),\ldots,F_{k+1}(x)),$$

are functionally independent and $\mu(F_1,\ldots,F_k,\widetilde{F}_{k+1})=0$. The last equality implies that the rational functions $F_1^0,\ldots,F_k^0,\widetilde{F}_{k+1}^0$ are functionally independent. Furthermore, the generalized rational functions

$$F_1(x),\ldots,F_k(x),\widetilde{F}_{k+1}(x),F_{k+2}(x),\ldots,F_m(x),$$

are functionally independent, because $\widetilde{F}_{k+1}(x)$ involves only $F_1,\ldots,F_{k+1}$, and $F_1,\ldots,F_k,\widetilde{F}_{k+1}$ are functionally independent, and also $F_1,\ldots,F_m$ are functionally independent. This can also be obtained by direct calculations as follows

$$\begin{aligned}
&\det\left(\frac{\partial(F_1(x),\ldots,F_k(x),\widetilde{F}_{k+1}(x),F_{k+2}(x),\ldots,F_m(x))}{\partial(x_1,\ldots,x_n)}\right)\\
=\ &\det\begin{pmatrix}
\frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n}\\
\vdots & \vdots & \vdots\\
\frac{\partial F_k}{\partial x_1} & \cdots & \frac{\partial F_k}{\partial x_n}\\
\frac{\partial \widetilde{P}}{\partial z_1}\frac{\partial F_1}{\partial x_1}+\ldots+\frac{\partial \widetilde{P}}{\partial z_{k+1}}\frac{\partial F_{k+1}}{\partial x_1} & \cdots & \frac{\partial \widetilde{P}}{\partial z_1}\frac{\partial F_1}{\partial x_n}+\ldots+\frac{\partial \widetilde{P}}{\partial z_{k+1}}\frac{\partial F_{k+1}}{\partial x_n}\\
\frac{\partial F_{k+1}}{\partial x_1} & \cdots & \frac{\partial F_{k+2}}{\partial x_n}\\
\vdots & \vdots & \vdots\\
\frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n}
\end{pmatrix}\\
=\ &\frac{\partial \widetilde{P}}{\partial z_{k+1}}(x)\ \det\left(\frac{\partial(F_1(x),\ldots,F_m(x))}{\partial(x_1,\ldots,x_n)}\right)\neq 0.
\end{aligned}$$

If $k+1=m$, the proof is completed. Otherwise we can continue the above procedure, and finally we get the functionally independent generalized rational functions $F_1(x),\ldots,F_k(x),\widetilde{F}_{k+1}(x)=\widetilde{P}_{k+1}(F_1(x),\ldots,F_{k+1}(x)),\ldots,$ $\widetilde{F}_m(x)=\widetilde{P}_m(F_1(x),\ldots,F_m(x))$ such that their lowest order rational functions $F_1^0(x),F_k^0(x),\widetilde{F}_{k+1}^0(x),\ldots,\widetilde{F}_m^0(x)$ are functionally independent, where $\widetilde{P}_j$ for $j=k+1,\ldots,m$ are polynomials in $F_1,\ldots,F_j$.

The proof of the lemma is completed. □

The next result characterizes rational first integrals of system (1). A *rational monomial* is by definition the ratio of two monomials, i.e. of the form $x^{\mathbf{k}}/x^{\mathbf{l}}$ with $\mathbf{k},\mathbf{l}\in(\mathbb{Z}^+)^n$. The rational monomial $x^{\mathbf{k}}/x^{\mathbf{l}}$ is *resonant* if $\langle\lambda,\mathbf{k}-\mathbf{l}\rangle=0$. A rational function is *homogeneous* if its denominator and numerator are both homogeneous polynomials. A rational homogeneous function is *resonant* if the ratio of any two elements in the set of all its monomials in both denominator and numerator is a resonant rational monomial.

In system (1) we assume without loss of generality that A is in its Jordan normal form and is a lower triangular matrix. Set $f(x) = Ax + g(x)$ with $g(x) = O(x^2)$. The vector field associated to (1) is written in

$$\mathcal{X} = \mathcal{X}_1 + \mathcal{X}_h := \langle Ax, \partial_x \rangle + \langle g(x), \partial_x \rangle.$$

Lemma 7. *If $F(x) = G(x)/H(x)$ is a generalized rational first integral of the vector field $\mathcal{X}$ defined by* (1)*, then $F^0(x) = G^0(x)/H^0(x)$ is a resonant rational homogeneous first integral of the linear vector field $\mathcal{X}_1$, where we assume that F^0 is non–constant, otherwise if $F^0(x) \equiv a \in \mathbb{C}$, then we consider $(F - a)^0$, which is not a constant.*

For proving this last lemma we will use the following result (see Lemma 1.1 of [2], for a different proof see for example [14]).

