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ON THE PERIODIC ORBITS AND THE INTEGRABILITY OF THE REGULARIZED HILL LUNAR PROBLEM

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Abstract. The classical Hill's problem is a simplified version of the restricted three–body problem where the distance of the two massive bodies (say, primary for the largest one and secondary for the smallest) is made infinity through the use of Hill's variables. The Levi-Civita regularization takes the Hamiltonian of the Hill lunar problem into the form of two uncoupled harmonic oscillators perturbed by the Coriolis force and the Sun action, polynomials of degree 4 and 6, respectively. In this paper we study periodic orbits of the planar Hill problem using the averaging theory. Moreover we provide information about the C^1 integrability or non–integrability of the regularized Hill lunar problem.

1. Introduction and statement of the main results

In this paper we study periodic orbits of the Hill lunar problem and its C^1 non-integrability. The Hill lunar problem is a limit version of the restricted three–body problem, it is a model originally based on the Moon's motion under the joint action of the Earth and the Sun [4]. In the rotating frame the Hamiltonian of Hill lunar problem is

$$H_{\text{Hill}}(\mathbf{x}) = \frac{1}{2}(x_3^2+x_4^2) + x_2x_3 - x_1x_4 - \frac{1}{\sqrt{x_1^2+x_2^2}} - x_1^2 + \frac{1}{2}x_2^2, \tag{1}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$. For more details on this Hamiltonian see [13].

To avoid the difficulties due to the collision is performed the Levi–Civita regularization as follows. We do the change of variables in the positions given by

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \hat{x}_1 & -\hat{x}_2 \\ \hat{x}_2 & \hat{x}_1 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix}, \tag{2}$$

and the induced change in the conjugate momenta are

$$\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = \frac{2}{\hat{r}^2} \begin{pmatrix} \hat{x}_1 & -\hat{x}_2 \\ \hat{x}_2 & \hat{x}_1 \end{pmatrix} \begin{pmatrix} \hat{x}_3 \\ \hat{x}_4 \end{pmatrix}, \tag{3}$$

where $\hat{r}^2 = \hat{x}_1^2 + \hat{x}_2^2 = r = \sqrt{x_1^2+x_2^2}$. To complete the regularization procedure is necessary to rescale the time doing $d\tau = \dfrac{4dt}{(\hat{x}_1^2+\hat{x}_2^2)}$. Applying theses changes of variables in (1), and considering the Hamiltonian $\hat{H}(\hat{\mathbf{x}}) = \dfrac{r}{4}(H_{\text{Hill}}(\mathbf{x}(\hat{\mathbf{x}})) + p_0)$, after omitting the hat of the variables we get that the new Hamiltonian for the Hill problem becomes

$$H(\mathbf{x}) = \left(\frac{p_0}{2}\right)\frac{x_1^2+x_2^2}{2} + \frac{1}{2}(x_3^2+x_4^2) + \frac{1}{2}\left(x_1^2+x_2^2\right)(x_2x_3 - x_1x_4) - \frac{1}{4}\left(x_1^6 - 3x_1^4x_2^2 - 3x_1^2x_2^4 + x_2^6\right). \tag{4}$$

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where $\frac{1}{2}p_0 = -\frac{1}{2}h = c$, h being the value of the Hamiltonian (1). The Hamiltonian (4) is still dependent on the parameter c which can be eliminated doing the canonical change of variables

$$x_1 = 2c^{1/4}\bar{x}_1, \quad x_2 = 2c^{1/4}\bar{x}_2, \quad x_3 = 2c^{3/4}\bar{x}_3, \quad x_4 = 2c^{3/4}\bar{x}_4, \tag{5}$$

and the Hamiltonian of the Hill problem that we shall use is

$$\begin{aligned} H_{\mathrm{Reg}} &= \frac{1}{4c}H(2c^{1/4}x_1, 2c^{1/4}x_2, 2c^{3/4}x_3, 2c^{3/4}x_4) \\ &= \frac{1}{2}\left(x_1^2+x_2^2+x_3^2+x_4^2\right) + 2\left(x_1^2+x_2^2\right)(x_2x_3 - x_1x_4) - 4\left(x_1^6 - 3x_1^4x_2^2 - 3x_1^2x_2^4 + x_2^6\right), \end{aligned} \tag{6}$$

where the bar has been suppressed from the variables. The Hamiltonian H_{Reg} is called the regularized Hamiltonian of Hill lunar problem, for more details see subsection 2.2 in [16].

