Non-landing hairs in Sierpiński curve Julia sets of transcendental entire maps

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July 20, 2011

Abstract

We consider the family of transcendental entire maps given by $f_a(z) = a(z - (1 - a)) \exp(z + a)$ where a is a complex parameter. Every map has a superattracting fixed point at $z = -a$ and an asymptotic value at $z = 0$. For $a > 1$ the Julia set of f_a is known to be homeomorphic to the Sierpiński universal curve [19], thus containing embedded copies of any one-dimensional plane continuum. In this paper we study subcontinua of the Julia set that can be defined in a combinatorial manner. In particular, we show the existence of non-landing hairs with prescribed combinatorics embedded in the Julia set for all parameters $a \geq 3$. We also study the relation between non-landing hairs and the immediate basin of attraction of $z = -a$. Even as each non-landing hair accumulates onto the boundary of the immediate basin at a single point, its closure, nonetheless, becomes an indecomposable subcontinuum of the Julia set.

Keywords: Transcendental entire maps, Julia set, non-landing hairs, indecomposable continua.

Mathematics Subject Classification (2000): 37F10, 37F20.

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1 Introduction

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire map. The *Fatou set* $\mathcal{F}(f)$ is the largest open set where iterates of f form a normal family. Its complement in $\mathbb{C}$ is the *Julia set* $\mathcal{J}(f)$ and it is a non-empty and unbounded subset of the plane. When the set of singular values is bounded, we say f is of *bounded singular type* and denote this class of maps by $\mathcal{B}$. It has been shown in [1] and [22] that the Julia set of a hyperbolic map in $\mathcal{B}$ contains uncountably many unbounded curves, usually known as *hairs*, [10]. A hair is said to *land* if it is homeomorphic to the half-closed ray $[0, +\infty)$. The point corresponding to $t = 0$ is known as the *endpoint* of the hair. In contrast, if its accumulation set is a non-trivial continuum, we obtain a *non-landing* hair.

In this paper we study a particular class of non-landing hairs in the Julia set of transcendental entire maps given by

$$f_a(z) = a(z - (1 - a)) \exp(z + a),$$

when a is a real parameter. For all complex values of a, the map f_a has a superattracting fixed point at $z = -a$ and an asymptotic value at the origin whose dynamics depends on the parameter a. If the orbit of the asymptotic value escapes to $+\infty$, we say a is an *escaping parameter*. For example, when $a > 1$, the orbit of the asymptotic value escapes to $+\infty$ along the positive real axis.

To our knowledge, the family f_a was first introduced by Morosawa, [19], as an example of a transcendental entire map whose Julia set is homeomorphic to the *Sierpiński curve* continuum when $a > 1$. Any planar set that is compact, connected, locally connected, nowhere dense, and has the property that any two complementary domains are bounded by disjoint simple closed curves is homeomorphic to the Sierpiński curve continuum (Whyburn, [26]). It is also a *universal* continuum, in the sense that it contains a homeomorphic copy of every one-dimensional plane continuum (Kuratowski, [17]). We take advantage of this property to combinatorially construct subsets of $\mathcal{J}(f_a)$, $a > 1$, that in turn are *indecomposable continua*. An *indecomposable continuum* is a compact, connected set that cannot be written as the union of two proper connected and closed subsets. Observe that a landing hair together with the point at infinity is in fact a decomposable continuum.

Every known example in the literature of indecomposable subsets of Julia sets arises from a single family of maps, namely the exponential family $E_\lambda(z) = \lambda \exp(z)$. The first example was given by

Devaney [5] when $\lambda = 1$ so the asymptotic value escapes to infinity and the Julia set is the whole plane. Under the assumption that either the asymptotic value escapes to infinity or has a preperiodic orbit (thus holding again $\mathcal{J}(E_\lambda) = \mathbb{C}$), several authors have been able to construct topologically distinct indecomposable continua embedded in $\mathcal{J}(E_\lambda)$ (see among other works, [7], [9], and [23], where a generalization of previous results for a large set of λ-parameters can be found).

Our work provides examples of indecomposable subcontinua of Julia sets outside the exponential family and without the assumption that $\mathcal{J}(f_a)$ equals $\mathbb{C}$, since f_a has a superattracting fixed point for all $a \in \mathbb{C}$. Denote by $\mathcal{A}(-a)$ the *basin of attraction* of $-a$, that is, the set of points with forward orbits converging to $-a$. Also denote by $\mathcal{A}^*(-a)$ the *immediate basin of attraction* of $-a$ which is the connected component of $\mathcal{A}(-a)$ containing $-a$. In [19] Morosawa showed that all connected components of $\mathcal{A}(-a)$ are bounded Jordan domains. Moreover, whenever $a > 1$, the orbit of the free asymptotic value escapes to infinity. Since there are no other singular values, $\mathcal{F}(f_a)$ cannot contain another attracting basin, or a parabolic basin, or a Siegel disk as these components must be associated with a non-escaping singular value. Maps with a finite number of singular values do not exhibit neither wandering domains ([13, 16]) nor Baker domains ([13]). Hence $\mathcal{F}(f_a) = \mathcal{A}(-a)$. In Figure 1, we display the dynamical plane of f_a for different values of $a > 1$. The basin of attraction of $-a$ is shown in black, while points in the Julia set are shown in white.

