This is a preprint of: "On the Lefschetz periodic point free continuous self-maps on connected compact manifolds", Juan Luis Garcia Guirao, Jaume Llibre, *Topology Appl.*, vol. 158, 2165–2169, 2011.
DOI: [10.1016/j.topol.2011.07.007]

ON THE LEFSCHETZ PERIODIC POINT FREE CONTINUOUS SELF-MAPS ON CONNECTED COMPACT MANIFOLDS

JUAN LUIS GARCÍA GUIRAO[1] AND JAUME LLIBRE[2]

ABSTRACT. We characterize the Lefschetz periodic point free self–continuous maps on the following connected compact manifolds: $\mathbb{C}\mathrm{P}^n$ the n–dimensional complex projective space, $\mathbb{H}\mathrm{P}^n$ the n–dimensional quaternion projective space, $\mathbb{S}^n$ the n–dimensional sphere and $\mathbb{S}^p \times \mathbb{S}^q$ the product space of the p–dimensional with the q–dimensional spheres.

1. Introduction and statement of the main results

We consider the discrete dynamical system $(\mathbb{M}, f)$ where $\mathbb{M}$ is a topological space and $f : \mathbb{M} \to \mathbb{M}$ be a continuous map. A point x is called *fixed* if $f(x) = x$, and *periodic* of *period* k if $f^k(x) = x$ and $f^i(x) \neq x$ if $0 < i < k$. By $\mathrm{Per}(f)$ we denote the *set of periods* of all the periodic points of f.

If $x \in \mathbb{M}$ the set $\{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$ is called the *orbit* of the point x. Here f^n means the composition of n times f with itself. To study the dynamics of the map f is to study all the different kind of orbits of f. Of course if x is a periodic point of f of period k, then its orbit is $\{x, f(x), f^2(x), \ldots, f^{k-1}(x)\}$, and it is called a *periodic orbit*.

The periodic orbits play an important role in the general dynamics of the system, for studying them we can use topological information. Perhaps the best known example in this direction are the results contained in the seminal paper entitle *Period three implies chaos* for continuous self–maps on the interval, see [15].

Let $\mathbb{M}$ be a connected compact manifold. Our aim would be characterize classes of continuous self–maps f on $\mathbb{M}$ which are *periodic point free*, i.e. for which $\mathrm{Per}(f) = \emptyset$.

There are only two 1–dimensional connected compact manifolds, the interval and the circle. It is well known that any continuous self–map

Key words and phrases. Periodic point free, Lefschetz zeta function, Lefschetz number, sphere, complex projective space, quaternion projective space, product of spheres.
2000 Mathematics Subject Classification: 37C05, 37C25, 37C30.

on the interval has fixed points, so there are no periodic point free maps on the interval. The circle admits periodic point free maps but then the degree of such maps must be 1, this is a necessary condition but not sufficient in order that a continuous self–map of the circle be periodic point free, see for definitions and details [1].

There are several papers studying different classes of periodic point free self–maps on the annulus see [10, 12], or on the 2–dimensional torus see [3, 11, 14], ... But in general to characterize the periodic point free maps on a connected compact manifold $\mathbb{M}$ is a very hard problem. Here we will consider a more easier problem, and then we will get results for continuous self–maps on some connected compact manifolds of arbitrary dimension. First we need to introduce some definitions.

Probably the main contribution of the Lefschetz's work in 1920's was to link the homology class of a given map with an earlier work on the indices of Brouwer on the continuous self–maps on compact manifolds. These two notions provide equivalent definitions for the Lefschetz numbers, and from their comparison, can be obtained information about the existence of fixed points.