Notice that the Hamiltonian (6) is polynomial. This is very convenient for numerical computations. The terms of degree 2 of the polynomial Hamiltonian for the regularized Hill problem takes the form of two uncoupled harmonic oscillators. The fourth degree terms are due to the Coriolis force because we have changed the inertial reference frame at the center of mass to a rotating frame centered at the secondary. The sixth degree terms are due to the action of the primary. The Hamiltonian equations of motion become

$$\begin{aligned} \dot{x}_1 &= x_3 + 2x_2\left(x_1^2+x_2^2\right), \\ \dot{x}_2 &= x_4 - 2x_1\left(x_1^2+x_2^2\right), \\ \dot{x}_3 &= -x_1 + 2x_4\left(x_1^2+x_2^2\right) + 4\left(6x_1^5 - 12x_1^3x_2^2 - 6x_1x_2^4\right) - 4x_1(x_2x_3 - x_1x_4), \\ \dot{x}_4 &= -x_2 - 2x_3\left(x_1^2+x_2^2\right) + 4\left(-6x_1^4x_2 - 12x_1^2x_2^3 + 6x_2^5\right) - 4x_2(x_2x_3 - x_1x_4). \end{aligned} \tag{7}$$

As usual the dot denotes derivative with respect to the independent variable t, the time.

In this work we use the averaging method of first order to compute periodic orbits, see appendix 4.2. This method allows to find analytically periodic orbits of the Hill lunar problem (7) at any positive values of the energy. Roughly speaking this method reduces the problem of finding periodic solutions of some differential system to the one of finding zeros of some convenient finite dimensional function. This method was also used by Koslov in [8], Llibre and Jiménez-Lara in [6, 7] and LIibre and Roberto in [9].

The following theorem is the main result.

Theorem 1. *At every positive energy level the regularized Hill lunar problem has at least two periodic orbits.*

The periodic orbits for Hill lunar problem was also studied by Maciejewski and Rybicki in [10]. They described global bifurcations of non-stationary periodic orbits of the regularized Hill lunar problem which emanate from stationary ones. The symmetric periodic orbits of this problem were studied by Henon in [3] and Howison and Meyer in [5] established the existence of a new family of periodic solutions for the spatial Hill's lunar problem.

Theorem 1 states that at any positive energy level there exist at least two periodic orbits and we can use these particular periodic orbits to prove our second main result about the C^1 integrability or non–integrability in the sense of Liouville–Arnold of the regularized Hill lunar problem.

Theorem 2. *The regularized Hill lunar problem* (7) *is*

(a) *either Liouville–Arnold integrable and the gradients of the two constants of motion are linearly dependent on some points of the periodic orbits found in Theorem 1,*
(b) *or it is not Liouville–Arnold integrable with any second first integral of class $\mathcal{C}^1$.*

The proof of Theorem 2 is based on a result of Poincaré that allows to prove the Liouville–Arnold non-integrability independently of the class of differentiability of the second first integral, see appendix 4.1. The main difficulty for applying Poincaré's non–integrability method to a given Hamiltonian system is to find for such system periodic orbits having multipliers different from 1.

The non-integrability of Hill lunar problem was studied by some authors. Winterberg and Meletlidou in [11] and [12] proved the analytic non–integrability of the Hill lunar problem, that is, the problem does not possess a second analytic integral of motion, independent of H. Morales–Ruiz, Simó and Simon in [14] presented an algebraic proof of meromorphic non–integrability. But in our case we present some result on the C^1 integrability .

In section 2 we prove Theorem 1, and Theorem 2 is proved in section 3.

