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# THE INDEX OF SINGULARITIES OF VECTOR FIELDS AND FINITE JETS

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Abstract. We describe when the index of a singularity of a smooth vector field is determined by a finite jet at the singularity. We also give some criteria to determine some terms from the formal series expansion of the vector field at the singularity which determine the index.

## 1. Introduction and statement of results

There exist a large literature regarding whether some finite jet of a smooth vector field determines the phase portrait near a singularity up to $C^0$ conjugacy (or even smooth conjugacies), starting with Hartman-Grobman Theorem for 1-jets (Sternberg for higher regularity), continuing with results by Takens, Dumortier and others for higher order jets (see [4],[5],[9]). There is also some work in determining whether some terms in the formal series expansion at a singularity determine the local phase portrait, using homogeneous or quasi-homogeneous components, or more general the Newton diagram and the principal part (see [2],[6],[7]). The aim of this paper is to obtain similar results, when instead of determining the phase portrait, we only want to determine the index of the singularity.

Let $\mathcal{V}^n$ be the space of $C^\infty$ vector fields on $\mathbb{R}^n$ with a singularity (a zero of the vector field, or a fixed point for the corresponding flow) at the origin, and $J^n_k$ the space of $k$-jets of (germs of) such vector fields, where $k \in \mathbb{N} \cup \{\infty\}$. Abusing notations, we denote by $\pi_k : \mathcal{V}^n, J^n_l \to J^n_k$ the natural projection from the respective spaces to the space of $k$-jets, for $l \geq k$. Let $\mathcal{J}$ be the ring of jets of (germs of) $C^\infty$ maps from $\mathbb{R}^n$ to $\mathbb{R}$. Most of the following definitions and results can be extended to vector fields with enough regularity, or to finite jets, but for simplicity we restrict ourselves to the smooth case. Also we will see later that the concept of index stability is equivalent with the fact that the origin is an isolated zero of the vector field in a robust way; with this in mind most of these results can be applied also for studying whether the zeros of functions $f : \mathbb{R}^n \to \mathbb{R}^m$ are (stable) isolated. Similar results can be obtained for the index of fixed points of differentiable functions $f : \mathbb{R}^n \to \mathbb{R}^n$, in this case one has to study the zeros of $f - Id$.

A vector field $f \in \mathcal{V}^n$ (or a jet in $J^n_\infty$) is called *index finitely determined* if there exists $k \in \mathbb{N}$ such that all the $C^k$ vector fields having the $k$-jet equal to $\pi_k(f)$ have an isolated singularity at the origin with the same index. If

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we want to emphasize the value of $k$, we call the vector field or the $C^\infty$ jet *index $k$-determined*, and we call its $k$-jet *index determining*.

A jet $f \in J^n_k$ is called *index stabilisable* if there exist $l \geq k$ and $g \in J^n_l$ such that $\pi_k(g) = f$ and $g$ is index determining (or $l$-determined).

A vector field $f \in \mathcal{V}^n$ (or a jet in $J^n_\infty$) is called *$k$-Lojasiewicz* if there exists $c, \delta > 0$ such that $\|f(x)\| \geq c\|x\|^k,\ \forall x \in \mathbb{R}^n, \|x\| < \delta$. It is called simply *Lojasiewicz* if it is $k$-Lojasiewicz for some $k \in \mathbb{N}$.

We remark that the vector field $f \in \mathcal{V}^n$ is index $k$-determined, or $k$-Lojasiewicz, if and only if the jet $\pi_\infty f \in J^n_\infty$ is index $k$-determined, respectively $k$-Lojasiewicz. In fact, this is just a property of the $k$-jet $\pi_k(f)$, as we see from the following proposition.

**Proposition 1.** *Let $f \in \mathcal{V}^n$ (or $J^n_\infty$) and $k \in \mathbb{N}$. The following statements are equivalent.*
*(a) $f$ is index $k$-determined.*
*(b) The origin is an isolated singularity for every $C^k$ vector fields $g$ with $\pi_k(g) = \pi_k(f)$.*
*(c) $f$ is $k$-Lojasiewicz.*
*(d) $\pi_k(f)$ satisfies a $k$-Lojasiewicz condition.*

The same results of Proposition 1 hold for $k$ replaced by $\infty$ (index $k$-determined replaced by index finitely determined, and $k$-Lojasiewicz replaced by Lojasiewicz). The only implication which is not completely trivial is the following.

**Proposition 2.** *If a vector field $f \in \mathcal{V}^n$ (or a jet in $J^n_\infty$) is not Lojasiewicz, then there exists a vector field $g \in \mathcal{V}^n$, with the same infinite jet as $f$, and for which the origin is not an isolated singularity.*

We obtain that most of the jets are index finitely determined.

**Proposition 3.** *Every finite jet is index stabilisable. The subset of jets in $J^n_\infty$ which are not index finitely determined has codimension infinity.*

In dimension two the result is similar to the one regarding the determination of the phase portrait instead of the index. However in higher dimension the situation is different. Although every jet is index stabilisable, there exist finite jets in dimension greater than two which are non-stabilisable for $C^0$ conjugacy (Dumortier, see [5]).

The following is an abstract condition which guarantees that a vector field is Lojasiewicz. An ideal $I$ of $\mathcal{J}$ is said to have *finite codimension* if it contains a power of the unique maximal ideal.

**Proposition 4.** *Let $f = (f_1, f_2, \ldots, f_n) \in \mathcal{V}^n$, and let $I$ be the ideal generated by $f_1, f_2, \ldots, f_n$ in $\mathcal{J}$ in $\mathcal{J}$. If $I$ has finite codimension, then $f$ is Lojasiewicz, or index finitely determined.*

The converse of this is not necessarily true, one has for example the vector field $f = (x^2+y^2, x^2+y^2)$ on $\mathbb{R}^2$ which is Lojasiewicz, but the ideal generated by its components does not have finite codimension in $\mathcal{J}$.