Let us summarize our main results. Since $\mathcal{J}(f_a)$ is homeomorphic to the Sierpiński universal curve, it must contain embedded copies of planar indecomposable continua, so we obtain some of them in terms of its combinatorics. To do so, we first characterize the topology and dynamics of the boundary of $\mathcal{A}^*(-a)$ by a polynomial-like construction (Proposition 2.4). Then, using general results of transcendental entire maps, we obtain curves in the Julia set contained in the far right plane and with specific combinatorics (Proposition 2.9). By a controlled process of consecutive pullbacks of some of these curves, we extend them into non-landing hairs that limit upon themselves at every point (Theorem 3.6). Using a result due to Curry, [4], we show the closure of such hairs are indecomposable continua (Theorem 4.2). Finally, we study the relation between each indecomposable continuum and the boundary of $\mathcal{A}^*(-a)$ showing that the intersection between these two sets reduces to a unique point (Theorem 4.4). As a

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consequence of these results, we show the existence of a dense set of points in $\partial\mathcal{A}^*(-a)$ that are landing points of a unique hair (in particular there are no *pinchings* that arise as in other maps in class $\mathcal{B}$ having a superattracting basin), while there is a residual set of points in $\partial\mathcal{A}^*(-a)$ that, even though they belong to the accumulation set of a certain ray, they are not landing points of hairs.

The outline of this paper is as follows: in §2 we describe the dynamical plane of f_a for $a \geq 3$. §3 contains most of our technical results while in §4 we provide the proofs of our main results.
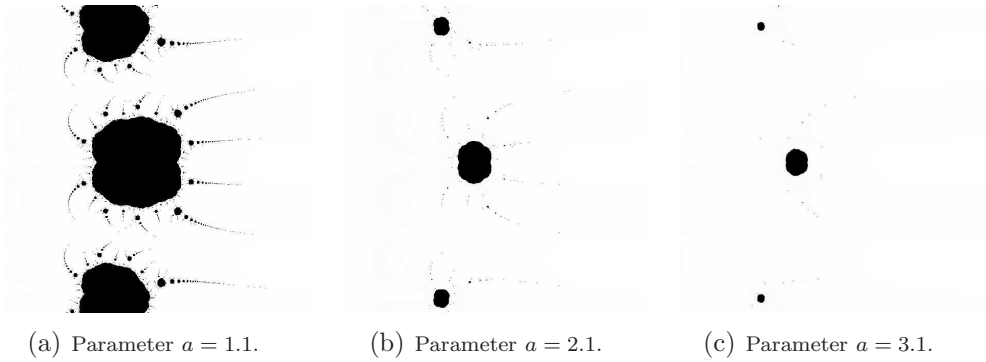

(a) Parameter $a = 1.1$.   (b) Parameter $a = 2.1$.   (c) Parameter $a = 3.1$.

Figure 1: The Julia set for f_a (and a an escaping parameter) is shown in white, the Fatou set, in black.

Notation and terminology.

$B_\varepsilon(x) = \{z \in \mathbb{C} \mid |z - x| < \varepsilon\}$.

$\overline{U}$ denotes the closure of a set U.

Connected components will be refered to as components.

A curve γ *cuts across* a

1. line L if the intersection $\gamma \cap L$ is not tangential,

2. rectangle R if γ cuts across both vertical boundaries of R so $\gamma \cap R$ contains a component with endpoints joining those sides,

3. semi-annular region A if γ cuts across the inner and outer semi-circular boundaries of A so $\gamma \cap A$ contains a connected component with endpoints joining those boundaries.
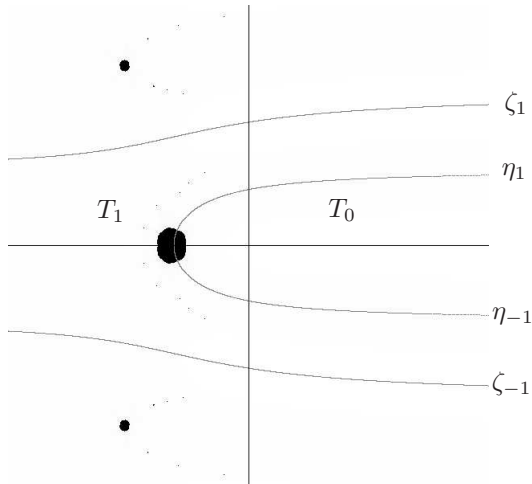
Figure 2: Dynamical plane of $f_{3.1}$ and regions $T_0 \cup T_1$. The boundary curves ζ_1 and ζ_{-1} extend from $-\infty$ to $+\infty$ while curves η_1 and η_{-1} meet at $-a = -3.1$ and extend to $+\infty$.

2 Dynamical plane for escaping real parameters

Consider escaping parameters of the form $a > 1$ for the family of transcendental entire maps

$$f_a(z) = a(z - (1 - a)) \exp(z + a),$$

which have a unique asymptotic value at $z = 0$ and a superattracting fixed point at $z = -a$. For $a > 1$, the asymptotic value escapes to infinity along the positive real line and the Fatou set reduces to the basin of attraction of $-a$, $\mathcal{A}(-a)$. Our first aim in this section is to provide a partition of the complex plane that will allow us to combinatorially analyze the dynamics of points in the Julia set.

