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POLYNOMIAL INTEGRABILITY OF THE HAMILTONIAN SYSTEMS WITH HOMOGENEOUS POTENTIAL OF DEGREE -2

JAUME LLIBRE[1], ADAM MAHDI[2] AND CLAUDIA VALLS[3]

ABSTRACT. We characterize the analytic integrability of Hamiltonian systems with Hamiltonian $H = \frac{1}{2}\sum_{i=1}^{2} p_i^2 + V(q_1, q_2)$, having homogeneous potential $V(q_1, q_2)$ of degree -2.

1. Introduction

We consider $\mathbb{C}^4$ as a symplectic linear space with canonical variables $q = (q_1, q_2)$ and $p = (p_1, p_2)$. We are interested in Hamiltonian systems defined by the Hamiltonian function

$$H = \frac{1}{2}\sum_{i=1}^{2} p_i^2 + V(q), \tag{1}$$

where $V(q) = V(q_1, q_2)$ is a homogeneous function of degree k. To be more precise we consider the following system of four differential equations

$$\dot{q}_i = p_i, \quad \dot{p}_i = -\frac{\partial V}{\partial q_i}, \qquad i = 1, 2. \tag{2}$$

Let $A = A(q, p)$ and $B = B(q, p)$ be two functions. Then their *Poisson bracket* $\{A, B\}$ is given by

$$\{A, B\} = \sum_{i=1}^{2} \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \right).$$

We say that functions A and B are *in involution* if $\{A, B\} = 0$. We say that a non–constant function $F = F(q, p)$ is a *first integral* for the Hamiltonian system (2) if it commutes with the Hamiltonian function H, i.e. $\{H, F\} = 0$. Since the Poisson bracket is antisymmetric it is clear that H itself is always a first integral. We say that a 2–degree of freedom Hamiltonian system (2) is *completely* or *Liouville integrable* if it has 2 functionally independent first integrals: H, and an additional one F, which are in involution. As usual H and F are *functionally independent* if their gradients are linearly independent at all points of $\mathbb{C}^4$ except perhaps in a zero Lebesgue set.

First we recall basic properties of system (2). Let $\mathrm{PO}_2(\mathbb{C})$ denote the group of 2×2 complex matrices A such that $AA^T = \alpha I$, where I is the identity matrix and $\alpha \in \mathbb{C} \setminus \{0\}$. We say that potentials $V_1(q)$ and $V_2(q)$ are *equivalent* if there exists

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Tel. 34 935811303 and Fax 34 935812790 of the corresponding author, J. Llibre.

Case	Potential
V_1	q_1^3
V_2	$q_1^3/3 + cq_2^3/3$
V_3	$aq_1^3/3 + q_1^2 q_2/2 + q_2^3/6$
V_4	$q_1^2 q_2/2 + q_2^3$
V_5	$\pm i7q_1^3/15 + q_1^2 q_2/2 + q_2^3/15$
V_6	$q_1^2 q_2/2 + 8q_2^3/3$
V_7	$\pm i17\sqrt{14}q_1^3/90 + q_1^2 q_2/2 + q_2^3/45$
V_8	$\pm i\sqrt{3}q_1^3/18 + q_1^2 q_2/2 + q_2^3$
V_9	$\pm i3\sqrt{3}q_1^3/10 + q_1^2 q_2/2 + q_2^3/45$
V_{10}	$\pm i11\sqrt{3}q_1^3/45 + q_1^2 q_2/2 + q_2^3/10$

TABLE 1. All nonequivalent integrable homogeneous potentials of degree 3.

a matrix $A \in \mathrm{PO}_2(\mathbb{C})$ such that $V_1(q) = V_2(Aq)$. So we divide all potentials into equivalent classes. Here a potential means a class of equivalent potentials in the above sense. This definition of equivalent potentials is motivated by the following simple lemma. For a proof see [8].

Lemma 1. *Let V_1 and V_2 be two equivalent potentials. If Hamiltonian system* (2) *is integrable with potential V_1 then it is also integrable with V_2.*

In the beginning of 80's all integrable Hamiltonian systems (1) with homogeneous polynomial potential of degree at most 5 and having a second polynomial first integral up to degree 4 in the variables p_1 and p_2 were found, see [14, 5, 3, 6, 2] and also [7] for the list of corresponding additional first integrals. We remark that all these first integrals are polynomials in the variables p_1, p_2, q_1 and q_2. The main tools used there in order to identify these integrable systems were Painlevé test [4] and direct methods [8].