There is the following conjecture, attributed by Dumortier to R. Thom (see [4]).

**Conjecture** ($C^\infty$ Curve Selection Lemma)**.** *Let $f \in \mathcal{V}^n$ such that the origin is not an isolated singularity. Then there exists a curve $\gamma : [-1, 1] \to \mathbb{R}^n$, $\gamma(0) = 0$, with nontrivial $C^\infty$ jet at zero, such that the $C^\infty$ jet of $f \circ \gamma$ is trivial.*

As we saw before, the condition that the origin is not an isolated singularity is equivalent to the fact that $f$ is not Lojasiewicz, which is in fact an algebraic condition on the $C^\infty$ jet of $f$. We have the following partial result in this direction.

**Proposition 5.** *Let $f \in \mathcal{V}^n$ such that the origin is not an isolated singularity, or is not Lojasiewicz. Then for any integer $k > 0$, there exist a curve $\gamma : [-1, 1] \to \mathbb{R}^n$ and an integer $l > 0$ such that $\gamma(0) = 0$, the $l$ jet of $\gamma$ at zero is nontrivial while the $(l - 1)$ jet of $\gamma$ at zero is trivial, and the $lk$ jet of $f \circ \gamma$ at zero is trivial (identically zero).*

Now we turn our attention to finding the terms from the expansion of a smooth vector field which determine the index of the singularity at the origin. First we have a weighted (or quasi-homogeneous) version of Proposition 1.

**Proposition 6** (Weighted Lojasiewicz Condition)**.** *Let $f = (f_1, f_2, \dots, f_n) \in \mathcal{V}^n$, $k > 0$ and $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ be strictly positive real numbers. Assume that there exist $C, \delta > 0$ such that $\forall x \in \mathbb{R}^n$, $\|x\| < \delta$ we have*

$$|f_1(x)|^{b_1} + |f_2(x)|^{b_2} + \cdots + |f_n(x)|^{b_n} \geq C(|x_1|^{a_1} + |x_2|^{a_2} + \cdots + |x_n|^{a_n})^k;$$

*Then the index at the origin is determined by the terms from the expansion of $f_i$ containing $x_1^{c_1} x_2^{c_2} \dots x_n^{c_n}$ with $\frac{c_1}{a_1} + \frac{c_2}{a_2} + \cdots + \frac{c_n}{a_n} \leq \frac{k}{b_i}$, for all $1 \leq i \leq n$.*

This proposition suggests a strategy which can be used to find the index of a singularity of a vector field. Let $f = (f_1, f_2, \dots, f_n) \in \mathcal{V}^n$ and the weights $a = (a_1, a_2, \dots, a_n) \in [0, \infty)^n$. Each monomial $Cx_1^{c_1} x_2^{c_2} \dots x_n^{c_n}$ has a *quasi-homogeneous degree with respect to the weights $a$* given by

$$\deg_a(Cx_1^{c_1} x_2^{c_2} \dots x_n^{c_n}) = \sum_{i=1}^{n} \frac{c_i}{a_i}.$$

We make the convention that if some $a_i = c_i = 0$ then $\frac{c_i}{a_i} = 0$, and if $a_i = 0$ and $c_i > 0$ then $\frac{c_i}{a_i} = \infty$ (we allow the possibility to have infinite quasi-homogeneous degree). The notions of weights and degree which we use are a bit different from other papers (where the corresponding weights would be $\frac{1}{a_i}$), but this allows us to consider together the cases when some weights are equal to zero. One can decompose each $f_i$ into quasi-homogeneous components (putting together the terms from the formal series expansion with the same quasi-homogeneous degree with respect to the weights $a$), and let $f_{i,a}$ be the nonzero such component with the lowest quasi-homogeneous degree (if it does not exist we take $f_{i,a} = 0$). This means that the terms in $f_{i,a}$

are of type $Cx_1^{c_1}x_2^{c_2}\dots x_n^{c_n}$, with $\frac{c_1}{a_1}+\frac{c_2}{a_2}+\dots+\frac{c_n}{a_n}=d_{i,a}$ for some $d_{i,a}\geq 0$, and all the remaining terms from the expansion have quasi-homogeneous degree strictly greater than $d_{i,a}$. In the case when all the components of $a$ are nonzero, this is equivalent to the fact that $|f_i(x)-f_{i,a}(x)| = o(|x_1|^{a_i}+|x_2|^{a_2}+\dots+|x_n|^{a_n})^{d_{i,a}}$ near the origin. Let $f_a=(f_{1,a},f_{2,a},\dots,f_{n,a})$ be the *$a$-principal part of $f$*. We recover the following result from Theorem 1.1 in [3], with a different proof.

**Theorem 7.** *Let $f, a$ and $f_a$ be as above, with all the weights $a_i$ strictly positive. If the origin is an isolated singularity for $f_a$, then the index of the origin for $f$ and $f_a$ coincide.*

If we want to apply this result to determine the index at a singularity of a given vector field, then the candidates for the weights $a_i$ should correspond to the faces of the respective Newton polyhedron (see the definition bellow). Of course this strategy does not always work, and one will have to consider some additional higher order terms. The fact that in $f_a$ we choose only the terms with the lowest (quasi-homogeneous) degree is essential, without it the result is not true. For instance, the vector field $f=(x-y^2,x-y^2+x^2-y^4)$ has a curve of singularities through the origin, $x=y^2$, while $\pi_2(f)=(x-y^2,x-y^2+x^2)$ has an isolated singularity at the origin.