We start by taking preimages of the forward invariant set $\mathbb{R}^+$. Any point $z = x + iy$ in the complex plane whose image under f_a is a real positive number must satisfy

$$
\begin{aligned}
(x - (1 - a)) \cos y - y \sin y > 0, \\
(x - (1 - a)) \sin y + y \cos y = 0.
\end{aligned}
\tag{1}
$$

From these conditions, the preimages of $\mathbb{R}^+$ are infinitely many analytic curves parametrized by $(x, \zeta_j(x))$, with $j \in \mathbb{Z}$. For $j = 0$,

$\zeta_0(x) = 0$ and is defined for all $x \in (1 - a, +\infty)$ while the rest of the ζ_j's are strictly monotonic functions of x defined for all $x \in \mathbb{R}$. When $j \neq 0$, each ζ_j has two horizontal asymptotes, given by

$$\lim_{x \to -\infty} \zeta_j(x) = \text{sign}(j)(2|j| - 1)\pi i \quad \text{and} \quad \lim_{x \to +\infty} \zeta_j(x) = 2j\pi i.$$

For our purposes, we need to consider the preimage of the interval $(-\infty, -a)$ inside the region bounded by ζ_1 and ζ_{-1} (see Figure 2). We obtain two strictly monotonic curves $\eta_1(x)$ and $\eta_{-1}(x)$ defined for $x \in [-a, +\infty)$, satisfying

$$\lim_{x \to -a^+} \eta_{\pm 1}(x) = 0, \qquad \lim_{x \to +\infty} \eta_1(x) = \pi i \quad \text{and} \quad \lim_{x \to +\infty} \eta_{-1}(x) = -\pi i.$$

Let T_0 denote the open and connected set containing $z = 0$ and bounded by $\eta_1 \cup \eta_{-1}$. Similarly, let T_1 be the open and connected set bounded by $\zeta_{-1} \cup \eta_{-1} \cup \eta_1 \cup \zeta_1$. Far to the right, T_1 consists of two unbounded and disjoint strips, one above and one below the positive real line. Since most of our results involve the dynamics of points in $T_0 \cup T_1$, we construct a refinement of this region. For $j = 0, 1$, denote by T_{j_1} and T_{j_2} the proper and disjoint domains in $T_j \setminus \mathbb{R}$ with negative and positive imaginary part, respectively. Finally, for each $j \in \mathbb{Z}, j \neq 0, 1$, denote by T_j the open and connected strip bounded by the curves ζ_{j-1} and ζ_j as $\text{Im}(z)$ increases. Then $\{T_j \mid j \in \mathbb{Z}\}$ defines the partition of the complex plane sought, while $\{T_{j_i} \mid j = 0, 1, \ i = 1, 2\}$ defines a refinement of the region $T_0 \cup T_1$.

It is straightforward to verify that

$$f_a : T_0 \to \mathbb{C} \setminus (-\infty, -a], \quad \text{and}$$
$$f_a : T_1 \to \mathbb{C} \setminus ((-\infty, -a] \cup [0, +\infty)),$$

are one-to-one maps. Define $g_a^0 = f_a^{-1}|T_0$ and $g_a^1 = f_a^{-1}|T_1$ the corresponding inverse branches of f_a taking values in T_0 and T_1, respectively. As for the refinement of $T_0 \cup T_1$, we denote by $g_a^{j_1}$ and $g_a^{j_2}$ the appropiate restrictions of g_a^j mapping into T_{j_1} and T_{j_2}, respectively.

Assume z is a point of the Julia set whose orbit is entirely contained in $\cup_{j \in \mathbb{Z}} T_j$. We can naturally associate to z the *itinerary* $s(z) = (s_0, s_1, \ldots)$, with $s_j \in \mathbb{Z}$, if and only if $f_a^j(z) \in T_{s_j}$. Let us concentrate on the space $\Sigma_B = \{0, 1\}^{\mathbb{N}}$ of *binary sequences* (in what follows *B-sequences*). With respect to the refinement of $T_0 \cup T_1$, consider the space of *extended sequences* given by $\Sigma_E = \{0_1, 0_2, 1_1, 1_2\}^{\mathbb{N}}$.

Since f_a is a one-to-one map in T_0 and T_1, not all extended sequences are *allowable*, that is, f_a behaves as a subshift of finite type over the set of points with full orbits inside $T_{0_1} \cup T_{0_2} \cup T_{1_1} \cup T_{1_2}$. Its transition matrix is given by

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix},$$

and determines the space of *allowable extended sequences* (in what follows *A-sequences*) given by

$$\Sigma_A = \{(s_0, s_1, \ldots) \in \Sigma_E \mid s_i \in \{0_1, 0_2, 1_1, 1_2\}, a_{s_i s_{i+1}} = 1, \forall i\}.$$

Denote by $\pi : \Sigma_A \to \Sigma_B$ the *projection map* that transforms an A-sequence into a B-sequence by erasing all subscripts. The form of the matrix A makes evident that π is a 2-to-1 map. For a given B-sequence t, denote by t^1 and t^2 the unique A-sequences so that $\pi(t^j) = t$. Observe that by interchanging all subscripts in t^1 we obtain t^2, and conversely.