An elegant result related with the integrability of Hamiltonian systems with a homogeneous polynomial potential was given by Morales and Ramis (see [13, p. 100] and references therein), which gives the necessary condition for the complete meromorphic integrability of such systems. Using the result of Morales–Ramis, Maciejewski and Przybylska [10] gave a necessary and sufficient condition for the complete meromorphic integrability of Hamiltonian systems with the homogeneous polynomial potential of degree 3. The list of nonequivalent integrable homogeneous potentials of degree 3 is given in Table 1. Later on in [11] the same authors studied, among other things, the meromorphic integrability of the class of Hamiltonian systems with a homogeneous polynomial potential of degree 4. They proved that except for the family of potentials

$$V = \frac{1}{2}aq_1^2(q_1 + iq_2)^2 + \frac{1}{4}(q_1^2 + q_2^2)^2, \tag{3}$$

Case	Potential
V_i	$\alpha(q_2 - iq_1)^i(q_2 + iq_1)^{4-i}$ for $i = 0, 1, 2, 3, 4.$
V_5	αq_2^4
V_6	$\alpha q_1^4/4 + q_2^4$
V_7	$4q_1^4 + 3q_1^2q_2^2 + q_2^4/4$
V_8	$2q_1^4 + 3q_1^2q_2^2/2 + q_2^4/4$
V_9	$(q_1^2 + q_2^2)^2/4$
V_{10}	$-q_1^2(q_1 + iq_2)^2 + (q_1^2 + q_2^2)^2/4$

Table 2. Nonequivalent integrable homogeneous potentials of degree 4.

only these systems with potentials V_i for $i = 0, 1, \dots, 8$ given in Table 2 are the nonequivalent integrable homogeneous potentials of degree 4. In [9] we proved that for the family (3) only the potentials V_9 and V_{10} of Table 2 are integrable.

In this paper we classify the analytic integrability of the Hamiltonian systems (2) with homogeneous potentials of degrees $k = 2$, $k = 1$, $k = 0$, $k = -1$ and $k = -2$. So at this moment the analytic integrability of the Hamiltonian systems (2) with homogeneous potentials of degrees $k = -2, -1, 0, 1, 2, 3, 4$ has been characterized.

2. Homogeneous potentials of degrees 2, 1, 0 and −1

For the sake of completeness we summarize here the trivial results related to the integrability of the Hamiltonian systems with the homogeneous potential of degree 2, 1, 0 and −1 being either a polynomial or an inverse of the polynomial. It turns out that all those systems are completely integrable, with a polynomial additional first integral.

Theorem 2. *Hamiltonian systems* (2) *with the homogeneous potential* V *and one corresponding additional polynomial first integral* I*:*

$$
\begin{aligned}
V &= aq_1^2 + bq_1q_2 + cq_2^2, & I &= b^2q_1^2 + 4bcq_2q_1 + (b^2 + 4c^2 - 4ac)q_2^2 - 2(a-c)p_2^2 + 2bp_1p_2,\\
V &= aq_1 + bq_2, & I &= ap_2 - bp_1,\\
V &= a, & I &= p_1,\\
V &= 1/(aq_1 + bq_2), & I &= ap_2 - bp_1,
\end{aligned}
$$

where $a, b, c \in \mathbb{C}$ *and* $V \not\equiv 0$.

Proof. The theorem follows by a straightforward computation. □

3. Homogeneous potential of degree −2

In this section we consider Hamiltonian systems (2) with a homogeneous potential of the form

$$
V = V(q) = \frac{1}{aq_1^2 + bq_1q_2 + cq_2^2} \quad \text{with } a,\ b, \text{ or } c \text{ nonzero.} \tag{4}
$$

As we shall see only few of these potentials of degree -2 will be analytically integrable, however all of them are rationally integrable with the additional well–known first integral

$$I = \frac{1}{2}(q_1p_2 - q_2p_1)^2 + (q_1^2 + q_2^2)V(q),$$

see for more details [1] and [12].

Our main results are the following two theorems (Theorems 3 and 4).