There is also a method to find whether the principal parts of the components of the vector field determine the index of the singularity. The result is similar to the one obtained by Brunella and Miari (see [2]) regarding the phase portrait near a singularity in dimension two, but instead of the blowing-up technique we use the Curve Selection Lemma for semi-algebraic sets (Milnor, see [8]). This allows us to use the principal parts of all the components of the vector field, and to extend the result to any dimension.

Let $f=(f_1,f_2,\dots,f_n)\in\mathcal{V}^n$, $\mathbb{N}$ the set of non-negative integers, and assume that each $f_i$ has the formal series expansion around the origin, or the $C^\infty$-jet

$$\bar{f}_i(x)=\sum_{b\in\mathbb{N}^n} c_{i,b}x_1^{b_1}x_2^{b_2}\dots x_n^{b_n},$$

where $b=(b_1,b_2,\dots,b_n)$. We say that $S_i=\{b\in\mathbb{N}^n : c_{i,b}\neq 0\}$ is the *support of $f_i$*, and $\Gamma_i=\overline{co}(\{S_i+[0,\infty)^n\})$ is the *Newton polyhedron of $f_i$*, where $\overline{co}(M)$ is the convex envelope of the set $M$. We call the *Newton diagram of $f_i$* the union $\gamma_i$ of the compact faces of the Newton polyhedron $\Gamma_i$, and then the *principal part of $f_i$* is $f_{i,P}=\sum_{b\in\gamma_i}c_{i,b}x^b$, and the *principal part of $f$* is $f_P=(f_{1,P},f_{2,P},\dots,f_{n,P})$ (the notations are similar to the ones in Brunella-Miari, but the principal part is defined in a different way). From this definition one can see that the principal part of $f$ ($f_P$) contains the $a$-principal parts of $f$ ($f_a$) for all the weights $a$ (and nothing more). We have the following result.

**Theorem 8.** *Let $f\in\mathcal{V}^n$ with the principal part $f_P$. Assume that for all the weights $a\in\mathbb{N}^n$, $f_a$ has singularities only on the coordinate axes.*

*Then $f_P$ determines the index of the origin for $f$.*

A similar result for analytic vector fields is obtained in [1] using different methods. We remark that each $f_a$ is actually determined by $f_P$, so the condition required is only on the principal part $f_P$, and not on $f$. Using the arguments from [2], one can easily prove that the condition $(i)$ is satisfied for an open and dense subset of the set of principal parts corresponding to $n$ given convenient Newton diagrams (for example Newton diagrams which intersect all the coordinate axes). In applications, obviously one does not have to check for all the possible weights $a \in \mathbb{N}^n$, it suffices to consider the ones corresponding to the faces and edges of the Newton diagrams (however, in high dimensions it may still be a lot); this is because once one quasi-homogeneous component $f_{i,a}$ is a monomial, the required condition is clearly satisfied.

The condition required in Theorem 8 is similar to the notion of (strongly) non-degenerate maps from [1]. We remark that some weights $a_i$ can be zero; for example if some $a_i$ is one and all the other ones are zero, the condition says that the principal part of $f$ (or the formal expansion of $f$) must contain a term which is just some power of $x_i$. This condition is actually necessary for the origin to be an isolated singularity for $f_P$, because otherwise the $i$-th coordinate axis would be a line of singularities.

In Section 2 we will give the proofs of these results. In Section 3 we will present some examples and discuss how the results can be generalized even further.

## 2. Proofs

*Proof of Proposition 1.* **(a)⇒(b)** It follows directly from the definition of index $k$-determinacy.

**(b)⇒(c)** Assume that $f$ is not $k$-Lojasiewicz. We will construct a sequence of $C^\infty$ vector fields $f_n$ converging in the $C^k$ topology to the $C^k$ vector field $g$, which has the same $k$-jet as $f$, and the origin is not an isolated singularity.

Suppose that we have the $C^\infty$ vector field $f_n$ which has singularities at $x_n \in \mathbb{R}^n$ with $\|x_{k+1}\| < \frac{1}{2}\|x_k\|$, $f_n = f$ on the ball of radius $\frac{1}{2}\|x_n\|$ centered at the origin, and $\|f_n - f_{n-1}\|_{C^k} < \frac{1}{2^n}$ (the construction of $f_1$ is easy, just create a singularity different from the origin). Let $\phi : \mathbb{R}^n \to [0, \infty)$ be a bump function such that $\phi(0) = 1$ and $\phi(x) = 0$ if $\|x\| \geq 1$. Let $C_n < \frac{1}{2^{n+k+1}\|\phi\|_{C^k}}$. Because $f$ is not Lojasiewicz, there exists $x_{n+1} \in \mathbb{R}^n$, $\|x_{n+1}\| < \frac{1}{4}\|x_n\|$, and $\|f(x_{n+1})\| < C_n\|x_{n+1}\|^k$. Let $f_{n+1}$ be equal to $f_n$ outside the ball of radius $\frac{1}{2}\|x_n\|$ centered at the origin, $f_{n+1}(x) = f(x) - \phi(\frac{2}{\|x_{n+1}\|}(x - x_{n+1}))f(x_{n+1})$ inside the ball of radius $\frac{1}{2}\|x_{n+1}\|$ centered at $x_{n+1}$, and $f_{n+1} = f$ in the rest.