Lemma 8. *Let $\mathcal{H}_n^m$ be the linear space of complex coefficient homogeneous polynomials of degree m in n variables. For any constant $c \in \mathbb{C}$, define a linear operator on $\mathcal{H}_n^m$ by*

$$L_c(h)(x) = \langle \partial_x h(x), Ax \rangle - c\, h(x), \qquad h(x) \in \mathcal{H}_n^m.$$

Then the spectrum of L_c is

$$\{\langle \mathbf{k}, \lambda \rangle - c : \ \mathbf{k} \in (\mathbb{Z}^+)^n, \ |\mathbf{k}| = k_1 + \ldots + k_n = m\},$$

where λ are the eigenvalues of A.

Proof of Lemma 7. As in (10) we write $F(x)$ in

$$F(x) = F^0(x) + \sum_{i=1}^{\infty} F^i(x),$$

where $F^0(x)$ is the lowest order rational homogeneous function and $F^i(x)$ for $i \in \mathbb{N}$ are rational homogeneous functions of order larger than $F^0(x)$. That $F(x)$ is a first integral in a neighborhood of $0 \in \mathbb{C}^n$ is equivalent to

$$\langle \partial_x F(x), f(x) \rangle \equiv 0, \qquad x \in (\mathbb{C}^n, 0).$$

Equating the lowest order rational homogeneous functions gives

$$\langle \partial_x F^0(x), Ax \rangle \equiv 0, \qquad i.e. \quad \left\langle \partial_x \left(\frac{G^0(x)}{H^0(x)} \right), Ax \right\rangle \equiv 0. \tag{12}$$

This shows that $F^0(x)$ is a rational homogeneous first integral of the linear system associated with (1).

Next we shall prove that $F^0(x)$ is resonant. From the equality (12) we can assume without loss of generality that $G^0(x)$ and $H^0(x)$ are relative prime. Now equation (12) can be written as

$$H^0(x) \left\langle \partial_x G^0(x), Ax \right\rangle - G^0(x) \left\langle \partial_x H^0(x), Ax \right\rangle \equiv 0.$$

So there exists a constant c such that

$$\big\langle \partial_x G^0(x), Ax \big\rangle - cG^0(x) \equiv 0, \quad \big\langle \partial_x H^0(x), Ax \big\rangle - cH^0(x) \equiv 0.$$

Set $\deg G^0(x) = l$, $\deg H^0(x) = m$ and L_c be the linear operator defined in Lemma 8. Recall from Lemma 8 that L_c has respectively the spectrums on $\mathcal{H}_n^l$

$$\mathcal{S}_l := \{\langle \mathbf{l}, \lambda\rangle - c : \ \mathbf{l} \in (\mathbb{Z}^+)^n, \ |\mathbf{l}| = l\},$$

and on $\mathcal{H}_n^m$

$$\mathcal{S}_m := \{\langle \mathbf{m}, \lambda\rangle - c : \ \mathbf{m} \in (\mathbb{Z}^+)^n, \ |\mathbf{m}| = m\}.$$

Separate $\mathcal{H}_n^l = \mathcal{H}_{n1}^l + \mathcal{H}_{n2}^l$ in such a way that for any $p(x) \in \mathcal{H}_{n1}^l$ its monomial $x^{\mathbf{l}}$ satisfies $\langle \mathbf{l}, \lambda\rangle - c = 0$, and for any $q(x) \in \mathcal{H}_{n2}^l$ its monomial $x^{\mathbf{l}}$ satisfies $\langle \mathbf{l}, \lambda\rangle - c \neq 0$. Separate $G^0(x)$ in two parts $G^0(x) = G_1^0(x) + G_2^0(x)$ with $G_1^0 \in \mathcal{H}_{n1}^l$ and $G_2^0 \in \mathcal{H}_{n2}^l$. Since A is in its Jordan normal form and is lower triangular, it follows that

$$L_c\mathcal{H}_{n1}^l \subset \mathcal{H}_{n1}^l, \quad \text{and} \quad L_c\mathcal{H}_{n2}^l \subset \mathcal{H}_{n2}^l.$$

Hence $L_cG^0(x) \equiv 0$ is equivalent to

$$L_cG_1^0(x) \equiv 0 \quad \text{and} \quad L_cG_2^0(x) \equiv 0.$$

Since L_c has the spectrum without zero element on $\mathcal{H}_{n2}^l$ and so it is invertible $\mathcal{H}_{n2}^l$, the equation $L_cG_2^0(x) \equiv 0$ has only the trivial solution, i.e. $G_2^0(x) \equiv 0$. This proves that $G^0(x) = G_1^0(x)$, i.e. each monomial, say $x^{\mathbf{l}}$, of $G^0(x)$ satisfies $\langle \mathbf{l}, \lambda\rangle - c = 0$.