Theorem 3. *The following statements hold.*

(a) *The polynomial integrability of the Hamiltonian system* (2) *with homogeneous potential* (4) *is equivalent to study the polynomial integrability of Hamiltonian system* (2) *with homogeneous potential* $V = 1/(aq_1^2 + cq_2^2)$.

(b) *The Hamiltonian system* (2) *with homogeneous potential* $V = 1/(aq_1^2 + cq_2^2)$ *is completely integrable with an additional polynomial first integral if and only if either* $c = 0$, *or* $c \neq 0$ *and* $a \in \{0, c\}$. *Moreover this additional first integral is* p_2 *if* $c = 0$; p_1 *if* $a = 0$ *and* $q_1p_2 - q_2p_1$ *if* $a = c$.

We consider polynomial differential systems of the form

$$\frac{dx}{dt} = \dot{x} = P(x), \quad x = (x_1, x_2, x_3, x_4) \in \mathbb{C}^4, \tag{5}$$

with $P(x) = (P_1(x), P_2(x), P_3(x), P_4(x))$ and $P_i \in \mathbb{C}[x_1, x_2, x_3, x_4]$ for $i = 1, 2, 3, 4$. We say that system (5) is *weight-homogeneous* if there exist $s = (s_1, s_2, s_3, s_4) \in \mathbb{Z}^4$ and $d \in \mathbb{Z}$ such that

$$P_i(\alpha^{s_1}x_1, \alpha^{s_2}x_2, \alpha^{s_3}x_3, \alpha^{s_4}x_4) = \alpha^{s_i-1+d}P_i(x_1, x_2, x_3, x_4), \quad i = 1, 2, 3, 4,$$

for arbitrary $\alpha \in \mathbb{R}^+ = \{\alpha \in \mathbb{R}, \alpha > 0\}$. We call $s = (s_1, s_2, s_3, s_4)$ the *weight exponent* of system (5) and d the *weight degree* with respect to the weight exponent s. We say that a polynomial $F(x_1, x_2, x_3, x_4)$ is a *weight-homogeneous* polynomial with *weight exponent* s and *weight degree* n if

$$F(\alpha^{s_1}x_1, \alpha^{s_2}x_2, \alpha^{s_3}x_3, \alpha^{s_4}x_4) = \alpha^n F(x_1, x_2, x_3, x_4).$$

We note that Hamiltonian system (2) with homogeneous potential (6) is a weight-homogeneous polynomial differential system with weight exponent $(s_1, s_2, s_3, s_4) = (-1, -1, 1, 1)$ and weight degree $d = 3$. Indeed with those values of d and s_i, $i = 1, 2, 3, 4$ we can easily show

$$\alpha^{s_1-1+d} = \alpha^{s_3}, \quad \alpha^{s_2-1+d} = \alpha^{s_4}, \quad \alpha^{s_3-1+d} = \alpha^{-3s_1}, \quad \alpha^{s_4-1+d} = \alpha^{-3s_2},$$

for an arbitrary $\alpha \in \mathbb{R}^+$. It is well-known (see for instance Proposition 1 of [9]) that the study of the existence of analytic first integrals of a weight–homogeneous polynomial differential system reduces to the study of the existence of a weight–homogeneous polynomial first integrals. This fact together with Theorem 3 states the following main theorem.

Theorem 4. *The Hamiltonian system* (2) *with homogeneous potential* (6) *is completely integrable with an additional analytic first integral if and only if either* $c = 0$, *or* $c \neq 0$ *and* $a \in \{0, c\}$.

The following lemma proves statement (a) of Theorem 3.

Lemma 5. *Let* $F(q) = aq_1^2 + bq_1q_2 + cq_2^2$. *Then there exists a change of variables* $q = A\bar{q}$, *where* $A \in \mathrm{PO}_2(\mathbb{C})$ *such that*

$$F(A\bar{q}) = \alpha\bar{q}_1^2 + \beta\bar{q}_2^2.$$

Proof. We can assume that $b \neq 0$, otherwise there is nothing to prove. Let

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix} \begin{pmatrix} \bar{q}_1 \\ \bar{q}_2 \end{pmatrix}.$$