Then

$$\begin{aligned}\|f_{n+1}-f_n\|_{C^k} &= \|\phi(\frac{2}{\|x_{n+1}\|}(x-x_{n+1}))f(x_{n+1})\|_{C^k}\\ &\leq \|f(x_{n+1})\|\frac{2^k}{\|x_{n+1}\|^k}\|\phi\|_{C^k}\leq C_n2^k\|\phi\|_{C^k}<\frac{1}{2^{n+1}}.\end{aligned}$$

Thus $f_n$ is a Cauchy sequence in the $C^k$ topology, so it is convergent to some $C^k$ vector field $g$. The $k$-jet of all $f_n$ at the origin is the same with the one of $f$, so the same must be true for the $k$-jet of $g$. Finally, $x_n$ is a sequence of singularities of $g$ which converges to the origin, which ends the proof.

**(c)⇒(d)** Since $f$ is Lojasiewicz, there exist $C,\delta>0$ such that $\|f(x)\|\geq C\|x\|^k$ if $\|x\|<\delta$. Because $f$ is $C^k$, there exist $\delta_1>0$ such that $\|f(x)-\pi_k(f)(x)\|\leq\frac{C}{2}\|x\|^k$ if $\|x\|<\delta_1$. But this implies that $\|\pi_k(f)(x)\|\geq\frac{C}{2}\|x\|^k$ if $\|x\|<\min\{\delta,\delta_1\}$, so the $k$-jet is also Lojasiewicz.

**(d)⇒(a)** It is enough to prove that if $\pi_k(f)$ is $k$-Lojasiewicz, then the index at the origin for $f$ and $\pi_k(f)$ are the same. Let $f_t(x)=t\pi_k(f)(x)+(1-t)f(x)$, for $t\in[0,1]$ be a homotopy between $\pi_k(f)$ and $f$. We know that for $\|x\|$ small we have $\|\pi_k(f)(x)\|\geq C\|x\|^k$ for some $C>0$ (the Lojasiewicz condition), and $\|f(x)-\pi_k(f)(x)\|\leq\frac{C}{2}\|x\|^k$ ($f$ is $C^k$). Then there exists $\delta<0$ such that for all $\|x\|<\delta$ we have

$$\|f_t(x)\|\geq\|\pi_k(f)(x)\|-(1-t)\|f(x)-\pi_k(f)(x)\|\geq\frac{C}{2}\|x\|^k.$$

Then the origin is the only singularity in the ball of radius $\frac{\delta}{2}$ around the origin for every $f_t$, and a standard homotopy argument implies that the indexes of the origin for $f$ and $\pi_k(f)$ coincide. □

*Proof of Proposition 2.* As in the previous proof, we will construct $g$ as the limit of a sequence $f_n$, such that there exists a sequence $x_n\to0$, $\|x_{n+1}\|<\frac{1}{4}\|x_n\|$, $f_n$ has singularities at $0,x_1,x_2,\ldots,x_n$, $f_n=f$ on the ball of radius $\frac{1}{2}\|x_n\|$ (in particular it has the same jet as $f$), and $\|f_{n+1}-f_n\|_{C^n}<\frac{1}{2^n}$. Standard arguments imply then that $f_n$ is Cauchy in every $C^m$ topology, $m\in\mathbb{N}$, so convergent to a $C^\infty$ function $g$, which has the same jet as $f$, and singularities at each $x_n$, which finishes the proof.

To construct $f_1$ it is again enough to make some $C^\infty$ perturbation of $f$ supported on a small neighborhood of $x_1$, such that $x_1$ becomes a singularity. Now assume that we constructed $f_1,f_2,\ldots,f_n$ with the required properties. Let $C_n<\frac{1}{2^{2n}\|\phi\|_{C^k}}$. Because $f$ is not Lojasiewicz, there exists $x_{n+1}\in\mathbb{R}^n$, $\|x_{n+1}\|<\frac{1}{4}\|x_n\|$, such that $\|f(x_{n+1})\|<C_n\|x_{n+1}\|^n$. Let $f_{n+1}$ be equal to $f_n$ outside the ball of radius $\frac{1}{2}\|x_n\|$ centered at the origin, $f_{n+1}(x)=f(x)-\phi(\frac{2}{\|x_{n+1}\|}(x-x_{n+1}))f(x_{n+1})$ inside the ball of radius $\frac{1}{2}\|x_{n+1}\|$ centered

at $x_{n+1}$, and $f_{n+1} = f$ in the rest. Then

$$\begin{aligned}\|f_{n+1} - f_n\|_{C^n} &= \|\phi(\frac{2}{\|x_{n+1}\|}(x - x_{n+1}))f(x_{n+1})\|_{C^n} \\ &\leq \|f(x_{n+1})\|\frac{2^n}{\|x_{n+1}\|^n}\|\phi\|_{C^n} \leq C_n 2^n \|\phi\|_{C^n} < \frac{1}{2^n}.\end{aligned}$$

The other required conditions of the induction are clearly satisfied. □

*Proof of Proposition 3.* Let $f \in J^n_k$ be a finite jet. Let $l > k$ be an even integer. Let $F : \mathbb{R}^{2n} \to \mathbb{R}^n$ given by $F_i(x, y) = f_i(x) + y_i(x_1^l + x_2^l + \cdots + x_n^l)$. On the set $A_r = \{(x, y) \in \mathbb{R}^{2n} : \|x\| \geq r\}$ the derivative $DF$ has maximal rank. This implies that zero is a regular value, so $M_r = F^{-1}(0)$ is a smooth immersed manifold of dimension $n$. Let $p_r : M_r \to \mathbb{R}^n$ be the projection into the second coordinate. Let $Y_r \subset \mathbb{R}^n$ be the set of regular values of $p_r$, meaning that if $y \in Y_r$ then $p_r^{-1}(y) = \{(x, y) \in \mathbb{R}^{2n} : F(x, y) = 0\}$ is an isolated set in $M_r$, so in $A_r$. This means that the zeros of $f_y = F(\cdot, y)$ are isolated outside the ball of radius $r$ centered at the origin. Let $r_j \to 0$ and $Y = \cap_{j=1}^{\infty} Y_{r_j}$, which will have full measure in $\mathbb{R}^n$ by Sard's Theorem. For some $y \in Y$ we get that the zeros of $f_y$ must be isolated outside the origin. Then the origin must be an isolated zero for $f_y$, because otherwise the zero set must contain an algebraic variety passing through origin (because $f_y$ is polynomial), which contradicts the above statement.