2. Proof of Theorem 1

We shall use the theory averaging of first order to analyze the existence of periodic orbits for system (7). To apply Theorem 5 we need a small parameter ε, then we rescale system (7) doing $(x_1, x_2, x_3, x_4) = (\sqrt{\varepsilon}X_1, \sqrt{\varepsilon}X_2, \sqrt{\varepsilon}X_3, \sqrt{\varepsilon}X_4)$ and we obtain the Hamiltonian system

$$
\begin{aligned}
\dot{X}_1 &= X_3 + 2X_2\varepsilon\left(X_1^2 + X_2^2\right),\\
\dot{X}_2 &= X_4 - 2X_1\varepsilon\left(X_1^2 + X_2^2\right),\\
\dot{X}_3 &= -X_1 + 2\varepsilon\left(3X_1^2X_4 - 2X_1X_2X_3 + X_2^2X_4\right) + 24\varepsilon^2X_1\left(X_1^4 - 2X_1^2X_2^2 - X_2^4\right),\\
\dot{X}_4 &= -X_2 - 2\varepsilon\left(X_1^2X_3 - 2X_1X_2X_4 + 3X_2^2X_3\right) - 24\varepsilon^2X_2\left(X_1^4 + 2X_1^2X_2^2 - X_2^4\right),
\end{aligned}
\tag{8}
$$

with the Hamiltonian

(9)

$$
\frac{1}{2}\left(X_1^2 + X_2^2 + X_3^2 + X_4^2\right) - 2\varepsilon\left(X_1^2 + X_2^2\right)\left(X_1X_4 - X_2X_3\right) - 4\varepsilon^2\left(X_1^2 + X_2^2\right)\left(X_1^4 - 4X_1^2X_2^2 + X_2^4\right).
$$

Notice that system (8) has the same phase portrait that system (7) for all $\varepsilon \neq 0$.

Another change of variables is done because system (8) is not in the normal form (19) for applying the averaging theory. We do the transfomation

$$
X_1 = r\cos\theta, \quad X_2 = \rho\cos(\theta+\alpha), \quad X_3 = r\sin\theta, \quad X_4 = \rho\sin(\theta+\alpha).
$$

Recall that this is a change of variables when $r > 0$ and $\rho > 0$. Moreover doing this change of variables appear in the system the periodic variables θ and α. Later on the variable θ will be used for obtaining the periodicity necessary for applying the averaging theory.

With this change of variables we obtain the equations of motion

(10)

$$
\begin{aligned}
\dot{r} = \;& \varepsilon\rho\left(\cos\alpha(\cos 2\theta(r^2 + \rho^2\cos 2\alpha) + r^2 + \rho^2) - \sin\alpha\sin 2\theta(-2r^2 + \rho^2\cos 2\alpha + \rho^2)\right) +\\
& 24\varepsilon^2 r\sin\theta\cos\theta\left(r^4\cos^4\theta - 2r^2\rho^2\cos^2\theta\cos^2(\alpha+\theta) - \rho^4\cos^4(\alpha+\theta)\right),\\
\dot{\theta} = \;& -1 + \varepsilon\tfrac{\rho}{r}\sin\alpha\left(3r^2\cos 2\theta + 3r^2 + \rho^2\cos 2(\alpha+\theta) + \rho^2\right) +\\
& 24\varepsilon^2\cos^2\theta\left(r^4\cos^4\theta - 2r^2\rho^2\cos^2\theta\cos^2(\alpha+\theta) - \rho^4\cos^4(\alpha+\theta)\right),\\
\dot{\rho} = \;& -\tfrac{1}{2}\varepsilon r\left(r^2\cos(\alpha - 2\theta) + 2(r^2+\rho^2)\cos\alpha + (r^2 - \rho^2)\cos(\alpha + 2\theta) + 3\rho^2\cos(3\alpha + 2\theta)\right) +\\
& 24\varepsilon^2\rho\sin(\alpha+\theta)\cos(\alpha+\theta)\left(-r^4\cos^4\theta - 2r^2\rho^2\cos^2\theta\cos^2(\alpha+\theta) + \rho^4\cos^4(\alpha+\theta)\right),\\
\dot{\alpha} = \;& \varepsilon\tfrac{1}{r\rho}\sin\alpha\left(r^4 + \left(3r^2\rho^2 - \rho^4\right)\cos 2(\alpha+\theta) + r^2\left(r^2 - 3\rho^2\right)\cos 2\theta - \rho^4\right) -\\
& 24\varepsilon^2\big(r^4\cos^6\theta + r^2(r^2 - 2\rho^2)\cos^4\theta\cos^2(\alpha+\theta) - \rho^2(\rho^2 - 2r^2)\cos^2\theta\cos^4(\alpha+\theta) -\\
& \rho^4\cos^6(\alpha+\theta)\big).
\end{aligned}
$$

