POLYNOMIAL INTEGRABILITY OF THE HAMILTONIAN SYSTEMS WITH HOMOGENEOUS POTENTIAL OF DEGREE $-2$

JAUME LLIBRE$^1$, ADAM MAHDI$^2$ AND CLAUDIA VALLS$^3$

Abstract. We characterize the analytic integrability of Hamiltonian systems with Hamiltonian $H = \frac{1}{2} \sum_{i=1}^{2} p_i^2 + V(q_1, q_2)$, having homogeneous potential $V(q_1, q_2)$ of degree $-2$.

1. Introduction

We consider $\mathbb{C}^4$ as a symplectic linear space with canonical variables $q = (q_1, q_2)$ and $p = (p_1, p_2)$. We are interested in Hamiltonian systems defined by the Hamiltonian function

$$H = \frac{1}{2} \sum_{i=1}^{2} p_i^2 + V(q),$$

where $V(q) = V(q_1, q_2)$ is a homogeneous function of degree $k$. To be more precise we consider the following system of four differential equations

$$\dot{q}_i = p_i, \quad \dot{p}_i = -\frac{\partial V}{\partial q_i}, \quad i = 1, 2.$$

Let $A = A(q, p)$ and $B = B(q, p)$ be two functions. Then their Poisson bracket $\{A, B\}$ is given by

$$\{A, B\} = \sum_{i=1}^{2} \left( \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right).$$

We say that functions $A$ and $B$ are in involution if $\{A, B\} = 0$. We say that a non-constant function $F = F(q, p)$ is a first integral for the Hamiltonian system (2) if it commutes with the Hamiltonian function $H$, i.e., $\{H, F\} = 0$. Since the Poisson bracket is antisymmetric it is clear that $H$ itself is always a first integral. We say that a 2-degree of freedom Hamiltonian system (2) is completely or Liouville integrable if it has 2 functionally independent first integrals: $H$, and an additional one $F$, which are in involution. As usual $H$ and $F$ are functionally independent if their gradients are linearly independent at all points of $\mathbb{C}^4$ except perhaps in a zero Lebesgue set.

First we recall basic properties of system (2). Let $PO_2(\mathbb{C})$ denote the group of $2 \times 2$ complex matrices $A$ such that $AA^T = \alpha I$, where $I$ is the identity matrix and $\alpha \in \mathbb{C} \setminus \{0\}$. We say that potentials $V_1(q)$ and $V_2(q)$ are equivalent if there exists

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a matrix \( A \in \text{PO}_2(\mathbb{C}) \) such that \( V_1(q) = V_2(Aq) \). So we divide all potentials into equivalent classes. Here a potential means a class of equivalent potentials in the above sense. This definition of equivalent potentials is motivated by the following simple lemma. For a proof see [8].

**Lemma 1.** Let \( V_1 \) and \( V_2 \) be two equivalent potentials. If Hamiltonian system (2) is integrable with potential \( V_1 \) then it is also integrable with \( V_2 \).

In the beginning of 80’s all integrable Hamiltonian systems (1) with homogeneous polynomial potential of degree at most 5 and having a second polynomial first integral up to degree 4 in the variables \( p_1 \) and \( p_2 \) were found, see [14, 5, 3, 6, 2] and also [7] for the list of corresponding additional first integrals. We remark that all these first integrals are polynomials in the variables \( p_1, p_2, q_1 \) and \( q_2 \). The main tools used there in order to identify these integrable systems were Painlevé test [4] and direct methods [8].

An elegant result related with the integrability of Hamiltonian systems with a homogeneous polynomial potential was given by Morales and Ramis (see [13, p. 100] and references therein), which gives the necessary condition for the complete meromorphic integrability of such systems. Using the result of Morales–Ramis, Maciejewski and Przybylska [10] gave a necessary and sufficient condition for the complete meromorphic integrability of Hamiltonian systems with the homogeneous polynomial potential of degree 3. The list of nonequivalent integrable homogeneous potentials of degree 3 is given in Table 1. Later on in [11] the same authors studied, among other things, the meromorphic integrability of the class of Hamiltonian systems with a homogeneous polynomial potential of degree 4. They proved that except for the family of potentials

\[
V = \frac{1}{2} a q_1^2 (q_1 + i q_2)^2 + \frac{1}{4} (q_1^2 + q_2^2)^2.
\]