Similarly we can prove that each monomial, say $x^{\mathbf{m}}$, of $H^0(x)$ satisfies $\langle \mathbf{m}, \lambda\rangle - c = 0$. This implies that $\langle \mathbf{l} - \mathbf{m}, \lambda\rangle = 0$. The above proofs show that $F^0(x) = G^0(x)/H^0(x)$ is a resonant rational homogeneous first integral of $\mathcal{X}_1$. □

Having the above lemmas we can prove Theorem 1.

Proof of Theorem 1. Let

$$F_1(x) = \frac{G_1(x)}{H_1(x)}, \ldots, F_m(x) = \frac{G_m(x)}{H_m(x)},$$

be the m functionally independent generalized rational first integrals of $\mathcal{X}$. Since the polynomial functions of $F_i(x)$ for $i = 1, \ldots, m$ are also generalized rational first integrals of $\mathcal{X}$, so by Lemma 6 we can assume without loss of generality that

$$F_1^0(x) = \frac{G_1^0(x)}{H_1^0(x)}, \ldots, F_m^0(x) = \frac{G_m^0(x)}{H_m^0(x)},$$

are functionally independent.

Lemma 7 shows that $F_1^0(x), \ldots, F_m^0(x)$ are resonant rational homogeneous first integrals of the linear vector field $\mathcal{X}_1$, that is, these first integrals are rational functions in the variables given by resonant rational monomials. According to the linear algebra (see for instance [3]), the square matrix A

in $\mathbb{C}$ has a unique representation in the form $A = A_s + A_n$ with A_s semi–simple and A_n nilpotent and $A_sA_n = A_nA_s$. The semi–simple matrix A_s is similar to a diagonal matrix. Without loss of generality we assume that A_s is diagonal, i.e. $A_s = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Define $\mathcal{X}_s = \langle A_s x, \partial_x \rangle$ and $\mathcal{X}_n = \langle A_n x, \partial_x \rangle$. Separate $\mathcal{X}_1 = \mathcal{X}_s + \mathcal{X}_n$. Direct calculations show that any resonant rational monomial is a first integral of $\mathcal{X}_s$ (for example, let x^m be a resonant rational monomial, i.e. it satisfies $\langle \lambda, m \rangle = 0$. Then $\mathcal{X}_s(x^m) = \langle \lambda, m \rangle x^m = 0$). So $F_1^0(x), \ldots, F_m^0(x)$ are also first integrals of X_s. This means that m is less than or equal to the number of functionally independent resonant rational monomials. In addition, we can show that the number of functionally independent resonant rational monomials is equal to the maximum number of linearly independent vectors in $\mathbb{R}^n$ of the set $\{\mathbf{k} \in \mathbb{Z}^n : \langle \mathbf{k}, \lambda \rangle = 0\}$.

This completes the proof of the theorem. □

3. Proof of Theorem 2

We only consider system (1) to be positively semi–quasi–homogeneous. The negative case can be studied similarly, and its details are omitted. An analytic function $w(x)$ is *semi–quasi–homogeneous of degree* k with weight exponent $\mathbf{s}$ if $w(x) = w_k(x) + w_h(x)$, where w_k is quasi–homogeneous of degree k and w_h is the sum of quasi–homogeneous polynomials of degree larger than k. A rational function $G(x)/H(x)$ is *rational quasi–homogeneous* with weight exponent $\mathbf{s}$ if $G(x)$ and $H(x)$ are both quasi–homogeneous with weight exponent $\mathbf{s}$. In this section, for an analytic function $w(x)$ we denote by $w^{(q)}(x)$ its lowest degree quasi–homogeneous part. For a generalized rational function $F(x) = G(x)/H(x)$ we denote by $F^{(q)}(x)$ the rational quasi–homogeneous function $G^{(q)}(x)/H^{(q)}(x)$.

The following result is the key point for proving Theorem 2, which is a generalization of Lemma 6 to rational quasi–homogenous functions. Its proof can be obtained in the same way as that of Lemma 6, where we replace the usual degree by the weight degree. The details are omitted.