So the origin is an isolated zero for the $l$-jet $f_y$, which implies that $f_y$ is $m$-Lojasiewicz for some positive integer $m$, so $f_y$ is $m$-determined (viewed as an $m$-jet). Since the $k$-jet of $f_y$ is $f$, we obtain that $f$ is index stabilising. □

*Proof of Proposition 4.* If the ideal has finite codimension, then it contains a power of the maximal ideal, say $\mathcal{M}^k$ for some positive integer $k$. This implies that $x_1^k, x_2^k, \ldots, x_n^k$ belong to this ideal, or there exist jets $g_i^j$ such that $\sum_{i=1}^n g_i^j f_i = x_j^k$. Let $\tilde{f}_i, \tilde{g_i^j}$ be $C^\infty$ functions with the corresponding jets and $h_j = \sum_{i=1}^n \tilde{g_i^j} \tilde{f}_i$ the $C^\infty$ function with the jet $x_j^k$. From the Cauchy-Schwarz inequality we get that

$$|\sum_{i=1}^n \tilde{f_i}^2| \, |\sum_{i=1}^n (\tilde{g_i^j})^2| \geq |\sum_{i=1}^n \tilde{g_i^j} \tilde{f}_i|^2 = h_j^2.$$

Since $h_j$ is $C^\infty$ with the jet $x_j^k$, there exists $\delta > 0$ such that for $\|x\| < \delta$ we have $|h_j(x)| \geq \frac{|x_j|^k}{2}$. Let $A_j = \sup_{\|x\|<\delta, 1\leq j\leq n} |\tilde{g_i^j}|$. We get that

$$\|\tilde{f}(x)\|^2 = |\sum_{i=1}^n \tilde{f_i}^2(x)| \geq \frac{h_j^2}{|\sum_{i=1}^n (\tilde{g_i^j})^2|} \geq \frac{|x_j|^{2k}}{4A_j^2} = C_j |x_j|^{2k}$$

for $\|x\| < \delta$. A similar inequality holds for every $1 \leq j \leq n$, so $\|\tilde{f}(x)\| \geq \frac{C}{n^{k/2}}\|x\|^k$ for $\|x\| < \delta$ and $C = \min_{1\leq j\leq n} C_j$, so $\tilde{f}$ (or $f$) is Lojasiewicz. □

In order to prove Proposition 5 we need the following result (see [8])

**Lemma 9** (Analytic Curve Selection Lemma for Semi-algebraic Sets)**.** *Let $A \subset \mathbb{R}^n$ be a semi-algebraic set, and assume that the origin is an accumulation point for $A$. Then there exists an analytic curve $\gamma : [0,1] \to \mathbb{R}^n$ such that $\gamma(0) = 0$ and $\gamma(t) \in A$ for all $t \in (0,1)$.*

*Proof of Proposition 5.* We know that $f$ is not Lojasiewicz, so in particular it is not $(k+1)$-Lojasiewicz, or for some $C > 0$, the origin is an accumulation point of the semi-algebraic set $A = \{x \in \mathbb{R}^n : \|f(x)\| < C\|x\|^{k+1}\}$. From Lemma 9 we obtain that there exists a curve $\gamma : [0,1] \to \mathbb{R}^n$ such that $\gamma(0) = 0$, and $\gamma(t) \in A$ for all $t \in (0,1)$. Because $\gamma$ is analytic and non-constant, some $l$-jet must be the first non-zero one, so $\|\gamma(t)\| \leq C_1 t^l$ for some $C > 0$. But because $\gamma(t) \in A$ for $t \in (0,1)$ we have $\|f(\gamma(t))\| \leq C\|\gamma(t)\|^{k+1} \leq C_2 t^{l(k+1)}$ for all $t \in [0,1)$, which implies that the $lk$-jet of $f \circ \gamma$ is trivial. □

In order to prove Proposition 6 we will use the following lemma.

**Lemma 10.** *Let $x_i \in \mathbb{R}$, $a_i \in (0,\infty)$, and $c_i \in [0,\infty)$, for $1 \leq i \leq n$. Then*

$$|x_1|^{c_1}|x_2|^{c_2}\dots|x_n|^{c_n} \leq (|x_1|^{a_1} + |x_2|^{a_2} + \cdots + |x_n|^{a_n})^{\frac{c_1}{a_1}+\frac{c_2}{a_2}+\cdots+\frac{c_n}{a_n}}.$$

*Proof.* Let $|x_i|^{a_i} = y_i$ and $\frac{c_i}{a_i} = b_i$. Then evaluating the right hand side of the inequality we obtain

$$\begin{aligned} RHS &= (y_1 + y_2 + \cdots + y_n)^{b_1}(y_1 + y_2 + \cdots + y_n)^{b_2}\dots(y_1 + y_2 + \cdots + y_n)^{b_n} \\ &\geq y_1^{b_1} y_2^{b_2} \dots y_n^{b_n} \\ &= |x_1|^{c_1}|x_2|^{c_2}\dots|x_n|^{c_n} \quad q.e.d. \end{aligned}$$