Remark 2.1. *It is important to observe that points on the real line (or on any of its preimages), do not have well defined A-sequences. However, since $\mathbb{R}$ is forward invariant under f_a, its dynamics and combinatorics are completely understood. Based on the next result, from now on we will only consider B-sequences (and its two associated A-sequences) that do not end in all zeros.*

Lemma 2.2. *Assume $a > 1$ and let $w \in \mathcal{J}(f_a)$ such that $f_a^k(w) \in T_0 \cup T_1$ for all $k \geq 0$. Then, $w \in \mathbb{R} \cap T_j$ if and only if $s(w) = (j, 0, 0, \ldots)$, for $j = 0, 1$.*

Proof. The first implication follows easily by analyzing the action of f_a in $\mathbb{R}$. Whenever $a > 1$, the set $\mathcal{A}^*(-a)$ intersects the real line in an open interval (q_a, p_a), where p_a is a repelling fixed point and q_a is its only preimage in $\mathbb{R}$. Moreover $(-\infty, q_a) \cup (p_a, +\infty)$ consists of points that escape to $+\infty$ along $\mathbb{R}^+$ and hence, belong to $\mathcal{J}(f_a)$. Since f_a sends $(-\infty, q_a]$ onto $[p_a, +\infty)$ and this second interval is fixed by f_a, then w has a well defined itinerary given by $s(w) = (1, 0, 0, \ldots)$, if $w \in (-\infty, q_a] \subset T_1$, or $s(w) = (0, 0, \ldots)$ if $w \in [p_a, +\infty) \subset T_0$.

To see the second implication, it is enough to show the interval $[p_a, +\infty)$ represents the only set of points in the Julia set that remain inside T_0 for all positive iterates. To do so, we analyze the preimages

of η_1 inside T_{0_2} (the case η_{-1} and T_{0_1} is analogous). Since f_a maps T_{0_2} onto the upper half plane $\text{Im}(z) > 0$, then for each $k \geq 1$, the kth preimage of η_1 in T_{0_2}, namely $\eta_1^k = (g_a^{0_2})^k(\eta_1)$, lies completely inside T_{0_2} (except for its endpoint in $z = -a$) and extends towards infinity into the right half plane. In particular, it lies in the strip bounded by $[-a, +\infty)$ and η_1^{k-1} (from bottom to top). Also, note η_1^k and η_1^j meet only at $z = -a$ whenever $k \neq j$. We claim that η_1^k accumulates onto $[p_a, +\infty)$ as $k \to \infty$. For otherwise, we can find a point $x \in [-a, +\infty)$ and $\varepsilon > 0$ so $B_\varepsilon(x) \cap \eta_1^k = \emptyset$ for all $k \geq 1$. Nevertheless, since x belongs to the Julia set, it follows from Montel's Theorem the existence of an integer $N > 0$ for which $f_a^N(B_\varepsilon(x)) \cap \eta_1 \neq \emptyset$. Hence, $B_\varepsilon(x) \cap \eta_1^N \neq \emptyset$, a contradiction.

Finally, for any given point $w \in T_{0_2}$ such that $s(w) = (0, 0, \dots)$, there exists an integer $m > 0$ for which, either $w \in \eta_1^m$ or w lies in the interior of the strip bounded by η_1^{m+1} and η_1^m (from bottom to top). In both situations, $f_a^{m+1}(w)$ lies outside T_0. This finishes the proof. $\qquad\square$

The rest of this section is devoted to a combinatorial description of the dynamics of points with forward orbits contained in $T_0 \cup T_1$ using A- and B-sequences. First, we focus our study on the set $\mathcal{A}^*(-a)$ and then analyze points in the Julia set that lie far to the right in $T_0 \cup T_1$. Using known results in complex dynamics we will prove that points with forward orbits completely contained in a given right hand plane are organized into continuous curves and their combinatorics are governed by the transition matrix A. For future reference, we compute the image of a vertical segment bounded above and below by ζ_1 and ζ_{-1}, respectively.

Lemma 2.3. *Let $x \in \mathbb{R}$ be fixed and consider the vertical segment $L[x] = \{x + iy \,|\, \zeta_{-1}(x) \leq y \leq \zeta_1(x)\}$. Then $f_a(L[x])$ lies inside the closed round annulus $f_a(x) \leq |z| \leq f_a(x + \zeta_1(x))$.*

Proof. From the definition of the map f_a we have

$$|f_a(x + iy)| = a \exp(x + a)\sqrt{(x - (1 - a))^2 + y^2}.$$

Evidently, when restricted to $L[x]$ for a fixed x, the above expression reaches its minimum value when $y = 0$ while its maximum value is reached whenever $y = \zeta_1(x) = \zeta_{-1}(x)$. $\qquad\square$

2.1 Dynamics near $z = -a$

In [19] it was shown that for $a > 1$, each Fatou domain of f_a is a bounded, connected component of $\mathcal{A}(-a)$ whose boundary is a Jordan curve. Here we show that $\overline{\mathcal{A}^*(-a)}$ is in fact a quasiconformal image of the closed unit disk. Precisely, we describe a set of points with bounded orbits inside $T_0 \cup T_1$ through a polynomial-like construction (see [11]) around the unique and simple critical point $z = -a$. For technical reasons, we restrict to parameters $a \geq 3$ from now on.