Lemma 9. *Let*

$$F_1(x) = \frac{G_1(x)}{H_1(x)}, \ldots, F_m(x) = \frac{G_m(x)}{H_m(x)},$$

be functionally independent generalized rational functions in $(\mathbb{C}^n, 0)$ *with* G_i *and* H_i *semi–quasi–homogeneous for* $i = 1, \ldots, m$. *Then there exist polynomials* $P_i(z_1, \ldots, z_m)$ *for* $i = 2, \ldots, m$ *such that* $F_1(x), \widetilde{F}_2(x) = P_2(F_1(x), \ldots, F_m(x)), \ldots, \widetilde{F}_m(x) = P_m(F_1(x), \ldots, F_m(x))$ *are functionally independent generalized rational functions, and that* $F_1^{(q)}(x), \widetilde{F}_2^{(q)}(x), \ldots, \widetilde{F}_m^{(q)}(x)$ *are functionally independent rational quasi–homogeneous functions.*

Now we shall prove Theorem 2. Since system (1) is semi–quasi–homogeneous of degree $q > 1$, we take the change of variables

$$x \to \rho^{\mathbf{S}} x, \quad t \to \rho^{-(q-1)} t,$$

where $\rho^{\mathbf{S}} = \mathrm{diag}\,(\rho^{s_1}, \dots, \rho^{s_n})$. System (1) is transformed into

$$\dot{x} = f_q(x) + \widetilde{f}_h(x, \rho), \tag{13}$$

where $\widetilde{f}_h(x,\rho) = \sum_{i \geq 1} \rho^i \widetilde{f}_{q+i}(x)$ and $\rho^{\mathbf{E}-\mathbf{S}} \widetilde{f}_{q+i}(x)$ is quasi–homogeneous of weight degree $q + i$.

If $F(x) = G(x)/H(x)$ is a generalized rational first integral of system (1) with $G(x)$ and $H(x)$ semi–quasi–homogeneous of weight degree l and m with weight exponent $\mathbf{s}$ respectively, then

$$F(x, \rho) := \frac{\rho^m G(\rho^{\mathbf{S}} x)}{\rho^l H(\rho^{\mathbf{S}} x)} = F^{(q)}(x) + \dots,$$

is a generalized rational first integral of the semi–quasi–homogeneous system (13), where the dots denote the sum of the higher order rational quasi–homogeneous functions. Some easy calculations show that $F^{(q)}(x)$ is a rational quasi–homogeneous first integral of the quasi–homogeneous system

$$\dot{x} = f_q(x). \tag{14}$$

Let c_0 be a balance. Taking the change of variable $x = t^{-\mathbf{W}}(c_0 + u)$, then

$$F^{(q)}(x) = \frac{t^{-\frac{l}{q-1}} G^{(q)}(c_0 + u)}{t^{-\frac{m}{q-1}} H^{(q)}(c_0 + u)} = u_0^{l-m} F^{(q)}(c_0 + u),$$

where $u_0 = t^{-1/(q-1)}$ will be chosen as a new auxiliary variable. Define $F^{q0}(u_0, u) = u_0^{l-m} F^{(q)}(c_0 + u)$. System (14) is transformed into

$$u' = Ku + \overline{f}_q(u), \tag{15}$$

where the prime denotes the derivative with respect to $\tau = \ln t$ and

$$\overline{f}_q(u) = \mathbf{W} c_0 + f_q(c_0 + u) - \partial_x f_q(c_0) u.$$

We claim that $F^{q0}(u_0, u)$ is a first integral of

$$u_0' = -\frac{1}{q-1} u_0, \quad u' = Ku + \overline{f}_q(u). \tag{16}$$

Indeed, since $\left\langle \partial_x F^{(q)}(x), f_q(x) \right\rangle \equiv 0$ and $\mathbf{W} = \mathbf{S}/(q-1)$ we have

$$
\begin{aligned}
&\left.\frac{F^{q0}(u_0,u)}{d\tau}\right|_{(16)} \\
&= -\frac{l-m}{q-1} u_0^{l-m} F^{(q)}(c_0+u) + u_0^{l-m} \left\langle \partial_u F^{(q)}(c_0+u), Ku + \overline{f}_q(u) \right\rangle \\
&= -\frac{u_0^{l-m}}{q-1} \Big((l-m) F^{(q)}(c_0+u) - \langle \partial_u F^{(q)}(c_0+u), \mathbf{S}(c_0+u) \rangle \Big) \\
&= 0,
\end{aligned}
$$

where we have used the facts that $F^{(q)}(c_0+u) = G^{(q)}(c_0+u)/H^{(q)}(c_0+u)$, and the generalized Euler's formula:

$$
\left\langle \partial_x G^{(q)}(x), \mathbf{S}x \right\rangle = l G^{(q)}(x) \quad \text{and} \quad \left\langle \partial_x H^{(q)}(x), \mathbf{S}x \right\rangle = m H^{(q)}(x),
$$

because $G^{(q)}(x)$ and $H^{(q)}(x)$ are quasi–homogeneous polynomials of weight degree l and m with weight exponents $\mathbf{s}$, respectively. The claim follows.