□

*Proof of Proposition 6.* Let $g = (g_1, g_2, \dots, g_n)$, where $g_i$ contains the terms from the expansion of $f_i$ containing $x_1^{c_1}x_2^{c_2}\dots x_n^{c_n}$ with $\frac{c_1}{a_1} + \frac{c_2}{a_2} + \dots \frac{c_n}{a_n} \leq \frac{k}{b_i}$, for all $1 \leq i \leq n$. There is a homotopy from $g$ to $f$ defined by $f_t = tg+(1-t)f = f-t(f-g)$. Now $f$ satisfies the weighted Lojasiewicz condition, and $\sum_{i=1}^n |f-g|^{b_i}$ can be bounded in terms of $\frac{C}{2}(|x_1|^{a_1}+|x_2|^{a_2}+\cdots+|x_n|^{a_n})^k$ for $\|x\|$ small enough using Lemma 10 for a finite number of terms from the expansion of $f$, and the sufficient high differentiability of the remainder for each $f_i$. In conclusion, each $f_t$ satisfies the weighted Lojasiewicz condition with $C$ replaced by $\frac{C}{2}$ and eventually a smaller $\delta$, so the origin is the only singularity inside a small ball for all $f_t$, and the same homotopy argument as in the proof of Proposition 1 finishes the proof. □

*Proof of Theorem 7.* From Proposition 6, it is enough to prove that $f_a$ satisfies a weighted Lojasiewicz condition. Let $d > 0$ be an integer large enough such that $b_i = \frac{d}{d_{i,a}}$ are even integers, and let $F(x) = \sum_{i=1}^n |f_{i,a}(x)|^{b_i} = \sum_{i=1}^n f_{i,a}(x)^{b_i}$. We remark that $F$ is a quasi-homogeneous polynomial of (quasi-homogeneous) degree $d$, with respect to the weights $a$, so

$$F(t^{\frac{1}{a_1}}x_1, t^{\frac{1}{a_2}}x_2, \dots, t^{\frac{1}{a_n}}x_n) = t^d F(x_1, x_2, \dots, x_n).$$

If the origin is an isolated singularity for $f$, then it is also an isolated zero for $F$, and because of the above relation $F$ must be strictly positive outside the origin. Let $S_a = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : |x_1|^{a_1} + |x_2|^{a_2} + \cdots + |x_n|^{a_n} = 1\}$. $S_a$ is a compact set which does not contain the origin, so there exists $C = \min_{x \in S_a} F(x) > 0$. If $x \in \mathbb{R}^n \setminus \{0\}$ with $|x_1|^{a_1} + |x_2|^{a_2} + \cdots + |x_n|^{a_n} = \frac{1}{t}$ then $y = (t^{\frac{1}{a_1}} x_1, t^{\frac{1}{a_2}} x_2, \ldots, t^{\frac{1}{a_n}} x_n) \in S_a$, so $F(y) = t^d F(x) \geq C$ so $F(x) \geq C\frac{1}{t^d} = C(|x_1|^{a_1} + |x_2|^{a_2} + \cdots + |x_n|^{a_n})^d$, or $f$ satisfies a Lojasiewicz condition, and Proposition 6 finishes the proof. $\square$

*Proof of Theorem 8.* Assume that the conclusion is not true, $f_P$ does not determine the index of $f$. Let $C > 0$ fixed. Let $l > \max\{\sum_{j=1}^{n} b_j : b \in \gamma_i, 1 \leq i \leq n\}$, and $k = l^n$. Let $P_k$ be the set of polynomial vector fields on $\mathbb{R}^n$ of degree $k$ which have the principal part equal to $f_P$.

**Step 1.** There exists $g_0 \in P_k$, $g_n \to g_0$, $x_n \to 0$, such that $\|g_n(x_n)\| < C\|x_n\|^k$.

If this is not true, then for every $g_0 \in P_k$, there is some open connected neighborhood $U \subset P_k$ of $g_0$ and $\delta > 0$ such that $\|g(x)\| \geq C\|x\|^k$ for every $g \in U$, $\|x\| \leq \delta$ (locally uniformly $k$-Lojasiewicz). This implies that all $g \in U$ have no other singularity inside the ball of radius $\delta$ around the origin, so the index of the origin is locally constant in $P_k$, so it is globally constant because $P_k$ is connected. Since all the polynomials of $P_k$ are $k$-Lojasiewicz, this would imply that the index of the origin is constant for all the vector fields with the principal part $f_P$, or that $f_P$ determines the index of $f$, which is a contradiction.

**Step 2.** The application of the Analytic Curve Selection Lemma.

Let $I_i \subset \mathbb{R}^n$ be the set of indexes $j = (j_1, j_2, \ldots, j_n)$ such that $j_1 + j_2 + \cdots + j_n \leq k$ and $j \in \Gamma_i \setminus \gamma_i$. Assume that the cardinality of $I_1 \times I_2 \times \cdots \times I_n$ is $l$ and consider the polynomial with $n + l$ variables

$$g(x, \mu) = f_P + \Big(\sum_{j \in I_1} \mu_j^1 x^j, \sum_{j \in I_2} \mu_j^2 x^j, \ldots, \sum_{j \in I_n} \mu_j^n x^j\Big),$$

where $x^j = x_1^{j_1} x_2^{j_2} \ldots x_n^{j_n}$ and $\mu = (\mu_j^i)_{1 \leq i \leq n,\ j \in I_i} \in \mathbb{R}^l$. There is a one-to-one correspondence between the elements $\mu \in \mathbb{R}^l$ and the polynomials in $P_k$, given by $\mu \to g(\cdot, \mu)$. Assume that $g_0 = g(\cdot, \mu_0)$, and consider the semi-algebraic set

$$A = \{(x, \mu) \in \mathbb{R}^{n+l} : \ \|g(x, \mu + \mu_0)\| < C\|x\|^k\}.$$

From the previous step we know that $(0,0)$ is an accumulation point for $A$, so from Lemma 9 we get that there exists an analytic curve $\gamma : [0,1] \to \mathbb{R}^{n+l}$ such that $\gamma(0) = (0,0)$ and $\gamma(t) = (x(t), \mu(t)) \in A$, $\forall t \in (0,1]$. Consequently $\|g(x(t), \mu(t) + \mu_0)\| < C\|x(t)\|^k$, $\forall t \in (0,1)$.