Proposition 2.4. *For any $a \geq 3$, there exist open, bounded and simply connected domains U_a and V_a with $-a \in \overline{U}_a \subset V_a$, such that (f_a, U_a, V_a) is a quadratic-like mapping. Furthermore, the filled Julia set of (f_a, U_a, V_a) is the image under a quasiconformal mapping of the closed unit disk and coincides with $\overline{\mathcal{A}^*(-a)}$.*
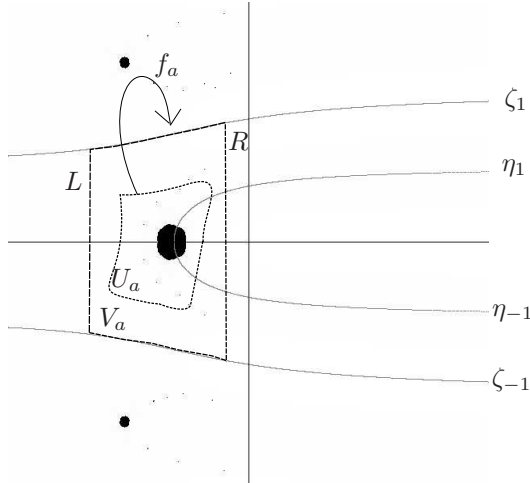

Figure 3: A sketch of the domains U_a and V_a found in Proposition 2.4

Proof. Define V_a as the open, simply connected pseudo-rectangle given by

$$V_a = \{z \in \mathbb{C} \mid -a - 6\ln a < \operatorname{Re}(z) < \frac{1-a}{2}, \ \zeta_{-1}(\operatorname{Re}(z)) < \operatorname{Im}(z) < \zeta_1(\operatorname{Re}(z))\}.$$

First, we show that V_a maps outside itself. Indeed, the top and bottom boundaries of V_a map into a segment lying on the positive

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real line, thus outside V_a as $(1 - a)/2 \leq -1$. Also, note that V_a lies in the interior of the annulus

$$\frac{|1-a|}{2} \leq |z| \leq |-a - 6\ln a + i2\pi|.$$

Following the notation in Lemma 2.3, $L = L[-a - 6\ln a]$ and $R = L[(1-a)/2]$ are the left and right hand boundaries of V_a. We show next the images of L and R lie in the complementary components of the annulus. First, for $z \in L$ we have

$$|f_a(z)| = \frac{1}{a^5}\sqrt{(1 + 6\ln a)^2 + y^2} < \frac{1}{a^5}\sqrt{(1 + 6\ln a)^2 + 4\pi^2} < \frac{1}{2} < \frac{|1-a|}{2}.$$

Similarly, if $z \in R$

$$|f_a(z)| = ae^{\frac{a+1}{2}}\sqrt{\left(\frac{1-a}{2}\right)^2 + y^2} > ae^{\frac{a+1}{2}}\frac{|1-a|}{2} > \sqrt{(a + 6\ln a)^2 + 4\pi^2}$$

for all $a \geq 3$, proving thus that V_a is mapped outside itself under f_a.

Now, we define U_a to be the connected component of $f_a^{-1}(V_a)$ containing $-a$. Since $-a$ is a superattracting fixed point with multiplicity one, and there are no other critical points, it follows that $\overline{U_a} \subset V_a$ and the map $f_a : U_a \to V_a$ sends ∂U_a to ∂V_a with degree 2, as $-a$ is a simple critical point. We conclude that (f_a, U_a, V_a) is a quadratic-like mapping.

What is left to verify is that the filled Julia set of (f_a, U_a, V_a) is a quasi-disk. Recall that the filled Julia set of a polynomial-like mapping is defined as the set $\{z \in U_a \mid f_a^n(z) \in U_a \text{ for all } n \geq 0\}$. Being (f_a, U_a, V_a) a quadratic-like mapping, there exists a quasiconformal conjugacy with a polynomial of degree two that has a superattracting fixed point. Thus, the polynomial must be $z \mapsto z^2$ after a holomorphic change of variables, if necessary. So the filled Julia set of (f_a, U_a, V_a) is the image under a quasiconformal mapping of the closed unit disk. $\square$

Proposition 2.5. *Let $a \geq 3$. The following statements hold.*

(a) *The map f_a restricted to the boundary of $\mathcal{A}^*(-a)$ is conjugate to the map $\theta \mapsto 2\theta$ in the unit circle.*

(b) *Let $t \in \Sigma_E$ be an extended sequence that does not end in 0_i's. Then, t is an A-sequence if and only if there exists a unique point $z \in \partial\mathcal{A}^*(-a)$ that realizes t as its itinerary.*

Proof. Statement (a) is a direct consequence of the previous proposition since the map f_a is conjugate in $\partial\mathcal{A}^*(-a)$ to $z \mapsto z^2$ acting on the unit circle.

We prove statement (b) by defining a partition of the boundary of $\mathcal{A}^*(-a)$ that coincides with the refinement of the partition $T_0 \cup T_1$ discussed before. For simplicity, angles are measured by $[0,1]$. Denote by $z(0), z(1/4), z(1/2)$ and $z(3/4)$ the points in $\partial\mathcal{A}^*(-a)$ corresponding under the conjugacy between f_a and $z \mapsto z^2$ to points in S^1 of angle $\theta = 0, 1/4, 1/2$ and $3/4$. Now label points in $\partial\mathcal{A}^*(-a)$ in the following way: traveling along $\partial\mathcal{A}^*(-a)$ in a clockwise direction, associate the symbol 0_1 to the arc joining $z(0)$ and $z(3/4)$, the symbol 1_1 to the arc joining $z(3/4)$ and $z(1/2)$, 1_2 to the arc joining $z(1/2)$ and $z(1/4)$, and 0_2 to the arc joining $z(1/4)$ and $z(0)$. We leave it to the reader to verify the transition matrix for this partition under the action of f_a is exactly A and the labeling is consistent with the one defined by the T_{j_i}'s. □

Theorem 2.6. *Let z be a point such that $f_a^n(z) \in \overline{T_0 \cup T_1}$ for all $n \geq 0$. If z belongs to the Julia set and has bounded orbit, then $z \in \partial\mathcal{A}^*(-a)$.*

Proof. If $z \in \mathcal{J}(f_a)$ satisfies the hypotheses, we can find $m < 0 < M$ so that $m < \mathrm{Re}(f_a^n(z)) < M$ for all $n \geq 0$.