Assume that system (1) has the maximal number, say r, of functionally independent generalized rational first integrals

$$
F_1(x) = \frac{G_1(x)}{H_1(x)}, \ldots, F_r(x) = \frac{G_r(x)}{H_r(x)}.
$$

By Lemma 9 we can assume that

$$
F_1^{(q)}(x) = \frac{G_1^{(q)}(x)}{H_1^{(q)}(x)}, \ldots, F_r^{(q)}(x) = \frac{G_r^{(q)}(x)}{H_r^{(q)}(x)},
$$

are functionally independent, and they are rational quasi–homogeneous first integrals of (14).

Assume that $G_i^{(q)}(x)$ and $H_i^{(q)}(x)$, $i = 1, \ldots, r$, have weight degree l_i and m_i, respectively. Then it follows from the last claim that

$$
F_1^{q0}(u_0,u) = u_0^{l_1-m_1} F_1^{(q)}(c_0+u), \ldots, F_r^{q0}(u_0,u) = u_0^{l_r-m_r} F_r^{(q)}(c_0+u),
$$

are functionally independent rational quasi–homogeneous first integrals of (16).

The linear part of (16) has the eigenvalues $-1/(q-1), \lambda_{c_0} = (\lambda_1^0, \ldots, \lambda_n^0)$. Since the Kowalevskaya matrix has always the eigenvalue -1 (see for instance [22]), we set $\lambda_1^0 = -1$. Then for $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ we have

$$
-k_0 + (q-1)\langle \lambda_{c_0}, \mathbf{k} \rangle = \langle \lambda_{c_0}, (k_0 + (q-1)k_1, (q-1)k_2, \ldots, (q-1)k_n) \rangle.
$$

Recall that d_{c_0} is the dimension of the minimum linear subspace of $\mathbb{R}^n$ containing the set $\{\mathbf{k} \in \mathbb{Z}^n : \langle \lambda_{c_0}, \mathbf{k} \rangle = 0\}$. Hence we get from Theorem 1 that equation (16) has at most d_{c_0} functionally independent first integrals. These proofs imply that $r \le d_{c_0}$, and consequently $r \le d = \min_{c \in \mathcal{B}} d_c$.

This completes the proof of the theorem.

4. Proof of Theorem 3

Assume that system (1) has the maximal number, say r, of functionally independent generalized rational first integrals in a neighborhood U of the given periodic orbit, denoted by

$$F_1(x) = \frac{G_1(x)}{H_1(x)}, \ldots, F_r(x) = \frac{G_r(x)}{H_r(x)},$$

where G_i and H_i are analytic functions. Let $P(x)$ be the Poincaré map defined in a neighborhood of the periodic orbit. Then $F_i(x)$ for $i = 1, \ldots, r$ are also first integrals of $P(x)$. Recall that a continuous function $C(x)$ is a *first integral* of a homeomorphism $M(x)$ defined in an open subset U of $\mathbb{C}^n$ if $C(M^m(x)) = C(x)$ for all $m \in \mathbb{Z}$ and $x \in U$.

We now turn to study the maximal number of functionally independent generalized rational first integrals of the Poincaré map. Since a polynomial function of generalized rational first integrals of a map is also a generalized rational first integral of the map, so by Lemma 6 we can assume without loss of generality that $F_1^0(x), \ldots, F_r^0(x)$ are functionally independent.

Since system (1) is analytic, the Poincaré map $P(x)$ is analytic (see for instance, [5]). Set

$$P(x) = Bx + P_h(x), \tag{17}$$

where $P_h(x)$ is the higher order terms of $P(x)$. We can assume without loss of generality that B is in its lower triangular Jordan normal form. Let B_s be the semi–simple part of B. Since $F_i(x)$ for $i = 1, \ldots, r$, are first integrals of $P(x)$, we have

$$\frac{G_i(P(x))}{H_i(P(x))} \equiv \frac{G_i(x)}{H_i(x)}, \qquad x \in U. \tag{18}$$

As in (10) we expand $G_i(P(x))/H_i(P(x))$ via (17) as the sum of rational homogeneous functions, and equating the lowest order rational homogeneous terms of (18), we get that

$$\frac{G_i^0(Bx)}{H_i^0(Bx)} \equiv \frac{G_i^0(x)}{H_i^0(x)}, \qquad x \in U. \tag{19}$$