Then

$$F(A\bar{q}) = \bar{\alpha}\bar{q}_1^2 + \bar{\beta}\bar{q}_1\bar{q}_2 + \bar{\gamma}\bar{q}_2^2,$$

with

$$\begin{aligned} \bar{\alpha} &= (aa_1^2 - a_1a_2b + a_2^2c), \\ \bar{\beta} &= 2aa_1a_2 + a_1^2b - a_2^2b - 2a_1a_2c, \\ \bar{\gamma} &= (aa_2^2 + a_1a_2b + a_1^2c). \end{aligned}$$

Taking

$$a_1 = \frac{a_2(c-a) + \sqrt{a_2^2(b^2 + (a-c)^2)}}{b},$$

we get $\bar{\beta} = 0$. □

The above lemma implies that we can work with a homogeneous potential of the form

$$V = \frac{1}{aq_1^2 + cq_2^2}, \quad \text{with } a \text{ or } c \text{ nonzero}. \tag{6}$$

First we consider the case $ac(c-a) \neq 0$. We recall that we have the Hamiltonian system

$$\dot{q}_1 = p_1,\ \dot{q}_2 = p_2,\ \dot{p}_1 = \frac{2aq_1}{(aq_1^2 + cq_2^2)^2},\ \dot{p}_2 = \frac{2cq_2}{(aq_1^2 + cq_2^2)^2}, \tag{7}$$

where the dot in (7) denotes the derivative with respect to t. Now we take the new independent variable τ defined by $dt = (aq_1^2 + cq_2^2)^2 d\tau$. Then system (7) becomes

$$\dot{q}_1 = p_1(aq_1^2 + cq_2^2)^2,\ \dot{q}_2 = p_2(aq_1^2 + cq_2^2)^2,\ \dot{p}_1 = 2aq_1,\ \dot{p}_2 = 2cq_2, \tag{8}$$

where now the dot denotes the derivative with respect to τ. Changing the variables $(q_1, q_2, p_1, p_2) \to (q_1, q_2, p_1, T)$, where $T = q_2p_1 - q_1p_2$, system (8) writes

$$\begin{aligned} \dot{q}_1 &= p_1(aq_1^2 + cq_2^2)^2, \\ \dot{q}_2 &= \frac{q_2p_1 - T}{q_1}(aq_1^2 + cq_2^2)^2, \\ \dot{p}_1 &= 2aq_1, \\ \dot{T} &= 2(a-c)q_1q_2. \end{aligned} \tag{9}$$

With this change of variables we put in evidence the first integral when $a = c$.

If we denote by $F(q_1, q_2, p_1, p_2) \in \mathbb{C}[q_1, q_2, p_1, p_2]$ a polynomial first integral of (8), then in the variables (q_1, q_2, p_1, T) it writes

$$F(q_1, q_2, p_1, T) = \sum_{j=-n}^{n} f_j(q_2, p_1, T)q_1^j, \tag{10}$$

where $f_j(q_2,p_1,T) \in \mathbb{C}[q_2,p_1,T]$. By definition F is a first integral of (9) if and only if F is non-constant and

$$(aq_1^2+cq_2^2)^2\left(\frac{\partial F}{\partial q_1}p_1+\frac{\partial F}{\partial q_2}\frac{q_2p_1-T}{q_1}\right)+2q_1\left(a\frac{\partial F}{\partial p_1}+\frac{\partial F}{\partial T}(a-c)q_2\right)=0. \tag{11}$$

We define the following differential operators that act on $f_j = f_j(q_2,p_1,T) \in \mathbb{R}[q_2,p_1,T]$:

$$\mathcal{A}[f_j]:=jp_1f_j+(q_2p_1-T)\frac{\partial f_j}{\partial q_2},$$
$$\mathcal{B}[f_j]:=cq_2^2a(q_2p_1-T)\frac{\partial f_j}{\partial q_2}+a\frac{\partial f_j}{\partial p_1}+(a-c)q_2\frac{\partial f_j}{\partial T}+jacq_2^2p_1f_j.$$