Let $\mathbb{M}$ be an n–dimensional manifold. We denote by $H_k(\mathbb{M}, \mathbb{Q})$ for $k = 0, 1, ..., n$ the homological groups with coefficients in $\mathbb{Q}$. Each of these groups is a finite dimensional linear space over $\mathbb{Q}$. Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ there exist $n+1$ induced linear maps $f_{*k} : H_k(\mathbb{M}, \mathbb{Q}) \to H_k(\mathbb{M}, \mathbb{Q})$ for $k = 0, 1, \ldots, n$ by f. Every linear map f_{*k} is given by an $n_k \times n_k$ matrix with integer entries, where n_k is the dimension of $H_k(\mathbb{M}, \mathbb{Q})$.

Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ on a compact n–dimensional manifold $\mathbb{M}$, its *Lefschetz number* $L(f)$ is defined as

$$L(f) = \sum_{k=0}^{n} (-1)^k \mathrm{trace}(f_{*k}). \tag{1}$$

One of the main results connecting the algebraic topology with the fixed point theory is the *Lefschetz Fixed Point Theorem* which establishes the existence of a fixed point if $L(f) \neq 0$, see for instance [4]. We can consider the Lefschetz number of f^m but, in general, it is not true that $L(f^m) \neq 0$ implies that f has a periodic point of period m; it only implies the existence of a periodic point with period a divisor of m.

Clearly a necessary condition in order that a map be periodic point free is that all Lefschetz numbers $L(f^m)$ be zero for $m = 1, 2, 3, \ldots$ This is the case of the continuous self–maps of the circle with degree 1, because the Lefschetz number of a continuous self–map of the circle with degree d is

$1-d$, and all the iterates f^m of f have degree 1 if f is of degree 1, for more details see again [1]. Then we define that a continuous self–map f on the compact manifold $\mathbb{M}$ is *Lefschetz periodic point free* if $L(f^m) = 0$ for $m = 1, 2, 3, \ldots$ For additional information about the sequence $\{L(f^m)\}$ see [2].

As far as we know the unique class of continuous self–maps for which have been characterized the ones which are Lefschetz periodic point free are: *The continuous self–maps on the n–dimensional torus are Lefschetz periodic point free if and only if* 1 *is an eigenvalue of the induced homology homomorphism* f_{*1}, see for instance [13].

Our objective will be to characterize some classes of continuous self–maps on connected compact manifolds which be Lefschetz periodic point free, but recall that this does not imply that they are periodic point free. As we have seen this is a necessary condition but not sufficient as it is shown by the continuous self–maps on the circle of degree 1, see [1]. If we restrict our attention to the class of Morse–Smale diffeomorphisms on the 2–dimensional sphere and on the n–dimensional torus we have characterize the ones which are Lefschetz periodic point free, see [7, 8, 9].

Our main results are the following.

Theorem 1. *Let f be a continuous self–map on $\mathbb{C}P^n$, the n–dimensional complex projective space, such that $f_{*2} = (a)$. Then f is Lefschetz periodic point free if and only if $a = 0$.*

Of course the notation $f_{*2} = (a)$ means the action by the 1×1 matrix (a), that is, just multiplication by a in the homological group $H_2(\mathbb{C}\mathrm{P}^n, \mathbb{Q}) \approx \mathbb{Q}$.

Theorem 2. *Let f be a continuous self–map on $\mathbb{H}P^n$, the n–dimensional quaternion projective space, such that $f_{*4} = (a)$. Then f is Lefschetz periodic point free if and only if $a = 0$.*

Theorems 1 and 2 are proved in section 3

Theorem 3. *Let f be a continuous self–map on $\mathbb{S}^n$, the n–dimensional sphere, such that $f_{*n} = (d)$, i.e. d is the degree of f. Then f is Lefschetz periodic point free if and only if n is odd and $d = 1$.*

Theorem 4. *Let f be a continuous self–map on $\mathbb{S}^p \times \mathbb{S}^q$ with $1 \leq p < q$, and let $f_{*p} = (a)$, $f_{*q} = (b)$ and $f_{*p+q} = (c)$ be. Then f is Lefschetz periodic point free if and only if one of the following conditions holds:*

(i) *p odd, q odd, $a = 1$ and $b = c$;*
(ii) *p odd, q odd, $a = c$ and $b = 1$;*
(iii) *p even, q odd, $a = c$ and $b = 1$;*