This change of variables is not canonical, so the system lost the Hamiltonian structure. In these news variables Hamiltonian (9) becomes into the first integral

(11)
$$\begin{aligned} H = \; & \tfrac{1}{2}(r^2+\rho^2) - \varepsilon r\rho\sin\alpha\left(r^2\cos 2\theta + r^2 + \rho^2\cos 2(\alpha+\theta) + \rho^2\right) - \\ & 4\varepsilon^2\left(r^2\cos^2\theta + \rho^2\cos^2(\alpha+\theta)\right)\left(r^4\cos^4\theta - 4r^2\rho^2\cos^2\theta\cos^2(\alpha+\theta) + \rho^4\cos^4(\alpha+\theta)\right). \end{aligned}$$

However the derivatives of the left hand side of equations (10) are with respect to the time variable t, which is not periodic. So we change to the variable θ as the new independent one, and we denote by a prime the derivative with respect to θ. Moreover we write the system as a Taylor series in powers of ε and obtain

(12)
$$\begin{aligned} r' &= \varepsilon\left(\rho\sin\alpha\sin 2\theta(-2r^2+\rho^2\cos 2\alpha+\rho^2) - \rho\cos\alpha(\cos 2\theta(r^2+\rho^2\cos 2\alpha)+r^2+\rho^2)\right) + O(\varepsilon^2), \\ \rho' &= \tfrac{1}{2}\varepsilon r\left(r^2\cos(\alpha-2\theta) + 2(r^2+\rho^2)\cos\alpha + (r^2-\rho^2)\cos(\alpha+2\theta) + 3\rho^2\cos(3\alpha+2\theta)\right) + O(\varepsilon^2), \\ \alpha' &= \varepsilon\tfrac{1}{r\rho}\sin\alpha\left(-r^4 + (\rho^4-3r^2\rho^2)\cos 2(\alpha+\theta) - r^2(r^2-3\rho^2)\cos 2\theta + \rho^4\right) + O(\varepsilon^2). \end{aligned}$$

Now system (12) is $2\pi-$periodic in the variable θ.

We shall apply Theorem 5 in the Hamiltonian level $H = h$ for $h > 0$, H given by (11). Solving $H = h$ for $\rho = \rho_0 + \varepsilon\rho_1 + O(\varepsilon^2)$ we have

(13)
$$\rho = \sqrt{2h-r^2} + \varepsilon(2hr\sin\alpha\cos 2(\alpha+\theta) + 2hr\sin\alpha + r^3\sin\alpha\cos 2\theta - r^3\sin\alpha\cos 2(\alpha+\theta)) + O(\varepsilon^2).$$

Substituting ρ in equation (12) and developing in power series of ε we have just two differential equations

(14)
$$\begin{aligned} r' &= -\tfrac{1}{2}\varepsilon\sqrt{2h-r^2}\left(8h\cos\alpha\cos^2(\alpha+\theta) + 2r^2\sin\alpha(\sin 2(\alpha+\theta) + 3\sin 2\theta)\right) + O(\varepsilon^2), \\ \alpha' &= \tfrac{2\varepsilon}{r\sqrt{2h-r^2}}\sin\alpha((2h^2-5hr^2+2r^4)\cos 2(\alpha+\theta) + r^2(3h-2r^2)\cos 2\theta + 2h(h-r^2)) + O(\varepsilon^2). \end{aligned}$$