<table>
<thead>
<tr>
<th>Case</th>
<th>Potential</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_1 )</td>
<td>( q_1 )</td>
</tr>
<tr>
<td>( V_2 )</td>
<td>( q_1^3/3 + cq_2^3/3 )</td>
</tr>
<tr>
<td>( V_3 )</td>
<td>( a q_1^3/3 + q_1^2 q_2/2 + q_2^2/6 )</td>
</tr>
<tr>
<td>( V_4 )</td>
<td>( q_1^2 q_2/2 + q_2^2 )</td>
</tr>
<tr>
<td>( V_5 )</td>
<td>( \pm i 7 q_1^3/15 + q_1^2 q_2/2 + q_2^2/15 )</td>
</tr>
<tr>
<td>( V_6 )</td>
<td>( q_1^2 q_2/2 + 8q_2^3/3 )</td>
</tr>
<tr>
<td>( V_7 )</td>
<td>( \pm i 17\sqrt{14}q_1^3/90 + q_1^2 q_2/2 + q_2^3/45 )</td>
</tr>
<tr>
<td>( V_8 )</td>
<td>( \pm i\sqrt{3}q_1^3/18 + q_1^2 q_2/2 + q_2^2 )</td>
</tr>
<tr>
<td>( V_9 )</td>
<td>( \pm i 3\sqrt{3}q_1^3/10 + q_1^2 q_2/2 + q_2^3/10 )</td>
</tr>
<tr>
<td>( V_{10} )</td>
<td>( \pm i 11\sqrt{3}q_1^3/45 + q_1^2 q_2/2 + q_2^3/10 )</td>
</tr>
</tbody>
</table>

Table 1. All nonequivalent integrable homogeneous potentials of degree 3.
only these systems with potentials $V_i$ for $i = 0, 1, \ldots, 8$ given in Table 2 are the nonequivalent integrable homogeneous potentials of degree 4. In [9] we proved that for the family (3) only the potentials $V_5$ and $V_{10}$ of Table 2 are integrable.

In this paper we classify the analytic integrability of the Hamiltonian systems (2) with homogeneous potentials of degrees $k = 2, 1, 0, -1$ and $k = -2$. So at this moment the analytic integrability of the Hamiltonian systems (2) with homogeneous potentials of degrees $k = -2, -1, 0, 1, 2, 3, 4$ has been characterized.

<table>
<thead>
<tr>
<th>Case</th>
<th>Potential</th>
</tr>
</thead>
<tbody>
<tr>
<td>$V_i$</td>
<td>$a(q_2 - i q_1)^i(q_2 + i q_1)^{4-i}$ for $i = 0, 1, 2, 3, 4$.</td>
</tr>
<tr>
<td>$V_5$</td>
<td>$a q_2^4$</td>
</tr>
<tr>
<td>$V_6$</td>
<td>$a q_4^4 / 4 + q_2^4$</td>
</tr>
<tr>
<td>$V_7$</td>
<td>$4 q_4^4 + 3 q_1^4 q_2^2 + q_2^4 / 4$</td>
</tr>
<tr>
<td>$V_8$</td>
<td>$2 q_4^4 + 3 q_1^4 q_2^2 / 2 + q_2^4 / 4$</td>
</tr>
<tr>
<td>$V_9$</td>
<td>$(q_1^4 + q_2^4)^2 / 4$</td>
</tr>
<tr>
<td>$V_{10}$</td>
<td>$- q_1^4 (q_1 + iq_2)^2 + (q_1^2 + q_2^2)^2 / 4$</td>
</tr>
</tbody>
</table>

Table 2. Nonequivalent integrable homogeneous potentials of degree 4.

For the sake of completeness we summarize here the trivial results related to the integrability of the Hamiltonian systems with the homogeneous potential of degree 2, 1, 0 and $-1$ being a polynomial or an inverse of the polynomial. It turns out that all those systems are completely integrable, with a polynomial additional first integral.

**Theorem 2.** Hamiltonian systems (2) with the homogeneous potential $V$ and one corresponding additional polynomial first integral $I$:

$V = a q_1^2 + b q_1 q_2 + c q_2^2$, \quad $I = b^2 q_1^2 + 4 b c q_2 q_1 + (b^2 + 4 c^2 - 4 a c) q_2^2 - 2 (a - c) p_1^2 + 2 b p_1 p_2$,

$V = a q_1 + b q_2$, \quad $I = a p_2 - b p_1$,

$V = a$, \quad $I = p_1$,

$V = 1/(a q_1 + b q_2)$, \quad $I = a p_2 - b p_1$,

where $a, b, c \in \mathbb{C}$ and $V \neq 0$.