From the last equality, we can assume without loss of generality that $G_i^0(x)$ and $H_i^0(x)$ are relatively prime. Recall that $A^0(x)$ is the lowest order homogeneous polynomial of a series $A(x)$. Equation (19) can be written as

$$\frac{G_i^0(Bx)}{G_i^0(x)} \equiv \frac{H_i^0(Bx)}{H_i^0(x)}, \qquad x \in U. \tag{20}$$

Since $G_i^0(Bx)$ is either identically zero or a homogeneous polynomial of the same degree than $G_i^0(x)$, and $G_i^0(x)$ and $H_i^0(x)$ are relative prime, there exists a constant c_i such that

$$G_i^0(Bx) \equiv c_i G_i^0(x) \quad \text{and} \quad H_i^0(Bx) = c_i H_i^0(x), \qquad x \in U. \tag{21}$$

For completing the proof of Theorem 3 we need the following result (see for instance, Lemma 11 of [14]).

Lemma 10. *Let $\mathcal{H}_n^m$ be the complex linear space of homogeneous polynomials of degree m in n variables, and let μ be the n–tuple of eigenvalues of B. Define the linear operator from $\mathcal{H}_n^m$ into itself by*

$$L_c(h)(x) = h(Bx) - ch(x), \qquad h(x) \in \mathcal{H}_n^m.$$

Then the set of eigenvalues of L_c is $\{\mu^{\mathbf{k}} - c : \mathbf{k} \in (\mathbb{Z}^+)^n, |\mathbf{k}| = m\}$.

Assume that $G_i^0(x)$ and $H_i^0(x)$ have respectively degrees l_i and m_i. Using the notations given in Lemma 10 we write equations (21) as

$$L_{c_i}(G_i^0)(x) \equiv 0 \quad \text{and} \quad L_{c_i}(H_i^0)(x) \equiv 0.$$

Working in a similar way as in the proof of Theorem 1 we obtain from Lemma 10 that each monomial, say $x^{\mathbf{l}}$, of $G_i^0(x)$ satisfies $\mu^{\mathbf{l}} - c_i = 0$, where $\mathbf{l} \in (\mathbb{Z}^+)^n$ and $|\mathbf{l}| = l_i$, and that each monomial, say $x^{\mathbf{m}}$, of $H_i^0(x)$ satisfies $\mu^{\mathbf{m}} - c_i = 0$, where $\mathbf{m} \in (\mathbb{Z}^+)^n$ and $|\mathbf{m}| = m_i$.

The above proof shows that the ratio of any two monomials in the numerator and denominator of $F_i^0(x)$, $i \in \{1, \ldots, r\}$, is a resonant monomial. Hence each of the ratios is a first integral of the vector field $B_s x$, and consequently $F_i^0(x)$, $i = 1, \ldots, r$, are first integrals of $B_s x$. In addition, we can check easily that the maximal number of functionally independent elements of $\{x^{\mathbf{k}} : \mu^{\mathbf{k}} = 1, \mathbf{k} \in \mathbb{Z}^n, \mathbf{k} \neq 0\}$ is equal to the dimension of the minimal linear subspace in $\mathbb{R}^n$ containing the set $\{\mathbf{k} \in \mathbb{Z}^n : \mu^{\mathbf{k}} = 1\}$. This proves the theorem.

5. Proof of Theorem 4

For proving Theorem 4 we need the following result, which is a modification of Lemma 6. Its proof can be got in the same way as the proof of Lemma 6, where we replace the field $\mathbb{C}$ by $\widetilde{\mathbb{C}}(t)$ the field of complex coefficient generalized rational functions in t. The details are omitted.

Lemma 11. *Let*

$$F_1(t,x) = \frac{G_1(t,x)}{H_1(t,x)}, \ldots, F_m(t,x) = \frac{G_m(t,x)}{H_m(t,x)}, \qquad (t,x) \in \mathbb{S}^1 \times (\mathbb{C}^n, 0),$$

be functionally independent generalized rational functions and 2π periodic in t. Then there exist polynomials $P_i(z_1, \ldots, z_m)$ for $i = 2, \ldots, m$ such that $F_1(t,x), \widetilde{F}_2(t,x) = P_2(F_1(t,x), \ldots, F_m(t,x)), \ldots, \widetilde{F}_m(t,x) = P_m(F_1(t,x), \ldots, F_m(t,x))$ are functionally independent generalized rational functions, and that $F_1^0(t,x), \widetilde{F}_2^0(t,x), \ldots, \widetilde{F}_m^0(t,x)$ are functionally independent rational homogeneous functions.

In the proof of Theorem 4 we also need the Floquet's Theorem. For readers' convenience we state it here.