Now system (14) satisfies all assumptions of Theorem 5 and it is in the normal form (19). We define the function

$$(15) \qquad \begin{aligned} f_1(r,\alpha) &= (f_{11}, f_{12}) \\ &= \frac{1}{2\pi}\int_0^{2\pi}(F_{11}, F_{12})d\theta \\ &= \left(-2h\sqrt{2h-r^2}\cos\alpha, \frac{4h(h-r^2)\sin\alpha}{r\sqrt{2h-r^2}}\right), \end{aligned}$$

where

$$F_{11} = -\frac{1}{2}\sqrt{2h-r^2}\left(8h\cos\alpha\cos^2(\alpha+\theta) + 2r^2\sin\alpha(\sin 2(\alpha+\theta) + 3\sin 2\theta)\right)$$

and

$$F_{12} = \frac{2}{r\sqrt{2h-r^2}}\left[\sin\alpha\left((2h^2-5hr^2+2r^4)\cos 2(\alpha+\theta) + r^2(3h-2r^2)\cos 2\theta + 2h(h-r^2)\right)\right].$$

To find the zeros of f_1 first we solve $f_{11} = 0$ with respect to r. These zeros are $r = \pm\sqrt{2h}$ that take f_2 indefinite. Then, we solve $f_{11} = 0$ for α and obtain $\alpha = \pm\pi/2$. Substituting in $f_{12} = 0$ these two values of α we have the following zeros for f_1: $s_1 = (\sqrt{h}, \pi/2)$ and $s_2 = (\sqrt{h}, -\pi/2)$. Of course $r = -\sqrt{h}$ is a zero for f_2 where $\alpha = \pm\pi/2$.

Now we shall verify at what zeros of f_1 we have a non–zero Jacobian. The Jacobian of f_1 is

$$
Jf_1 = \begin{vmatrix} \dfrac{2hr\cos\alpha}{\sqrt{2h-r^2}} & 2h\sqrt{2h-r^2}\sin\alpha \\ -\dfrac{8h^3\sin\alpha}{r^2(2h-r^2)^{3/2}} & \dfrac{4h(h-r^2)\cos\alpha}{r\sqrt{2h-r^2}} \end{vmatrix}, \tag{16}
$$

and the Jacobian restricted at each zero s_1 and s_2 takes the value $16h^2 > 0$. In short, by Theorem 5 the solutions s_1 and s_2 of $f_1(r,\alpha) = 0$ provide two periodic solutions of system (14), and consequently of the Hamiltonian system (7) on the level $h > 0$. This completes the proof of Theorem 1.

3. Proof of Theorem 2

We know by Theorem 1 that the regularized Hill lunar system at every positive Hamiltonian level has at least 2 periodic solutions corresponding to solutions s_1 and s_2, and that their associated Jacobians are non–zero, that is, $Jf_1(s_i) = 16h^2$ for $i = 1,2$. So the corresponding multipliers are not all equal to 1 (see for more details the last part of appendix 4.2). Hence, by Theorem 4, either the regularized Hill lunar problem cannot be Liouville–Arnold integrable with any second C^1 first integral G, or this system is Liouville-Arnold integrable and the vector gradient of H and G are linearly dependent on some points of these periodic orbits. Therefore Theorem 2 is proved.

4. Appendix

4.1. **Periodic orbits and the Liouville–Arnold integrability.** We shall summarize some facts on the Liouville–Arnold integrability of the Hamiltonian systems, and on the theory of the periodic orbits of the differential equations, for more details see [1, 2] and the subsection 7.1.2 of [2], respectively. We present these results for Hamiltonian systems of two degrees of freedom, because we are studying a Hamiltonian system with two degrees of freedom associated to regularized Hill lunar problem, *but these results work for an arbitrary number of degrees of freedom.*

We recall that a Hamiltonian system with Hamiltonian H of two degrees of freedom is *integrable in the sense of Liouville–Arnold* if it has a first integral G independent with H (i.e. the gradient vectors of H and G are *independent* in all the points of the phase space except perhaps in a set of zero Lebesgue measure), and in *involution* with H (i.e. the parenthesis of Poisson of H and G is zero). For Hamiltonian systems with two degrees of freedom the involution condition is redundant, because the fact that G is a first integral of the Hamiltonian system, implies that the mentioned Poisson parenthesis is always zero. A flow defined on a subspace of the phase space is *complete* if its solutions are defined for all time.

Now we shall state the Liouville–Arnold Theorem restricted to Hamiltonian systems of two degrees of freedom.