Computing the different coefficients of q_1^j in (11) for $j=-n-1,\ldots,n+3$ we get that F is a first integral of (9) if and only if

$$\begin{aligned}
c^2q_2^4\,\mathcal{A}[f_i]&=0, \quad \text{for } i=-n,-n+1,\\
2\,\mathcal{B}[f_i]+c^2q_2^4\,\mathcal{A}[f_{i+2}]&=0, \quad \text{for } i=-n,-n+1,\\
c^2q_2^4\,\mathcal{A}[f_i]+2\,\mathcal{B}[f_{i-2}]+a^2\,\mathcal{A}[f_{i-4}]&=0, \quad \text{for } i=-n+4,\ldots,n,\\
2\,\mathcal{B}[f_i]+a^2\,\mathcal{A}[f_{i-2}]&=0, \quad \text{for } i=n-1,n,\\
a^2\,\mathcal{A}[f_i]&=0, \quad \text{for } i=n-1,n.
\end{aligned} \tag{12}$$

We shall prove that if $ac(a-c)\neq 0$, then $F=const$ and consequently it is not a first integral. The proof will follow from the following two lemmas.

Lemma 6. *Let F be as in* (10) *and $ac(a-c)\neq 0$. If F is first integral of* (9)*, then $f_j(q_2,p_1,T)=0$ for $j=1,\ldots,n$.*

Proof. From (12) we consider $a^2\,\mathcal{A}[f_n]=0$. Using that $a\neq 0$, the solution is $f_n=\alpha/(T-q_2p_1)^n$, where $\alpha=\alpha(p_1,T)$. Since $f_n\in\mathbb{C}[q_2,p_1,T]$ we conclude that $f_n=0$. Similarly, from $a^2\mathcal{A}[f_{n-1}]$ we show that $f_{n-1}=0$. Now using that $\mathcal{B}[0]=0$, and $f_n=f_{n-1}=0$ the conditions

$$2\mathcal{B}[f_i]+a^2\mathcal{A}[f_{i-2}]=0 \quad \text{for} \quad i=n-1,n,$$

implies that $\mathcal{A}[f_{n-2}]=\mathcal{A}[f_{n-3}]=0$. Thus using the arguments for solving $a^2\mathcal{A}[f_n]=0$ we obtain that as long as $n-3\geq 1$ we get $f_{n-2}=f_{n-3}=0$. If $n=4$ we are done. If $n\geq 5$, then we proceed by induction. Assume that $f_n=f_{n-1}=\ldots=f_{j+1}=0$, where $j\geq 1$. We shall show that $f_j=0$. Now we consider condition (12) for $i=j+4$, that is,

$$c^2q_2^4\,\mathcal{A}[f_{j+4}]+2\,\mathcal{B}[f_{j+2}]+a^2\,\mathcal{A}[f_j]=0. \tag{13}$$

Since $f_{j+4}=f_{j+2}=0$, we have $\mathcal{A}[f_{j+4}]=\mathcal{B}[f_{j+2}]=0$. Thus condition (13) reduces to $\mathcal{A}[f_j]=0$. Since $j\geq 1$, the only polynomial solution of this differential equation is $f_j=0$. □

Lemma 7. *Let F be as in* (10) *and $ac(a-c)\neq 0$. If F is first integral of* (9)*, then $f_j(q_2,p_1,T)=0$ for $j=-n,-n+1\ldots,-1$ and $f_0(q_2,p_1,T)=constant$.*

Proof. Consider (12) for $i=-n,-n+1$, that is, $c^2q_2^4\,\mathcal{A}[f_{-n}]=0$ and $c^2q_2^4\,\mathcal{A}[f_{-n+1}]=0$. Since $c\neq 0$ this implies that $\mathcal{A}[f_{-n}]=\mathcal{A}[f_{-n+1}]=0$ and solving it we get

$$f_{-n}=(q_2p_1-T)^n\alpha_{-n}, \quad \text{and} \quad f_{-n+1}=(q_2p_1-T)^{n-1}\alpha_{-n+1}, \tag{14}$$

where $\alpha_{-n} = \alpha_{-n}(p_1, T)$ and $\alpha_{-n+1} = \alpha_{-n+1}(p_1, T)$ are polynomials. Now we consider the condition $2\mathcal{B}[f_i] + c^2 q_2^4 \mathcal{A}[f_{i+2}] = 0$, for $i = -n, -n+1$ and we shall use (14). Thus for $i = -n$ we get

$$f_{-n+2} = (q_2 p_1 - T)^{n-2}[\alpha_{-n+2} + \beta_{-n+2}],$$

where $\alpha_{-n+2} = \alpha_{-n+2}(p_1, T)$ is an integral constant and

$$\beta_{-n+2} = \frac{1}{3c^2 q_2^3}\left(3ncq_2\alpha_{-n} - 3(a-c)q_2(T - 2q_2p_1)\frac{\partial \alpha_{-n}}{\partial T} + a(3q_2p_1 - 2T)\frac{\partial \alpha_{-n}}{\partial p_1}\right).$$