(iv) *p even, q odd, $a = b$ and $c = 1$;*
(v) *p odd, q even, $a = 1$ and $b = c$;*
(vi) *p odd, q even, $a = b$ and $c = 1$.*

Theorem 5. *For $p \geq 1$ let f be a continuous self–map on $\mathbb{S}^p \times \mathbb{S}^p$ of degree e, and let $f_{*p} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then f is Lefschetz periodic point free if and only if p is odd, $a + d = 1 + e$ and $ad - bc = e$.*

Theorems 3, 4 and 5 are proved in section 4.

Consider the family F of all self-maps f of a given compact manifold with given maps induced in homology groups by iterates of f. Suppose that all Lefschetz numbers are 0, or equivalently $Z_f(t) \equiv 1$. *Does that mean that there is a map from F with no periodic point*? As it was mentioned before for a continuous self–maps f of $\mathbb{T}^n$ it is well known that the Lefschetz numbers $L(f^r)$ are zero for $r = 1, 2, \ldots$ if and only if 1 is an eigenvalue of f_{*1}. For the continuous self-maps of the circle $\mathbb{T}^1$ of degree 1, denoted simply as circle maps in what follows, all possible sets of periods have been characterized, see for instance the chapter 3 of the reference [1]. There are circle maps without periodic points, there are circle maps with periodic points of any period, and ... Using these circle maps identical set of periods to those of the circle maps can be obtained easily for continuous self–maps of $\mathbb{T}^n$ such that 1 is an eigenvalue of f_{*1}. These families of maps show that when a map has all its Lefschetz numbers $L(f^r)$ zero almost anything can occurs with its set of periods.

On the other hand the previous question seems very hard. We think that it is possible that if for a continuous self-map f of a compact manifold $\mathbb{M}$ its Lefschetz numbers $L(f^r)$ are zero for $r = 1, 2, \ldots$, then there exists a continuous self-map g of a compact manifold $\mathbb{N}$ with the same homology spaces than $\mathbb{M}$ (i.e. $H_k(\mathbb{M}, \mathbb{Q}) \approx H_k(\mathbb{N}, \mathbb{Q})$ for all k) and having g the same induced homology endomorphims than f (i.e. $g_{*k} = f_{*k}$ for all k) such that g is periodic point free. But for the moment we do not have a clear idea for proving or disproving this.

In section 2 we recall some basic results on the Lefschetz zeta function that we shall use in the proof of our main results.

2. The Lefschetz zeta function

The Lefschetz zeta function $\mathcal{Z}_f(t)$ will simplify the study of the periodic points of f, and in particular it also facilitates to determine when f is Lefschetz periodic point free. The function $\mathcal{Z}_f(t)$ is a generating function for all the Lefschetz numbers of all iterates of f. More precisely the

Lefschetz zeta function of f is defined as

$$\mathcal{Z}_f(t) = \exp\left(\sum_{m\geq 1} \frac{L(f^m)}{m} t^m\right). \tag{2}$$

This function keeps the information of the Lefschetz number for all the iterates of f, so this function gives information about the set of periods of f. There is the following alternative way to compute it

$$\mathcal{Z}_f(t) = \prod_{k=0}^{n} \det(Id_k - tf_{*k})^{(-1)^{k+1}}, \tag{3}$$

where $n = \dim M$ and Id_k is the identity map of $H_k(M, \mathbb{Q})$, and by convention $\det(Id_k - tf_{*k}) = 1$ if $n_k = 0$, for more details see [6]. Note that from (3) the Lefschetz zeta function is a rational function.

From the definition of Lefschetz zeta function follows immediately that f is Lefschetz periodic point free if and only if its Lefschetz zeta function $\mathcal{Z}_f(t) \equiv 1$. We shall use this fact as a key point for proving our results.

3. Proof of Theorems 1 and 2

Proof of Theorem 1. Assume that we are under the assumptions of Theorem 1.