Let $\varepsilon > 0$ small enough and denote by $B_\varepsilon = \overline{B_\varepsilon(0)}$. Since the orbit of the origin escapes monotonically along the positive real line, redefining M if necessary, there exists an integer $N = N(M) > 0$ for which $B_\varepsilon, f_a(B_\varepsilon), \ldots, f_a^N(B_\varepsilon)$ are pairwise disjoint compact domains, such that for all $0 \leq j \leq N-1$,

$$f_a^j(B_\varepsilon) \subset T_0 \cap \{z \,|\, \mathrm{Re}(z) < M\}, \text{ and}$$
$$f_a^N(B_\varepsilon) \subset T_0 \cap \{z \,|\, \mathrm{Re}(z) > M\}.$$

Moreover, we may choose ε small enough so $B_\varepsilon, f_a(B_\varepsilon), \ldots, f_a^N(B_\varepsilon)$ are all compact domains contained in T_0. Finally, select $m' \leq m < 0$ so the subset $\{z \in T_1 \mid \mathrm{Re}(z) \leq m'\}$ maps completely inside B_ε.

Denote by $\Phi_a : \mathbb{D} \to \mathcal{A}^*(-a)$ the Böttcher coordinates tangent to the identity at the origin. For $0 < r < 1$ let $\Delta_r = \Phi_a(B_r(0))$. Clearly $\Delta_r \subset \mathcal{A}^*(-a)$ and maps compactly into its own interior. Moreover, we can choose r small enough so for $j = 0, 1$, $T_j \setminus \Delta_r$ is a connected set and the intersection of Δ_r and $\mathbb{R}$ is an open interval (c, d), since Φ_a has been chosen to be tangent to the identity at the origin. We can now define the set E, illustrated in Figure 4, as follows

$$E = \{z \in \mathbb{C} \mid m' < \mathrm{Re}(z) < M, \, \zeta_{-1}(\mathrm{Re}(z)) < \mathrm{Im}(z) < \zeta_1(\mathrm{Re}(z))\} \setminus \Omega$$

where $\Omega = \bigcup_{k=0}^{N-1} f_a^k(B_\varepsilon) \cup \Delta_r \cup [c, M]$. It is easy to verify that E is an open, bounded, connected and simply connected set and $\partial \mathcal{A}^*(-a) \subset E$. Moreover, $E \subset f_a(E)$ although some boundary components map into ∂E. Indeed, $f_a(\partial E) \cap \partial E$ consists of segments along the real line and $f_a^j(\partial B_\varepsilon)$, for $j = 1, 2, \ldots, N - 1$. So after N iterations, the only boundary points mapping into ∂E are points over the real line, thus having B-itineraries $(1, 0, \ldots)$ or $(0, 0, \ldots)$. We show next that for $\ell > 0$ sufficiently large, $f_a^{-\ell}|E$ becomes an strict contraction.
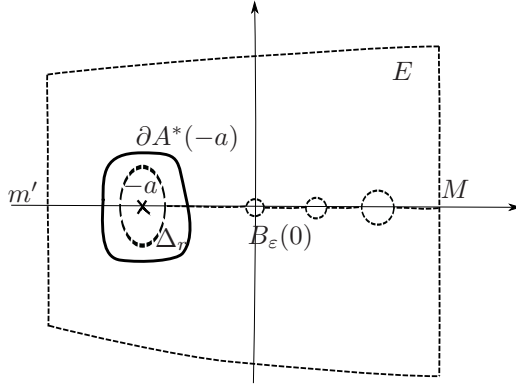

Figure 4: An schematic representation of the set E described in Theorem 2.6.

Let $s = (s_0, s_1, s_2, \ldots)$ be the A-sequence associated to z. If s has finitely many 1_i's, then its B-sequence ends with 0's and since z has bounded orbit inside $\overline{T_0 \cup T_1}$, then $z \in \partial \mathcal{A}^*(-a)$ by Lemma 2.2.

If s has infinitely many 1_i's there exists a first integer $n > N$ for which $s_n \in \{1_1, 1_2\}$, and thus $\overline{E \cap T_{s_n}} \subset f^n(E)$, as points in $f^n(\partial E)$ mapping into ∂E have by now itinerary $(0, 0, \ldots)$.