Floquet's Theorem *There exists a change of variables $x = B(t)y$ periodic of period 2π in t, which transforms the linear periodic differential system* (6) *into the linear autonomous one*

$$\dot{y} = \Lambda y, \qquad \Lambda \text{ is a constant matrix.}$$

Furthermore the characteristic multipliers μ of (6) *satisfy $\mu_i = \exp(2\pi\lambda_i)$ for $i = 1, \ldots, n$, where $\lambda = (\lambda_1, \ldots, \lambda_n)$ are the eigenvalues of Λ.*

Now we can prove Theorem 4. Assume that system (5) has the maximal number, say m, of functionally independent generalized rational first integrals, denoted by

$$F_1(t,x) = \frac{G_1(t,x)}{H_1(t,x)}, \ldots, F_m(t,x) = \frac{G_m(t,x)}{H_m(t,x)}.$$

By Lemma 11 we can assume without loss of generality that the rational homogeneous functions

$$F_1^0(t,x) = \frac{G_1^0(t,x)}{H_1^0(t,x)}, \ldots, F_m^0(t,x) = \frac{G_m^0(t,x)}{H_m^0(t,x)},$$

are functionally independent.

By the Floquet's Theorem, system (5) is transformed via a change of the form $x = B(t)y$ into

$$\dot{y} = \Lambda y + h(t,y), \tag{22}$$

where $h(t,y) = O(y^2)$ is 2π periodic in t. Therefore system (22) has the functionally independent generalized rational first integrals

$$\widetilde{F}_1(t,y) = \frac{\widetilde{G}_1(t,y)}{\widetilde{H}_1(t,x)} = \frac{G_1(t,B(t)y)}{H_1(t,B(t)y)}, \ldots, \widetilde{F}_m(t,y) = \frac{\widetilde{G}_m(t,y)}{\widetilde{H}_m(t,x)} = \frac{G_m(t,B(t)y)}{H_m(t,B(t)y)}.$$

and

$$\widetilde{F}_1^0(t,y) = \frac{\widetilde{G}_1^0(t,y)}{\widetilde{H}_1^0(t,x)}, \ldots, \widetilde{F}_m^0(t,y) = \frac{\widetilde{G}_m^0(t,y)}{\widetilde{H}_m^0(t,x)},$$

are also functionally independent. We can assume without loss of generality that $\widetilde{G}_i^0(t,y)$ and $\widetilde{H}_i^0(t,x)$ have respectively degrees l_i and m_i, and are relatively prime for $i = 1, \ldots, m$.

We expand $\widetilde{F}_i(t,y)$, $i = 1, \ldots, m$, in the way done in (10), and since $\widetilde{F}_i(t,y)$ are first integrals of (22), we get that

$$\partial_t \widetilde{F}_i^0(t,y) + \left\langle \partial_y \widetilde{F}_i^0(t,y), \Lambda y \right\rangle \equiv 0, \qquad i = 1, \ldots, m, \tag{23}$$

i.e., $\widetilde{F}_i^0(t,y)$, $i = 1, \ldots, m$, are functionally independent first integrals of the linear differential system

$$\dot{y} = \Lambda y. \tag{24}$$

Equations (23) are equivalent to

$$\begin{aligned}&\widetilde{H}_i^0(t,y)\left(\partial_t\widetilde{G}_i^0(t,y)+\left\langle\partial_y\widetilde{G}_i^0(t,y),\Lambda y\right\rangle\right)\\&\equiv\widetilde{G}_i^0(t,y)\left(\partial_t\widetilde{H}_i^0(t,y)+\left\langle\partial_y\widetilde{H}_i^0(t,y),\Lambda y\right\rangle\right),\qquad i=1,\ldots,m.\end{aligned}\tag{25}$$

So there exist constants, say c_i, such that

$$\partial_t\widetilde{G}_i^0(t,y)+\left\langle\partial_y\widetilde{G}_i^0(t,y),\Lambda y\right\rangle-c_i\widetilde{G}_i^0(t,y)\equiv 0,\tag{26}$$

and

$$\partial_t\widetilde{H}_i^0(t,y)+\left\langle\partial_y\widetilde{H}_i^0(t,y),\Lambda y\right\rangle-c_i\widetilde{H}_i^0(t,y)\equiv 0.\tag{27}$$

For the set of monomials of degree k, $\Upsilon_k:=\left\{y^{\mathbf{k}}:\mathbf{k}\in(\mathbb{Z}^+)^n,\ |\mathbf{k}|=k\right\}$, we define their order as follows: $y^{\mathbf{p}}$ is before $y^{\mathbf{q}}$ if $\mathbf{p}-\mathbf{q}\succ 0$, i.e., there exists an $i_0\in\{1,\ldots,n\}$ such that $p_i=q_i$ for $i=1,\ldots,i_0-1$ and $p_{i_0}>q_{i_0}$. Then Υ_k is a base of the set of homogeneous polynomials of degree k with the given order. According to the given base and order, each homogeneous polynomial of degree k is uniquely determined by its coefficients.