For $n \geq 1$ let $f : \mathbb{C}P^n \to \mathbb{C}P^n$ be a continuous map. The homological groups of $\mathbb{C}P^n$ over $\mathbb{Q}$ are of the form

$$H_q(\mathbb{C}P^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, 2, 4, ..., 2n\}, \\ 0 & \text{otherwise.} \end{cases}$$

The induced linear maps are $f_{*q} = (a^{q/2})$ for $q \in \{0, 2, 4, ..., 2n\}$ with $a \in \mathbb{Z}$, and $f_{*q} = (0)$ otherwise (see for more details [17, Corollary 5.28]). We recall that since we work with connected manifolds always $H_0(\mathbb{M}, \mathbb{Q}) = \mathbb{Q}$ and $f_{*0} = (1)$.

From (3) the Lefschetz zeta function of f has the form

$$\mathcal{Z}_f(t) = \left(\prod_q (1 - a^{q/2}t)\right)^{-1}, \tag{4}$$

where q runs over $\{0, 2, 4, ..., 2n\}$.

By section 2 the map f is Lefschetz periodic point free if and only if $\mathcal{Z}_f(t) \equiv 1$. From (4) $\mathcal{Z}_f(t) \equiv 1$ if and only if $a = 0$. So the theorem is proved. $\square$

Proof of Theorem 2. Assume that we are under the assumptions of Theorem 2.

For $n \geq 1$ let $f : \mathbb{H}P^n \to \mathbb{H}P^n$ be a continuous map. The homological groups of $\mathbb{H}P^n$ over $\mathbb{Q}$ are of the form

$$H_q(\mathbb{H}P^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, 4, 8, ..., 4n\}, \\ 0 & \text{otherwise.} \end{cases}$$

The induced linear maps are $f_{*q} = (a^{q/4})$ for $q \in \{0, 4, 8, ..., 4n\}$ with $a \in \mathbb{Z}$, and $f_{*q} = (0)$ otherwise (see for more details [17, Corollary 5.33]).

The rest of the proof can be made using the same arguments than in the proof of Theorem 1. □

4. Proof of Theorems 3, 4 and 5

Proof of Theorem 3. Assume that we are under the assumptions of Theorem 3.

Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a continuous map. The homological groups of $\mathbb{S}^n$ over $\mathbb{Q}$ are of the form

$$H_q(\mathbb{S}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

The induced linear maps are $f_{*0} = (1)$ and $f_{*n} = (d)$, where d is the degree of the map f and $f_{*i} = 0$ for $i \in \{1, ..., n-1\}$ (see for more details [16]).

From (3) we have that

$$\mathcal{Z}_f(t) = \frac{(1-dt)^{(-1)^{n+1}}}{1-t} = \begin{cases} \dfrac{1-dt}{1-t} & \text{if } n \text{ is odd,} \\ \dfrac{1}{(1-t)(1-dt)} \neq 1 & \text{if } n \text{ is even.} \end{cases}$$

So clearly $\mathcal{Z}_f(t) \equiv 1$ if and only if n is odd and $d = 1$. This completes the proof of the theorem. □

Proof of Theorem 4. Assume that we are under the assumptions of Theorem 4.

For $1 \leq p < q$, let $f : \mathbb{S}^p \times \mathbb{S}^q \to \mathbb{S}^p \times \mathbb{S}^q$ be a continuous map. The homological groups of $\mathbb{S}^p \times \mathbb{S}^q$ over $\mathbb{Q}$ are of the form

$$H_q(\mathbb{S}^p \times \mathbb{S}^q, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, p, q, p+q\}, \\ 0 & \text{otherwise.} \end{cases}$$

The induced linear maps are $f_{*0} = (1)$, $f_{*p} = (a)$, $f_{*q} = (b)$ with $a, b \in \mathbb{Z}$, $f_{*p+q} = (c)$, where c is the degree of the map f and $f_{*i} = (0)$ for $i \in \{1, ..., p+q\}$ and $i \neq p, q, p+q$ (see for more details [5]).