Theorem 3. *Suppose that a Hamiltonian system with two degrees of freedom defined on the phase space M has its Hamiltonian H and the function G as two independent first integrals in involution. If $I_{hc} = \{p \in M : H(p) = h \text{ and } C(p) = c\} \neq \emptyset$ and (h,c) is a regular value of the map (H,G), then the following statements hold.*

(a) *I_{hc} is a two dimensional submanifold of M invariant under the flow of the Hamiltonian system.*

(b) *If the flow on a connected component I^*_{hc} of I_{hc} is complete, then I^*_{hc} is diffeomorphic either to the torus $\mathbb{S}^1 \times \mathbb{S}^1$, or to the cylinder $\mathbb{S}^1 \times \mathbb{R}$, or to the plane $\mathbb{R}^2$. If I^*_{hc} is compact, then the flow on it is always complete and $I^*_{hc} \approx \mathbb{S}^1 \times \mathbb{S}^1$.*

(c) *Under the hypothesis* (b) *the flow on I^*_{hc} is conjugated to a linear flow on $\mathbb{S}^1 \times \mathbb{S}^1$, on $\mathbb{S}^1 \times \mathbb{R}$, or on $\mathbb{R}^2$.*

The main result of this theorem is that the connected components of the invariant sets associated with the two independent first integrals in involution are generically submanifolds of the phase space, and if the flow on them is complete then they are diffeomorphic to a torus, a cylinder or a plane, where the flow is conjugated to a linear one.

Using the notation of Theorem 3 when a connected component I^*_{hc} is diffeomorphic to a torus, either all orbits on this torus are periodic if the rotation number associated to this torus is rational, or they are quasi-periodic (i.e. every orbit is dense in the torus) if the rotation number associated to this torus is not rational.

We consider the autonomous differential system

$$\dot{x} = f(x), \tag{17}$$

where $f\colon U \to \mathbb{R}^n$ is C^2, U is an open subset of $\mathbb{R}^n$ and the dot denotes the derivative respect to the time t. We write its general solution as $\phi(t, x_0)$ with $\phi(0, x_0) = x_0 \in U$ and t belonging to its maximal interval of definition.

We say that $\phi(t, x_0)$ is *T-periodic* with $T > 0$ if and only if $\phi(T, x_0) = x_0$ and $\phi(t, x_0) \neq x_0$ for $t \in (0, T)$. The periodic orbit associated to the periodic solution $\phi(t, x_0)$ is $\gamma = \{\phi(t, x_0), t \in [0, T]\}$. The variational equation associated to the T-periodic solution $\phi(t, x_0)$ is

$$\dot{M} = \left(\frac{\partial f(x)}{\partial x}\Big|_{x=\phi(t,x_0)} \right) M, \tag{18}$$

where M is an $n \times n$ matrix. The *monodromy matrix* associated to the T-periodic solution $\phi(t, x_0)$ is the solution $M(T, x_0)$ of (18) satisfying that $M(0, x_0)$ is the identity matrix. The eigenvalues λ of the monodromy matrix associated to the periodic solution $\phi(t, x_0)$ are called the *multipliers* of the periodic orbit.

For an autonomous differential system, one of the multipliers is always 1, and its corresponding eigenvector is tangent to the periodic orbit.

A periodic orbit of an autonomous Hamiltonian system always has two multipliers equal to one. One multiplier is 1 because the Hamiltonian system is autonomous, and another is 1 due to the existence of the first integral given by the Hamiltonian.

Theorem 4. *If a Hamiltonian system with two degrees of freedom and Hamiltonian H is Liouville–Arnold integrable, and G is a second first integral such that the gradients of H and G are linearly independent at each point of a periodic orbit of the system, then all the multipliers of this periodic orbit are equal to* 1.

Theorem 4 is due to Poincaré [15]. It gives us a tool to study the non Liouville–Arnold integrability, independently of the class of differentiability of the second first integral. The main problem for applying this theorem is to find periodic orbits having multipliers different from 1.

4.2. **Averaging Theory of First Order.** Now we shall present the basic results from averaging theory that we need for proving the results of this paper.

The next theorem provides a first order approximation for the periodic solutions of a periodic differential system, for the proof see Theorems 11.5 and 11.6 of Verhulst [17].