We denote by $\widetilde{G}_i^0(t)$ the vector of dimension $\begin{pmatrix} l_i+n-1\\ n-1\end{pmatrix}$ formed by the coefficients of $\widetilde{G}_i^0(t,y)$. Let $\mathcal{L}_k$ be the linear operator on $\mathcal{H}_n^k(t)$, the linear space of homogeneous polynomials of degree k in y with coefficients 2π periodic in t, defined by

$$\mathcal{L}_k(h(t,y))=\langle\partial_y h(t,y),\Lambda y\rangle,\qquad h(t,y)\in\mathcal{H}_n^k(t).$$

Using these notations equations (26) and (27) can be written as

$$\partial_t\widetilde{G}_i^0(t)+(\mathcal{L}_{l_i}-c_i)\widetilde{G}_i^0(t)\equiv 0,\quad \partial_t\widetilde{H}_i^0(t)+(\mathcal{L}_{m_i}-c_i)\widetilde{H}_i^0(t)\equiv 0.$$

They have solutions

$$\widetilde{G}_i^0(t)=\exp\left((c_i\mathbf{E}_{1i}-\mathcal{L}_{l_i})t\right)\widetilde{G}_i^0(0),\quad \widetilde{H}_i^0(t)=\exp\left((c_i\mathbf{E}_{2i}-\mathcal{L}_{m_i})t\right)\widetilde{H}_i^0(0),$$

where $\mathbf{E}_{1i}$ and $\mathbf{E}_{2i}$ are two identity matrices of suitable orders. In order that $\widetilde{G}_i^0(t)$ and $\widetilde{H}_i^0(t)$ be 2π periodic, we should have

$$\left(\exp\left((c_i\mathbf{E}_{1i}-\mathcal{L}_{l_i})2\pi\right)-\mathbf{E}_{1i}\right)\widetilde{G}_i^0(0)=0,\tag{28}$$

and

$$\left(\exp\left((c_i\mathbf{E}_{2i}-\mathcal{L}_{m_i})2\pi\right)-\mathbf{E}_{2i}\right)\widetilde{H}_i^0(0)=0.\tag{29}$$

Recall that $\lambda=(\lambda_1,\ldots,\lambda_n)$ be the eigenvalues of Λ. Then it follows from Lemma 8 that $\exp((c_i\mathbf{E}_{2i}-\mathcal{L}_{l_i})2\pi)$ and $\exp((c_i\mathbf{E}_{2i}-\mathcal{L}_{m_i})2\pi)$ have respectively the eigenvalues

$$\left\{\exp((c_i-\langle\mathbf{l},\lambda\rangle)2\pi):\ \mathbf{l}\in(\mathbb{Z}^+)^n,\ |\mathbf{l}|=l_i\right\},$$

and

$$\left\{\exp((c_i-\langle\mathbf{m},\lambda\rangle)2\pi):\ \mathbf{m}\in(\mathbb{Z}^+)^n,\ |\mathbf{m}|=m_i\right\}.$$

In order that equations (28) and (29) have nontrivial solutions we must have

$$\exp(\langle \mathbf{l}, \lambda\rangle 2\pi) = \exp(c_i 2\pi) \quad \text{for all } \mathbf{l} \in (\mathbb{Z}^+)^n,\ |\mathbf{l}| = l_i,$$

and

$$\exp(\langle \mathbf{m}, \lambda\rangle 2\pi) = \exp(c_i 2\pi) \quad \text{for all } \mathbf{m} \in (\mathbb{Z}^+)^n,\ |\mathbf{m}| = m_i.$$

It follows that

$$\exp(\langle \mathbf{l} - \mathbf{m}, \lambda\rangle 2\pi) = 1 \quad \text{for all } \mathbf{l}, \mathbf{m} \in (\mathbb{Z}^+)^n,\ |\mathbf{l}| = l_i,\ |\mathbf{m}| = m_i,$$

i.e.,

$$\mu^{\mathbf{l}-\mathbf{m}} = 1 \quad \text{for } \mathbf{l}, \mathbf{m} \in (\mathbb{Z}^+)^n,\ |\mathbf{l}| = l_i,\ |\mathbf{m}| = m_i.$$

The above proof shows that the ratio of any two monomials in the denominator and numerator of each $\widetilde{F}_i^0(t, y)$ for $i \in \{1, \ldots, m\}$ is resonant. Hence working in a similar way to the proof of Theorem 1 we get that m is at most the dimension of Ξ.

This completes the proof of the theorem.

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