From (3) the Lefschetz zeta function of f is of the form

$$\mathcal{Z}_f(t) = \frac{(1-at)^{(-1)^{p+1}}(1-bt)^{(-1)^{q+1}}(1-ct)^{(-1)^{p+q+1}}}{1-t}. \tag{5}$$

Now we consider four cases.

Case 1: p and q even. Then

$$\mathcal{Z}_f(t) = \frac{1}{(1-t)(1-at)(1-bt)(1-ct)} \neq 1. \tag{6}$$

Case 2: p and q odd. Then

$$\mathcal{Z}_f(t) = \frac{(1-at)(1-bt)}{(1-t)(1-ct)}. \tag{7}$$

Therefore $\mathcal{Z}_f(t) \equiv 1$ if and only if either $a = 1$ and $b = c$, or $a = c$ and $b = 1$.

Case 3: p even and q odd.

$$\mathcal{Z}_f(t) = \frac{(1-bt)(1-ct)}{(1-t)(1-at)}. \tag{8}$$

So $\mathcal{Z}_f(t) \equiv 1$ if and only if either $b = 1$ and $a = c$, or $a = b$ and $c = 1$.

Case 4: p odd and q even.

$$\mathcal{Z}_f(t) = \frac{(1-at)(1-ct)}{(1-t)(1-bt)}. \tag{9}$$

Hence $\mathcal{Z}_f(t) \equiv 1$ if and only if either $a = 1$ and $b = c$, or $a = b$ and $c = 1$.

From the expressions (6) to (9) if follows easily the proof of the theorem. $\square$

Proof of Theorem 5. Assume that we are under the assumptions of Theorem 5.

Let $f : \mathbb{S}^p \times \mathbb{S}^p \to \mathbb{S}^p \times \mathbb{S}^p$ be a continuous map. The homological groups of $\mathbb{S}^p \times \mathbb{S}^p$ over $\mathbb{Q}$ are of the form

$$H_q(\mathbb{S}^p \times \mathbb{S}^p, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, 2p\}, \\ \mathbb{Q} \oplus \mathbb{Q} & \text{if } q = p, \\ 0 & \text{otherwise.} \end{cases}$$

The induced linear maps are $f_{*0} = (1)$, $f_{*p} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$, $f_{*2p} = (e)$ where e is the degree of the map f, and $f_{*i} = (0)$ for $i \in$

$\{1,...,2p\}$, $i \neq 1, p, 2p$ (see for more details [5]). From (3) the Lefschetz zeta function of f is

$$\mathcal{Z}_f(t) = \frac{(1-(a+d)t+(ad-bc)t^2)^{(-1)^{p+1}}}{(1-t)(1-et)}. \tag{10}$$

Clearly if p is even then $\mathcal{Z}_f(t) \not\equiv 1$, and if p is odd then $\mathcal{Z}_f(t) \equiv 1$ if and only if $a+d = 1+e$ and $ad-bc = e$. Therefore the theorem is proved. □

Acknowledgements

The authors want to thank to the referee his comments which help us to improve the presentation of this paper.

The first author was partially supported by MCYT/FEDER grant number MTM2008- 03679/MTM, Fundación Séneca de la Región de Murcia, grant number 08667/PI/08 and Junta de Comunidades de Castilla-La Mancha, grant number PEII09-0220-0222. The second author was partially supported by MCYT/FEDER grant number MTM2008-03437, by CICYT grant number 2009SGR 410 and ICREA Academia.

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[1] Departamento de Matemática Aplicada y Estadística. Universidad Politécnica de Cartagena, Hospital de Marina, 30203-Cartagena, Región de Murcia, Spain
E-mail address: juan.garcia@upct.es

[2]Departament de Matemàtiques. Universitat Autònoma de Barcelona, Bellaterra, 08193-Barcelona, Catalonia, Spain
E-mail address: jllibre@mat.uab.cat