**Proof.** The theorem follows by a straightforward computation. \hfill \Box

3. Homogeneous potential of degree $-2$

In this section we consider Hamiltonian systems (2) with a homogeneous potential of the form

$$V = V(q) = \frac{1}{a q_1^2 + b q_1 q_2 + c q_2^2} \quad \text{with } a, b, \text{ or } c \text{ nonzero.}$$
As we shall see only few of these potentials of degree \(-2\) will be analytically integrable, however all of them are rationally integrable with the additional well-known first integral
\[
I = \frac{1}{2}(q_1p_2 - q_2p_1)^2 + (q_1^2 + q_2^2)V(q).
\]
see for more details [1] and [12].

Our main results are the following two theorems (Theorems 3 and 4).

**Theorem 3.** The following statements hold.

(a) The polynomial integrability of the Hamiltonian system (2) with homogeneous potential (4) is equivalent to study the polynomial integrability of Hamiltonian system (2) with homogeneous potential \(V = 1/(aq_1^2 + cq_2^2)\).

(b) The Hamiltonian system (2) with homogeneous potential \(V = 1/(aq_1^2 + cq_2^2)\) is completely integrable with an additional polynomial first integral if and only if either \(c = 0\), or \(c \neq 0\) and \(a \in \{0, c\}\). Moreover this additional first integral is \(p_2\) if \(c = 0\); \(p_1\) if \(a = 0\) and \(q_1p_2 - q_2p_1\) if \(a = c\).

We consider polynomial differential systems of the form
\[
\frac{dx}{dt} = \dot{x} = P(x), \quad x = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4,
\]
with \(P(x) = (P_1(x), P_2(x), P_3(x), P_4(x))\) and \(P_i \in \mathbb{C}[x_1, x_2, x_3, x_4]\) for \(i = 1, 2, 3, 4\).

We say that system (5) is weight-homogeneous if there exist \(s = (s_1, s_2, s_3, s_4) \in \mathbb{Z}^4\) and \(d \in \mathbb{Z}\) such that
\[
P_i(\alpha^{s_1}x_1, \alpha^{s_2}x_2, \alpha^{s_3}x_3, \alpha^{s_4}x_4) = \alpha^{s_i-1+d}P_i(x_1, x_2, x_3, x_4), \quad i = 1, 2, 3, 4,
\]
for arbitrary \(\alpha \in \mathbb{R}^+ = \{\alpha \in \mathbb{R}, \alpha > 0\}\). We call \(s = (s_1, s_2, s_3, s_4)\) the weight exponent of system (5) and \(d\) the weight degree with respect to the weight exponent \(s\). We say that a polynomial \(F(x_1, x_2, x_3, x_4)\) is a weight-homogeneous polynomial with weight exponent \(s\) and weight degree \(n\) if
\[
F(\alpha^{s_1}x_1, \alpha^{s_2}x_2, \alpha^{s_3}x_3, \alpha^{s_4}x_4) = \alpha^nF(x_1, x_2, x_3, x_4).
\]

We note that Hamiltonian system (2) with homogeneous potential (6) is a weight-homogeneous polynomial differential system with weight exponent \((s_1, s_2, s_3, s_4) = (-1, -1, 1, 1)\) and weight degree \(d = 3\). Indeed with those values of \(d\) and \(s_i, i = 1, 2, 3, 4\) we can easily show
\[
\alpha^{s_1-1+d} = \alpha^{s_1}, \quad \alpha^{s_2-1+d} = \alpha^{s_2}, \quad \alpha^{s_3-1+d} = \alpha^{-3s_1}, \quad \alpha^{s_4-1+d} = \alpha^{-3s_2},
\]
for an arbitrary \(\alpha \in \mathbb{R}^+\). It is well-known (see for instance Proposition 1 of [9]) that the study of the existence of analytic first integrals of a weight-homogeneous polynomial differential system reduces to the study of the existence of a weight-homogeneous polynomial first integrals. This fact together with Theorem 3 states the following main theorem.

**Theorem 4.** The Hamiltonian system (2) with homogeneous potential (6) is completely integrable with an additional analytic first integral if and only if either \(c = 0\), or \(c \neq 0\) and \(a \in \{0, c\}\).