Since f_{-n+2} is a polynomial, β_{-n+2} also is a polynomial. In the expression of β_{-n+2} there are terms $(a-c)p_1 q_2^{-1}\frac{\partial \alpha_{-n}}{\partial T}$ and $-\frac{2}{3}aTq_2^{-3}\frac{\partial \alpha_{-n}}{\partial p_1}$. Since f_{n-2} is a polynomial we obtain that

$$\frac{\partial \alpha_{-n}}{\partial T} = \frac{\partial \alpha_{-n}}{\partial p_1} = 0.$$

So $\alpha_{-n} =$ constant. Moreover in β_{-n+2} we also have the term $nc^{-1}q_2^{-2}\alpha_{-n}$. Again since f_{-n+2} is a polynomial $\alpha_{-n} = 0$, thus $f_{-n} = 0$. Working with f_{-n+1} similarly as with f_{-n} we obtain

$$f_{-n+3} = (q_2 p_1 - T)^{n-3}[\alpha_{-n+3} + \beta_{-n+3}],$$

where $\alpha_{-n+3} = \alpha_{-n+3}(p_1, T)$ and

$$\beta_{-n+3} = \frac{1}{3c^2 q_2^3}\left(3(n-1)cq_2\alpha_{-n+1} - 3(a-c)q_2(T - 2q_2p_1)\frac{\partial \alpha_{-n+1}}{\partial T} + a(3q_2p_1 - 2T)\frac{\partial \alpha_{-n+1}}{\partial p_1}\right).$$

Similarly as in the previous case we conclude that $\alpha_{-n+1} = 0$. In summary we have proved that $f_{-n} = f_{-n+1} = 0$. Now we shall proceed by induction. Assume that $f_{-n} = \ldots = f_{-j-1} = 0$ and for $-j \le -3$. We shall prove that $f_{-j} = 0$. Consider (12) for $i = -j$, that is,

$$c^2 q_2^4 \mathcal{A}[f_{-j}] + 2\,\mathcal{B}[f_{-j-2}] + a^2\,\mathcal{A}[f_{-j-4}] = 0. \tag{15}$$

Since by induction hypothesis $f_{-j-4} = f_{-j-2} = 0$, we have $\mathcal{A}[f_{-j-4}] = \mathcal{B}[f_{-j-2}] = 0$. Thus, condition (15) reduces to $\mathcal{A}[f_{-j}] = 0$. This implies that $f_{-j} = (q_2p_1 - T)^j\alpha_{-j}$, where again $\alpha_{-j} = \alpha_{-j}(p_1, T)$. Now considering (12) for $i = -j+2$ we get

$$c^2 q_2^4 \mathcal{A}[f_{-j+2}] + 2\,\mathcal{B}[f_{-j}] + a^2\,\mathcal{A}[f_{-j-2}] = 0. \tag{16}$$

Taking into account that that $\mathcal{A}[f_{-j-2}] = 0$ as well as $f_{-j} = (q_2p_1 - T)^j\alpha_{-j}$ the solution of (16) writes

$$f_{-j+2} = (q_2 p_1 - T)^{j-2}[\alpha_{-j+2} + \beta_{-j+2}],$$

where

$$\beta_{-j+2} = \frac{1}{3c^2 q_2^3}\left(3jcq_2\alpha_{-j} - 3(a-c)q_2(T - 2q_2p_1)\frac{\partial \alpha_{-j}}{\partial T} + a(3q_2p_1 - 2T)\frac{\partial \alpha_{-j}}{\partial p_1}\right).$$

Again, since β_{-j+2} has to be a polynomial the same argument as before allows to deduce that $\alpha_{-j} = 0$, therefore $f_{-j} = 0$.