For each $k \in \{0_1, 0_2, 1_1, 1_2\}$, the set $E \cap T_k$ is an open, connected and simply connected set with a Riemann mapping given by $\psi_k : \mathbb{D} \to E \cap T_k$. Consider the mapping $\Psi_\ell : \mathbb{D} \to \mathbb{D}$ with $\ell > n$, given by

$$\Psi_\ell = \psi_{s_0}^{-1} \circ (g_a^{s_0} \circ \ldots \circ g_a^{s_{\ell-1}}) \circ \psi_{s_\ell}.$$

It follows that $\Psi_\ell(\mathbb{D})$ is compactly contained in $\mathbb{D}$, that is, for $\ell > n$, Ψ_ℓ is an strict contraction with respect to the Poincaré metric on the unit disk. Consequently, the sets $\overline{\Psi_\ell(\mathbb{D})}$ form a nested sequence of compact sets with diameters converging to zero as $\ell \to \infty$. This implies that for any $w \in \mathbb{D}$, $\lim_{\ell \to \infty} \Psi_\ell(w)$ exists and is independent of the point w. Therefore, by construction, $\psi_{s_0}(\lim_{\ell \to \infty} \Psi_\ell(0)) = z$ is the unique point in E with itinerary s. From Proposition 2.5(b) we conclude $z \in \partial \mathcal{A}^*(-a)$. $\qquad \square$

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2.2 Dynamics near infinity

Our first aim is to prove that for $R > 0$ sufficiently large and the region

$$H_R = \{z \in T_0 \cup T_1 \mid \mathrm{Re}(z) \geq R\},$$

there exist continuous curves in $\mathcal{J}(f_a) \cap H_R$ consisting of points whose orbits escape to $+\infty$ with increasing real part. These curves are usually known as *tails*. The existence of tails as disjoint components of the Julia set were first observed by Devaney and Tangerman [10] for certain entire transcendental maps and by Schleicher and Zimmer [25] for the exponential family $E_\lambda(z) = \lambda \exp(z)$ and all $\lambda \in \mathbb{C}$. In greater generality, Barański [1] and Rempe [22] have shown the existence of tails for hyperbolic maps belonging to the class $\mathcal{B}$.

For completeness, we analyze in detail some of their results in the setting of our work to obtain tails in $\mathcal{J}(f_a) \cap H_R$. Once each tail has been assigned an A-sequence, we describe a pullback process to compute the full set of points in $T_0 \cup T_1$ associated to such A-sequence. In the final section, we study the topological properties of that set.

Consider an entire transcendental map f in the class $\mathcal{B}$. The *escaping set of f*, denoted as $I(f)$, is the set of points whose orbits under f tend to infinity. For an entire transcendental map, Erëmenko [12] has shown the Julia set coincides with the boundary of $I(f)$. We say that two maps $f, g \in \mathcal{B}$ are *quasiconformally equivalent near infinity* if there exist quasiconformal maps $\phi_1, \phi_2 : \mathbb{C} \to \mathbb{C}$ that satisfy $\phi_1 \circ f = g \circ \phi_2$ in a neighborhood of infinity.

Theorem 2.7 (Rempe, 2009)**.** *Let $f, g \in \mathcal{B}$ be two entire transcendental maps which are quasiconformally equivalent near infinity. Then there exist $\rho > 0$ and a quasiconformal map $\theta : \mathbb{C} \to \mathbb{C}$ such that $\theta \circ f = g \circ \theta$ on*

$$A_\rho = \{z \in \mathbb{C} \mid |f^n(z)| > \rho, \ \forall n \geq 1\}.$$

Furthermore, the complex dilatation of θ on $I(f) \cap A_\rho$ is zero.

A straightforward computation shows that $f_a(z)$ is conjugate to the function $\tilde{f}_a(z) = az \exp(z + 1) - (1 - a)$ under the conformal isomorphism $\varphi_a(z) = z - (1 - a)$. Since $\tilde{f}_a(z) = \varphi_a \circ f_a \circ \varphi_a^{-1}(z)$, it is easily verified that $\tilde{f}_a$ has a free asymptotical value at $z = a - 1$ and a fixed critical point at $z = -1$. In turn, $\tilde{f}_a$ is (globally) conformally equivalent to $g_b(z) = bz \exp(z)$ via $\phi_1(z) = \alpha z + \alpha(1 - a)$ and $\phi_2(z) = z$. Indeed, it is easy to see that $\phi_1 \circ \tilde{f}_a = g_b \circ \phi_2$, where $b = ea\alpha$.

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For small values of b, the Fatou set of g_b consists solely of the completely invariant basin of attraction of the fixed point (and asymptotic value) $z = 0$. Thus, we can describe the Julia set of g_b by applying the following result found in [1].

Theorem 2.8 (Barański, 2007)**.** *Let g be an entire transcendental function of finite order so that all critical and asymptotic values are contained in a compact subset of a completely invariant attracting basin of a fixed point. Then $\mathcal{J}(g)$ consists of disjoint curves (hairs) homeomorphic to the half-line $[0, +\infty)$. Moreover, the hairs without endpoints are contained in the escaping set $I(g)$.*

These disjoint curves are usually known as hairs or *dynamic rays*. Several consequences are derived from the above theorem. Firstly, if γ denotes a hair, it can be parametrized by a continuous function $h(t)$, $t \in [0, +\infty)$, such that $\gamma = h([0, +\infty))$. The point $h(0)$ is called the *endpoint* of the hair. Secondly, each hair is a curve that extends to infinity and, for $r > 0$, we say that $\omega = h((r, +\infty))$ is the *tail of the hair*. Moreover, all points in a given hair share the same symbolic itinerary defined by a dynamical partition of the plane with respect to f, and for every point $z \in \gamma$ that is not the endpoint, we have $f^n(z) \to \infty$ as $n \to \infty$. Finally, if f and g are as in Theorem 2.7 and in addition, g satisfies hypotheses in Theorem 2.8, then near infinity the topological structure of the escaping set of f is also given by disjoint curves extending to infinity. We refer to [2] for a topological description of the Julia set in terms of what is known as *Cantor bouquets*. Furthermore, the dynamics of f in those curves is quasiconformally conjugate to the dynamics of g in the corresponding curves near infinity. We deduce the following result based on the previous theorems and the specific expression of f_a.