The following lemma proves statement (a) of Theorem 3.
Lemma 5. Let \( F(q) = aq_1^2 + bq_1q_2 + cq_2^2 \). Then there exists a change of variables 
\( q = Aq \), where \( A \in \text{PO}_2(\mathbb{C}) \) such that

\[
F(Aq) = \alpha q_1^2 + \beta q_2^2.
\]

Proof. We can assume that \( b \neq 0 \), otherwise there is nothing to prove. Let

\[
\left( \begin{array}{c}
q_1 \\
q_2
\end{array} \right) = \left( \begin{array}{cc}
a_1 & a_2 \\
a_2 & a_1
\end{array} \right) \left( \begin{array}{c}
\bar{q}_1 \\
\bar{q}_2
\end{array} \right).
\]

Then

\[
F(Aq) = \dot{\bar{q}}_1^2 \bar{q}_1^2 + \tilde{\beta} \dot{\bar{q}}_1 \bar{q}_2 + \tilde{\gamma} \dot{\bar{q}}_2^2,
\]

with

\[
\dot{\bar{q}} = (aa_1^2 - a_1a_2b + a_2^2c),
\]
\[
\tilde{\beta} = 2aa_1a_2 + a_1^2b - a_2^2b - 2a_1a_2c,
\]
\[
\tilde{\gamma} = (aa_1^2 + a_2a_2b + a_2^2c).
\]

Taking

\[
a_1 = \frac{a_2(c - a) + \sqrt{a_2^2(b^2 + (a - c)^2)}}{b},
\]

we get \( \tilde{\beta} = 0 \). \( \square \)

The above lemma implies that we can work with a homogeneous potential of the form

\[
(6) \quad V = \frac{1}{a_1q_1^2 + cq_2^2}, \quad \text{with } a \text{ or } c \text{ nonzero.}
\]

First we consider the case \( ac(c - a) \neq 0 \). We recall that we have the Hamiltonian system

\[
(7) \quad \dot{q}_1 = p_1, \quad \dot{q}_2 = p_2, \quad \dot{p}_1 = \frac{2a_1q_1}{(a_1q_1^2 + cq_2^2)^2}, \quad \dot{p}_2 = \frac{2cq_2}{(a_1q_1^2 + cq_2^2)^2},
\]

where the dot in (7) denotes the derivative with respect to \( t \). Now we take the new independent variable \( \tau \) defined by \( dt = (a_1q_1^2 + cq_2^2)^2d\tau \). Then system (7) becomes

\[
(8) \quad \dot{q}_1 = p_1(a_1q_1^2 + cq_2^2)^2, \quad \dot{q}_2 = p_2(a_1q_1^2 + cq_2^2)^2, \quad \dot{p}_1 = 2a_1q_1, \quad \dot{p}_2 = 2cq_2,
\]

where now the dot denotes the derivative with respect to \( \tau \). Changing the variables

\( (q_1, q_2, p_1, p_2) \to (q_1, q_2, p_1, T) \), where \( T = q_2p_1 - q_1p_2 \), system (8) writes

\[
\dot{q}_1 = p_1(a_1q_1^2 + cq_2^2)^2,
\]
\[
\dot{q}_2 = \frac{q_2p_1 - T}{q_1}(a_1q_1^2 + cq_2^2)^2,
\]
\[
\dot{p}_1 = 2a_1q_1,
\]
\[
\dot{T} = 2(a - c)q_1q_2.
\]

With this change of variables we put in evidence the first integral when \( a = c \).

If we denote by \( F(q_1, q_2, p_1, p_2) \in \mathbb{C}[q_1, q_2, p_1, p_2] \) a polynomial first integral of

(8), then in the variables \( (q_1, q_2, p_1, T) \) it writes

\[
(10) \quad F(q_1, q_2, p_1, T) = \sum_{j=-n}^{n} f_j(q_2, p_1, T)q_1^j,
\]
where \( f_j(q_2, p_1, T) \in \mathbb{C}[q_2, p_1, T] \). By definition \( F \) is a first integral of (9) if and only if \( F \) is non-constant and

\[
(11) \quad (aq_1^2 + cq_2^2)^2 \left( \frac{\partial F}{\partial q_1} p_1 + \frac{\partial F}{\partial q_2} q_1 + T \right) + 2q_1 \left( a \frac{\partial F}{\partial p_1} + \frac{\partial F}{\partial T} (a-c) q_2 \right) = 0.
\]