Following the induction steps we have proved that $f_{-n} = \ldots = f_{-3} = 0$, $f_{-2} = (T - q_2p_1)^2\alpha_{-2}$ and $f_{-1} = (T - q_2p_1)\alpha_{-1}$, where $\alpha_{-1} = \alpha_{-1}(p_1, T)$ and $\alpha_{-2} = \alpha_{-2}(p_1, T)$. Consider again (12) for $i = 1$, that is,

$$c^2 q_2^4 \mathcal{A}[f_1] + 2\,\mathcal{B}[f_{-1}] + a^2\,\mathcal{A}[f_{-3}] = 0. \tag{17}$$

By Lemma 6 $f_1 = 0$ so this implies $\mathcal{A}[f_1] = 0$. Since by the induction process $f_{-3} = 0$, and $f_{-1} = (T - q_2p_1)\alpha_{-1}$ the solution of (17) is given by

$$\alpha_{-1} = \gamma \left(T + \frac{c-a}{a} q_2p_1 \right) / (a(T - q_2p_1)), \quad \text{where} \quad \gamma \in \mathbb{C}.$$

Again since $\alpha_{-1} = \alpha_{-1}(p_1, T)$ is a polynomial and $(c-a)/a \neq 0$ we conclude that $\gamma = 0$, thus $\alpha_{-1} = 0$ which implies that $f_{-1} = 0$. To show that $f_{-2} = 0$ and that $f_0 = f_0(p_1, T)$, that is, f_0 does not depend on q_2 consider (12) for $i = 0$, that is,

$$c^2q_2^4\,\mathcal{A}[f_0] + 2\,\mathcal{B}[f_{-2}] + a^2\,\mathcal{A}[f_{-4}] = 0. \tag{18}$$

Since $\mathcal{A}[f_{-4}] = 0$, $f_{-2} = (T - q_2p_1)^2\alpha_{-2}$ and $\alpha_{-2} = \alpha_{-2}(p_1, T)$ solving (18) we get

$$f_0 = \alpha_0 + \beta_0,$$

where $\alpha_0 = \alpha_0(p_1, T)$ and

$$\beta_0 = \frac{1}{3cq_2^3} \left(6cq_2\alpha_{-2} - 3(a-c)q_2(T - 2q_2p_1)\frac{\partial \alpha_{-2}}{\partial T} + a(3q_2p_1 - 2T)\frac{\partial \alpha_{-2}}{\partial p_1} \right).$$

Since β_0 has to be a polynomial, using the same arguments for proving that $\beta_{-n+2} = 0$ we conclude that $\alpha_{-2} = 0$, and consequently $\beta_0 = 0$. Thus $f_{-2} = 0$, and therefore $f_0 = \alpha_0(p_1, T)$.

Finally we consider (12) for $i = 2$. Since by Lemma 6 we have that $f_2 = 0$, as well as $f_0 = f_0(p_1, T)$ we get

$$a\frac{\partial f_0}{\partial p_1} + (a-c)q_2\frac{\partial f_0}{\partial T} = 0.$$

Its solution is of the form

$$f_0(p_1, T) = F\left(T + \frac{c-a}{a}\, q_2p_1 \right).$$

Since $a(a-c) \neq 0$ and f_0 does not depend on q_2 we get that $f_0 =$ constant which ends the proof. □

Proof of Theorem 3. If $c = 0$ then p_2 is an additional polynomial first integral and the corresponding Hamiltonian system (2) with potential (6) is completely integrable. So we can assume that $c \neq 0$.

There are at least two values of a for which system (7) is completely integrable. These cases are $a = 0$ with additional first integral p_1 and $a = c$ with additional first integral $q_1p_2 - q_2p_1$. We note that in both cases the additional first integral is a polynomial. The rest of the proof follows directly from Lemmas 6 and 7. □

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[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

E-mail address: jllibre@mat.uab.cat, mahdi@mat.uab.cat

[2] Mathematics Department, University of North Carolina at Charlotte, Charlotte, North Carolina 28223, USA

E-mail address: adam.mahdi@uncc.edu

[2] Faculty of Applied Mathematics, AGH University of Science and Technology, al. Mickiewicza 30, 30-059 Kraków, Poland

[3] Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais 1049-001, Lisboa, Portugal

E-mail address: cvalls@math.ist.utl.pt