**Step 3.** Contradicting the quasi-homogeneous condition.

First we remark that $x(t)$ cannot be identically zero, because $\gamma(t)$ must be in $A$. Suppose that the first non-zero term in the expansion of $x_i(t)$ is $c_i t^{a_i}$, with the convention that if some $x_i(t)$ is identically zero then $a_i = \infty$, and all $c_i \neq 0$. Let $\beta = \min_{1\leq i\leq n} a_i$, and $\bar{a} \in \mathbb{R}^n$ be the weights $\bar{a}_i = \frac{1}{a_i}$. An easy computation shows that the first term in the expansion of $g_i(x(t), \mu(t)+\mu_0)$ is exactly $f_{i,\bar{a}}(c)t^{d_{i,\bar{a}}}$, while the first nonzero term in the expansion of $\|x(t)\|^k$ is $Ct^{\beta k}$. From the conclusion of the previous step, $\|g(x(t), \mu(t) + \mu_0)\| < C\|x(t)\|^k$, $\forall t \in (0,1)$, we obtain that if $d_{i,\bar{a}} < \beta k$, then $f_{i,\bar{a}}(c) = 0$. If this holds for all $1 \leq i \leq n$, we get that $f_{\bar{a}}(c) = 0$, and the condition from the hypothesis implies that at least one coefficient $c_i$ must be equal to zero, which is a contradiction with our assumption, so we are done.

Now assume that $d_{i,\bar{a}} \geq \beta k$ for some $i \in \{1, 2, \dots, n\}$. We will see that this happens when some weights $a_i$ are large comparing to the others, and somehow they can be disregarded.

It is easy to see that

$$l \max_{1\leq j\leq n} a_j > d_{i,\bar{a}} \geq \beta k,$$

so $\max_{1\leq j\leq n} a_j > \beta l^{n-1}$ (it can be infinity). Then there exists a partition of $\{1, 2, \dots, n\}$ into two non-empty subsets $A$ and $B$ such that
(i) for any $i \in A$, $j \in B$, we have $\frac{a_j}{a_i} > l$;
(ii) for any $i \in A$, we have $a_i \leq \beta l^{n-2}$.
Let $\tilde{a} \in \mathbb{R}^n$ be the weights $\tilde{a}_i = \bar{a}_i = \frac{1}{a_i}$ if $i \in A$, and $\tilde{a}_i = 0$ if $i \in B$. Let $\tilde{\gamma}_i = \gamma_i \cap \{b \in \mathbb{R}^n : \ b_j = 0, \forall j \in B\}$. If $\tilde{\gamma}_i = \emptyset$, then we have $f_{i,\tilde{a}} = 0$. If $\tilde{\gamma}_i \neq \emptyset$ then we claim that $f_{i,\bar{a}} = f_{i,\tilde{a}}$. For this it is enough to prove that for every $d \in \gamma_i \setminus \tilde{\gamma}_i$ we have $d_1a_1 + \cdots + d_na_n > d_{i,\tilde{a}}$. Since $d \notin \tilde{\gamma}_i$, there exists some $j_0 \in B$ such that $d_{j_0} \geq 1$, and consequently

$$\begin{aligned}
d_1a_1 + \cdots + d_na_n &\geq d_{j_0}a_{j_0} \\
&\geq a_{j_0} \\
&> l \max_{j\in A} a_j \\
&\geq \min_{b\in\tilde{\gamma}_i} \sum_{j\in A} b_ja_j \\
&= \min_{b\in\gamma_i} \frac{b_1}{\tilde{a}_1} + \frac{b_2}{\tilde{a}_2} + \cdots + \frac{b_n}{\tilde{a}_n} \\
&= d_{i,\tilde{a}},
\end{aligned}$$

so the claim is true. On the other hand we have that

$$d_{i,\bar{a}} = d_{i,\tilde{a}} = \min_{b\in\tilde{\gamma}_i} \sum_{j\in A} b_ja_j < \beta l^n.$$

In conclusion, we obtain in this case that $f_{\tilde{a}}(c) = 0$, and again the hypothesis implies that at least one $c_i$ is equal to zero, which is a contradiction. □

## 3. Examples and further remarks

In this section we will present some examples. The first one will be an application of Theorem 7.

**Example 1.** Let $f = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field given by

$$\begin{aligned} f_1(x,y,z) &= 2x + 3y + h.o.t. \\ f_2(x,y,z) &= x^2 - 2y^2 + x^2y + 3z^3 + h.o.t. \\ f_3(x,y,z) &= x^3 - xyz^2 + 2x^2y^2 + 3xz^4 - x^3y^2 + x^6 + x^2y^4 + h.o.t., \end{aligned}$$

where $h.o.t.$ means terms of (homogeneous) degree strictly greater than 1,3 and 6 respectively. If we consider the weights $a = (1,1,1)$ (the homogeneous case), we obtain the $a$-principal part

$$f_a(x,y,z) = (2x + 3y, x^2 - 2y^2, x^3),$$

and clearly the origin is not an isolated singularity (the entire $z$-axis is in the zero set of $f_a$). But if we consider the weights $b = (1,1,2)$, we obtain the $b$-principal part

$$f_b(x,y,z) = (2x + 3y, z^3, x^3 - xyz^2 + 3xz^4),$$

and in this case the origin is an isolated singularity, so the index of $f$ coincides with the index of $f_b$.