Consider the differential equation

$$\dot{\mathbf{x}} = \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{19}$$

with $\mathbf{x} \in D \subset \mathbb{R}^n,\ t \geq 0$. Moreover we assume that both $F_1(t,\mathbf{x})$ and $F_2(t,\mathbf{x},\varepsilon)$ are $T-$periodic in t. Separately we consider in D the averaged differential equation

$$\dot{\mathbf{y}} = \varepsilon f_1(\mathbf{y}), \quad \mathbf{y}(0) = \mathbf{x}_0, \tag{20}$$

where

$$f_1(\mathbf{y}) = \frac{1}{T}\int_0^T F_1(t,\mathbf{y})dt.$$

Under certain conditions, equilibrium solutions of the averaged equation turn out to correspond with $T-$periodic solutions of equation (19).

Theorem 5. *Consider the two initial value problems* (19) *and* (20). *Suppose:*

(i) *F_1, its Jacobian $\partial F_1/\partial x$, its Hessian $\partial^2 F_1/\partial x^2$, F_2 and its Jacobian $\partial F_2/\partial x$ are defined, continuous and bounded by a constant independent of ε in $[0,\infty) \times D$ and $\varepsilon \in (0,\varepsilon_0]$.*
(ii) *F_1 and F_2 are $T-$periodic in t (T independent of ε).*
(iii) *$y(t)$ belongs to Ω on the interval of time $[0,1/\varepsilon]$.*

Then the following statements hold.

(a) *For $t \in [1,\varepsilon]$ we have that $x(t) - y(t) = O(\varepsilon)$, as $\varepsilon \to 0$.*
(b) *If p is a singular point of the averaged equation* (20) *and*

$$\det\left(\frac{\partial f_1}{\partial \mathbf{y}}\right)\Big|_{\mathbf{y}=p} \neq 0,$$

then there exists a $T-$periodic solution $\varphi(t,\varepsilon)$ of equation (19) *which is close to p such that $\varphi(0,\varepsilon) \to p$ as $\varepsilon \to 0$.*
(c) *The stability or instability of the limit cycle $\varphi(t,\varepsilon)$ is given by the stability or instability of the singular point p of the averaged system* (20). *In fact, the singular point p has the stability behavior of the Poincaré map associated to the limit cycle $\varphi(t,\varepsilon)$.*

In the follow we use the idea of the proof of Theorem 5(c). For more details see the section 6.3 and 11.8 of [17]. Suppose that $\varphi(t,\varepsilon)$ is a periodic solution of (19) corresponding to $y = p$ a singular point of the averaged equation (20). We linearise equation (19) in a neighbourhood of the periodic solution $\varphi(t,\varepsilon)$ and obtain a linear equation with $T-$periodic coefficients

$$\dot{x} = \varepsilon A(T,\varepsilon)x, \tag{21}$$

with $A(t,\varepsilon) = \dfrac{\partial}{\partial x}\left[F_1(t,x) + \varepsilon F_2(t,x,\varepsilon)\right]_{x=\varphi(t,\varepsilon)}$.

We introduce the $T-$periodic matrix

$$B(t) = \frac{\partial F_1}{\partial x}(t,p).$$

From Theorem 5 we have $\lim_{\varepsilon\to 0} A(t,\varepsilon) = B(t)$. We shall use the matrices

$$B_1 = \frac{1}{T}\int_0^T B(t)dt$$

and

$$C(t) = \int_0^t \left[B(s) - B_1\right] ds.$$

Note that B_1 is the matrix of the linearized averaged equation. The matrix $C(t)$ is T–periodic and it has average zero. The near-identity transformation

$$x \mapsto y = (I - \varepsilon C(t))x \tag{22}$$

writes equation (21) as

$$\dot{y} = \varepsilon B_1 y + \varepsilon(A(t,\varepsilon) - B(t))y + O(\varepsilon^2). \tag{23}$$

We note that $A(t,\varepsilon) - B(t) \to 0$ as $\varepsilon \to 0$, and also that the characteristic exponents of equation (23) depend continuously on the small parameter ε. It follows that, for ε sufficiently small, if the determinant of B_1 is not zero, then 0 is not an eigenvalue of the matrix B_1 and then it is not a characteristic exponent of (23). By the near-identity transformation we obtain that system (21) has not multipliers equal to 1.

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