We define the following differential operators that act on \( f_j = f_j(q_2, p_1, T) \in \mathbb{R}[q_2, p_1, T] \):

\[
\mathcal{A}[f_j] := j p_1 f_j + (q_2 p_1 - T) \frac{\partial f_j}{\partial q_2},
\]

\[
\mathcal{B}[f_j] := cq_2^2 q_1 (q_2 p_1 - T) \frac{\partial f_j}{\partial q_2} + a \frac{\partial f_j}{\partial p_1} + (a-c) q_2 \frac{\partial f_j}{\partial T} + jacq_2 q_1 f_j.
\]

Computing the different coefficients of \( q_1^2 \) in (11) for \( j = -n - 1, \ldots, n + 3 \) we get that \( F \) is a first integral of (9) if and only if

\[
(12) \quad c^2 q_2^4 \mathcal{A}[f_j] = 0, \quad \text{for } i = -n, -n + 1, \quad 2 \mathcal{B}[f_j] + c^2 q_2^2 \mathcal{A}[f_{j+2}] = 0, \quad \text{for } i = -n, -n + 1, \quad c^2 q_2^4 \mathcal{A}[f_{j+2}] + 2 \mathcal{B}[f_j] + a^2 \mathcal{A}[f_{j-1}] = 0, \quad \text{for } i = -n + 4, \ldots, n, \quad 2 \mathcal{B}[f_j] + a^2 \mathcal{A}[f_{j-2}] = 0, \quad \text{for } i = n - 1, n, \quad a^2 \mathcal{A}[f_{j+1}] = 0, \quad \text{for } i = n - 1, n.
\]

We shall prove that if \( ac(a-c) \neq 0 \), then \( F = \text{const} \) and consequently it is not a first integral. The proof will follow from the following two lemmas.

**Lemma 6.** Let \( F \) be as in (10) and \( ac(a-c) \neq 0 \). If \( F \) is first integral of (9), then \( f_j(q_2, p_1, T) = 0 \) for \( j = 1, \ldots, n \).

**Proof.** From (12) we consider \( a^2 \mathcal{A}[f_n] = 0 \). Using that \( a \neq 0 \), the solution is \( f_n = \alpha/(T - q_2 p_1)^n \), where \( \alpha = \alpha(p_1, T) \). Since \( f_n \in \mathbb{C}[q_2, p_1, T] \) we conclude that \( f_n = 0 \). Similarly, from \( a^2 \mathcal{A}[f_{n-1}] \) we show that \( f_{n-1} = 0 \). Now using that \( \mathcal{B}[0] = 0 \), and \( f_n, f_{n-1} = 0 \) the conditions

\[
2 \mathcal{B}[f_j] + a^2 \mathcal{A}[f_{j-2}] = 0 \quad \text{for } i = n - 1, n,
\]

implies that \( \mathcal{A}[f_{n-2}] = \mathcal{A}[f_{n-3}] = 0 \). Thus using the arguments for solving \( a^2 \mathcal{A}[f_n] = 0 \) we obtain that as long as \( n - 3 \geq 1 \) we get \( f_n-2 = f_n-3 = 0 \). If \( n = 4 \) we are done. If \( n \geq 5 \), then we proceed by induction. Assume that \( f_n = f_{n-1} = \ldots = f_{j+1} = 0 \), where \( j \geq 1 \). We shall show that \( f_j = 0 \). Now we consider condition (12) for \( i = j + 4 \), that is,

\[
(13) \quad c^2 q_2^4 \mathcal{A}[f_{j+4}] + 2 \mathcal{B}[f_{j+2}] + a^2 \mathcal{A}[f_j] = 0.
\]

Since \( f_{j+4} = f_{j+2} = f_j = 0 \), we have \( \mathcal{A}[f_{j+4}] = \mathcal{A}[f_{j+2}] = 0 \). Thus condition (13) reduces to \( \mathcal{A}[f_j] = 0 \). Since \( j \geq 1 \), the only polynomial solution of this differential equation is \( f_j = 0 \). \( \square \)

**Lemma 7.** Let \( F \) be as in (10) and \( ac(a-c) \neq 0 \). If \( F \) is first integral of (9), then \( f_j(q_2, p_1, T) = 0 \) for \( j = -n, -n + 1, \ldots, -1 \) and \( f_0(q_2, p_1, T) = \text{constant} \).