The next example is an application of Theorem 8.

**Example 2.** Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field given by

$$\begin{aligned} f_1(x,y) &= x^2y - xy^2 + x^4 - 2x^5 - 3x^2y^3 - 2y^5 + h.o.t. \\ f_2(x,y) &= x^2y + xy^2 - 2x^5 - x^3y^2 + 2x^4y + y^5 + h.o.t., \end{aligned}$$

where $h.o.t.$ means terms of degree strictly greater than 5. Then the Newton diagrams of $f_1$ and $f_2$ are shown in Figure 1, and the principal part of $f$ is

$$f_P(x,y) = (x^2y - xy^2 + x^4 - 2y^5, x^2y + xy^2 - 2x^5 + y^5).$$

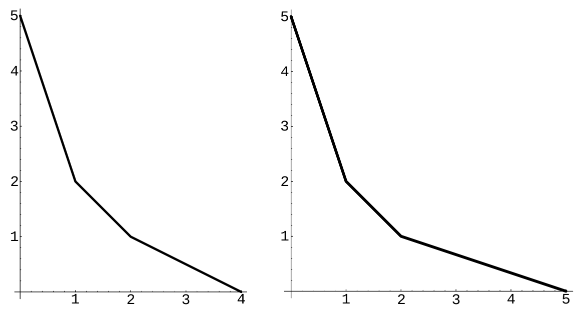


FIGURE 1. Newton diagrams for Example 2

We will find out if the principal part $f_P$ determines the index of the origin for $f$. As we remarked in the proof of Theorem 8, it is enough to check for the weights corresponding to common edges from the Newton diagrams of $f_1$ and $f_2$ (otherwise some $f_{i,a}$ is a monomial and the roots have to be on the axes). Thus it is enough to check for $a = (1,1)$ and $b = (1,3)$. We obtain

$f_a(x,y) = (x^2y - xy^2, x^2y + xy^2)$ and $f_b(x,y) = (-xy^2 - 2y^5, xy^2 + y^5)$, and clearly both of them satisfy the required condition from Theorem 8, so indeed the index of the origin for $f$ coincides with the index for $f_P$.

The next example shows how to generalize slightly Theorem 8 in order to find the index of more complicated vector fields.

**Example 3.** Let $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field given by

$$\begin{aligned} f_1(x,y) &= x^2y - xy^2 + x^4 - 2x^5 - 3x^2y^3 - 2y^5 + h.o.t. \\ f_2(x,y) &= x^2y - xy^2 - 2x^5 - x^3y^2 + 2x^4y + y^5 + h.o.t., \end{aligned}$$

where $h.o.t.$ means terms of degree strictly greater than 5. Then the Newton diagrams of $f_1$ and $f_2$ are again the ones from Figure 1, and the principal part of $f$ is

$$f_P(x,y) = (x^2y - xy^2 + x^4 - 2y^5, x^2y - xy^2 - 2x^5 + y^5).$$

However, in this case for the weights $a = (1,1)$ we have $f_a(x,y) = (x^2y - xy^2, x^2y - xy^2)$, which has zeros outside of the coordinate axes (the line $x = y$), so we cannot apply Theorem 8. But we can use similar ideas and try to add some other terms to $f_P$, and check if this new polynomial determines the index of the origin for $f$.

Let $g(x,y) = f_P + (x^4, 0)$ (we added the terms with the next degree for the weights (1,1) which created problems). Repeating the proof of Theorem 8, if $g$ does not determine the index of the origin, then there exist a nonzero analytic curve $\gamma : [0,1] \to \mathbb{R}^2$ such that $\|g(\gamma(t))\| \le ct^{5a}$ for some positive number $c$ and $t$ sufficiently close to zero. Furthermore the curve $\gamma$ must be of the type

$$\gamma(t) = (t^a + c_1t^{a+1} + c_2t^{a+2} + \dots, t^a + d_1t^{a+1} + d_2t^{a+2} + \dots) = t^a(h_1(t), h_2(t)).$$

This follows from the fact that the first term in the expansion of $g(\gamma(t))$ must vanish (after eventually rescaling). But all we have to do now is to look at the next terms from this expansion. We have

$$\begin{aligned} g_1(\gamma(t)) &= t^{3a}h_1(t)h_2(t)(h_1(t) - h_2(t)) + t^{4a}h_1^4(t) + O(t^{5a}), \\ g_2(\gamma(t)) &= t^{3a}h_1(t)h_2(t)(h_1(t) - h_2(t)) + O(t^{5a}). \end{aligned}$$

Then

$$g_1(\gamma(t)) - g_2(\gamma(t)) = t^{4a}h_1^4(t) + O(t^{5a}) = O(t^{5a}),$$

which is a contradiction, because $h_1(t) = 1 + h.o.t.$, so indeed $g$ must determine the index of the origin for $f$.

**Question**. Is the converse of Theorem 8 true? In other words, assume that the polynomial $g$ has the following property: for any polynomial $f$ such that the principal part of $f$ is $f_P = g$ (and there exists at least one such $f$), the index of the origin for $f$ and $g$ coincide. Is it true that for any weights $a$, the zeros of $g_a$ are all on the coordinate axes?

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