**Proof.** Consider (12) for \( i = -n, -n + 1 \), that is, \( c^2 q_2^4 \mathcal{A}[f_{-n}] = 0 \) and \( c^2 q_2^4 \mathcal{A}[f_{-n+1}] = 0 \). Since \( c \neq 0 \) this implies that \( \mathcal{A}[f_{-n}] = \mathcal{A}[f_{-n+1}] = 0 \) and solving it we get

\[
(14) \quad f_{-n} = (q_2 p_1 - T)^n \alpha_{-n}, \quad \text{and} \quad f_{-n+1} = (q_2 p_1 - T)^{n-1} \alpha_{-n+1},
\]
where $\alpha_{-n} = \alpha_{-n}(p_1,T)$ and $\alpha_{-n+1} = \alpha_{-n+1}(p_1,T)$ are polynomials. Now we consider the condition $2B[f_i] + c^2q_2^4 A[f_{i+2}] = 0$, for $i = -n, -n+1$ and we shall use (14). Thus for $i = -n$ we get
\[
\alpha_{-n+2} = \alpha_{-n+2}(p_1,T),
\]
where $\alpha_{-n+2} = \alpha_{-n+2}(p_1,T)$ is an integral constant and
\[
\beta_{-n+2} = \frac{1}{3c^2q_2^4} \left( 3ncq_2 \alpha_{-n} - 3(a-c)q_2(T - 2q_2p_1) \frac{\partial \alpha_{-n}}{\partial T} + a(3q_2p_1 - 2T) \frac{\partial \alpha_{-n}}{\partial p_1} \right).
\]
Since $f_{-n+2}$ is a polynomial, $\beta_{-n+2}$ also is a polynomial. In the expression of $\beta_{-n+2}$ there are terms $(a-c)p_1q_2^{-1} \frac{\partial \alpha_{-n}}{\partial T}$ and $-\frac{3}{2}aTq_2^{-3} \frac{\partial \alpha_{-n}}{\partial p_1}$. Since $f_{-n+2}$ is a polynomial we obtain that
\[
\frac{\partial \alpha_{-n}}{\partial T} = \frac{\partial \alpha_{-n}}{\partial p_1} = 0.
\]
So $\alpha_{-n} = \text{constant}$. Moreover in $\beta_{-n+2}$ we also have the term $nc^{-1}q_2^{-2} \alpha_{-n}$. Again since $f_{-n+2}$ is a polynomial $\alpha_{-n} = 0$, thus $f_{-n} = 0$. Working with $f_{-n+1}$ similarly as with $f_{-n}$ we obtain
\[
f_{-n+3} = (q_2p_1 - T)^{n-3}[\alpha_{-n+3} + \beta_{-n+3}],
\]
where $\alpha_{-n+3} = \alpha_{-n+3}(p_1,T)$ and
\[
\beta_{-n+3} = \frac{1}{3c^2q_2^4} \left( 3(n-1)ncq_2 \alpha_{-n+1} - 3(a-c)q_2(T - 2q_2p_1) \frac{\partial \alpha_{-n+1}}{\partial T} + a(3q_2p_1 - 2T) \frac{\partial \alpha_{-n+1}}{\partial p_1} \right).
\]
Similarly as in the previous case we conclude that $\alpha_{-n+1} = 0$. In summary we have proved that $f_{-n} = f_{-n+1} = 0$. Now we shall proceed by induction. Assume that $f_{-n} = \ldots = f_{-j+1} = 0$ and for $-j \leq -3$. We shall prove that $f_{-j} = 0$. Consider (12) for $i = -j$, that is,
\[
c^2q_2^4 A[f_{-j}] + 2B[f_{-j-2}] + a^2 A[f_{-j-4}] = 0.
\]
Since by induction hypothesis $f_{-j-4} = f_{-j-2} = 0$, we have $A[f_{-j-4}] = A[f_{-j-2}] = 0$. Thus, condition (15) reduces to $A[f_{-j}] = 0$. This implies that $f_{-j} = (q_2p_1 - T)^{\alpha_{-j}}$, where again $\alpha_{-j} = \alpha_{-j}(p_1,T)$. Now considering (12) for $i = -j+2$ we get
\[
\]
Taking into account that $A[f_{-j-2}] = 0$ as well as $f_{-j} = (q_2p_1 - T)^{\alpha_{-j}}$ the solution of (16) writes
\[
\alpha_{-j+2} = (q_2p_1 - T)^{\alpha_{-j+2} + \beta_{-j+2}},
\]
where
\[
\beta_{-j+2} = \frac{1}{3c^2q_2^4} \left( 3jcq_2 \alpha_{-j} - 3(a-c)q_2(T - 2q_2p_1) \frac{\partial \alpha_{-j}}{\partial T} + a(3q_2p_1 - 2T) \frac{\partial \alpha_{-j}}{\partial p_1} \right).
\]
Again, since $\beta_{-j+2}$ has to be a polynomial the same argument as before allows to deduce that $\alpha_{-j} = 0$, therefore $f_{-j} = 0$.

Following the induction steps we have proved that $f_{-n} = \ldots = f_{-3} = 0$, $f_{-2} = (T - q_2p_1)^{\alpha_{-2}}$ and $f_{-1} = (T - q_2p_1)^{\alpha_{-1}}$, where $\alpha_{-1} = \alpha_{-1}(p_1,T)$ and $\alpha_{-2} = \alpha_{-2}(p_1,T)$. Consider again (12) for $i = 1$, that is,
\[
\]
By Lemma 6 $f_1 = 0$ so this implies $A[f_1] = 0$. Since by the induction process $f_{-1} = 0$, and $f_{-2} = (T - q_2 p_1)\alpha_{-1}$ the solution of (17) is given by

$$\alpha_{-1} = \gamma \left( T + \frac{c - a}{a} q_2 p_1 \right) / (a(T - q_2 p_1)), \text{ where } \gamma \in \mathbb{C}. $$

Again since $\alpha_{-1} = \alpha_{-1}(p_1, T)$ is a polynomial and $(c - a)/a \neq 0$ we conclude that $\gamma = 0$, thus $\alpha_{-1} = 0$ which implies that $f_{-1} = 0$. To show that $f_{-2} = 0$ and that $f_0 = f_0(p_1, T)$, that is, $f_0$ does not depend on $q_2$ consider (12) for $i = 0$, that is,

$$c^2 q_2^4 A[f_0] + 2 B[f_{-2}] + a^2 A[f_{-4}] = 0. $$

Since $A[f_{-4}] = 0$, $f_{-2} = (T - q_2 p_1)^2 \alpha_{-2}$ and $\alpha_{-2} = \alpha_{-2}(p_1, T)$ solving (18) we get

$$f_0 = \alpha_0 + \beta_0, $$

where $\alpha_0 = \alpha_0(p_1, T)$ and

$$\beta_0 = \frac{1}{3c q_2^2} \left( 6c q_2 \alpha_{-2} - 3(a - c) q_2 (T - 2q_2 p_1) \frac{\partial \alpha_{-2}}{\partial T} + a(3q_2 p_1 - 2T) \frac{\partial \alpha_{-2}}{\partial p_1} \right). $$

Since $\beta_0$ has to be a polynomial, using the same arguments for proving that $\beta_{-n+2} = 0$ we conclude that $\alpha_{-2} = 0$, and consequently $\beta_0 = 0$. Thus $f_{-2} = 0$, and therefore $f_0 = \alpha_0(p_1, T)$.

Finally we consider (12) for $i = 2$. Since by Lemma 6 we have that $f_2 = 0$, as well as $f_0 = f_0(p_1, T)$ we get

$$a \frac{\partial f_0}{\partial p_1} + (a - c) q_2^2 \frac{\partial f_0}{\partial T} = 0. $$

Its solution is of the form

$$f_0(p_1, T) = F \left( T + \frac{c - a}{a} q_2 p_1 \right). $$

Since $a(a - c) \neq 0$ and $f_0$ does not depend on $q_2$ we get that $f_0 = \text{constant}$ which ends the proof.

**Proof of Theorem 3.** If $c = 0$ then $p_2$ is an additional polynomial first integral and the corresponding Hamiltonian system (2) with potential (6) is completely integrable. So we can assume that $c \neq 0$.

There are at least two values of $a$ for which system (7) is completely integrable. These cases are $a = 0$ with additional first integral $p_1$ and $a = c$ with additional first integral $q_3 p_2 - q_2 p_1$. We note that in both cases the additional first integral is a polynomial. The rest of the proof follows directly from Lemmas 6 and 7. \hfill \Box

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