Proposition 2.9. *Let $f_a(z) = a(z - (1 - a)) \exp(z + a)$, $g_b(z) = bz \exp(z)$, $a \geq 3$, and b a complex parameter.*

(a) *If $|b|$ is small enough, the Julia set of g_b is given by the union of disjoint hairs. Each hair lands at a distinguished endpoint. Moreover, hairs without endpoints are contained in $I(g_b)$.*

(b) *Let $R > 0$ large enough. The set of points with forward f_a-orbits that are always contained in H_R are given by the union of disjoint curves extending to infinity to the right. All points in those curves belong to $I(f_a)$. Precisely, these curves are quasiconformal copies of connected components of hairs described in (a).*

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(c) *To each curve in (b) that does not coincide with $\mathbb{R}$ or one of its preimages, we can assign a unique sequence t in Σ_A. We denote this curve by ω_t. All points in ω_t escape to infinity under the action of f_a following the itinerary t.*

(d) *For any $t \in \Sigma_A$, if $R > 0$ is large enough, there exists a unique curve ω_t in H_R. Moreover, for each $r \geq R$, $\omega_t \cap \{z \mid \mathrm{Re}(z) = r\}$ is a unique point. In particular ω_t is the graph of a Lipschitz function.*

(e) *ω_t is a tail, i.e., a quasiconformal copy of a tail in (a).*

Proof. Statement (a) follows directly from Theorem 2.8. To see statement (b) note that for any value of $a \neq 0$ and b, there exists $\alpha = b/(ea)$ so f_a and g_b are (globally) conformally equivalent. By Theorem 2.7, there exists $\rho > 0$ such that f_a and g_b are conjugate on the set $A_\rho = \{z \mid |f_a^n(z)| > \rho, \forall n \geq 1\}$. Let $R > \rho$. Denote by S the set of points in H_R with forward f_a-orbits contained in H_R. Clearly, each point in S must belong to $\mathcal{J}(f_a)$ (since a point in the Fatou set eventually maps into $\mathcal{A}^*(-a)$) and in particular, $S \subset \mathcal{J}(f_a) \cap A_R$. Moreover, far enough to the right, a point $z \in H_R$ whose forward orbit remains forever in H_R must satisfy $\mathrm{Re}\big(f_a^{k+1}(z)\big) > \mathrm{Re}\big(f_a^k(z)\big)$ for all $k > 0$ (see Lemma 2.3). Hence from Theorems 2.7 and 2.8 we know that S is the union of disjoint curves extending to infinity (that is, quasiconformal copies of components of the hairs in (a)) belonging to the escaping set.

To prove statement (c) we start by assigning to each of these curves, ω, a unique sequence in Σ_A. Let z_0 be a point in ω and let $s(z_0)$ be its itinerary in Σ_A in terms of the partition $T_{0_1}, T_{0_2}, T_{1_1}$ and T_{1_2}. By assumption this itinerary is well defined since ω is not $\mathbb{R}$ or one of its preimages. Let z_1 be another point in ω. We claim that $s(z_1) = s(z_0)$ and proceed by contradiction. Let $\mathcal{C}$ be the connected component of ω joining z_0 and z_1. If $s(z_1) \neq s(z_0)$, there exists an integer $k \geq 0$ for which the kth entries in both itineraries are the first ones to differ.

Hence there is a point $q \in \mathcal{C}$ such that $f_a^k(q)$ belongs to either $[R, \infty)$, η_1 or η_{-1}. Clearly $f_a^k(q)$ cannot belong to $\eta_{\pm 1}$, since otherwise $f_a^{k+1}(q) \in \mathbb{R}^-$ and by hypothesis the forward orbit of q belongs to H_R. On the other hand $f_a^k(q)$ cannot belong to $\mathbb{R}$ since by item (b), $f_a^k(\omega)$ and $\mathbb{R}$ are disjoint curves of the escaping set.

Finally, there cannot be two curves having the same itinerary as this will imply the existence of an open set of points following the same itinerary, which is impossible. Thus, we may now denote by ω_t

the unique curve in H_R formed by escaping points with itinerary t.

To prove statement (d) fix $t = (t_0, t_1, \ldots, t_n, \ldots) \in \Sigma_A$ and R large enough. Let $r \geq R$ and let $L[r] = \{r + iy \,|\, \zeta_{-1}(r) \leq y \leq \zeta_1(r)\}$. We denote by $I_{t_0}(r)$ the set of points in $L[r] \cap \overline{T}_{s_0}$. We know that the image of the vertical segment $L[r]$ cuts across H_R in two almost vertical lines (see Lemma 2.3). Recall that when f_a is restricted over the set of points with forward orbits in H_r, it behaves as a subshift of finite type governed by the matrix A, and when restricted to T_0 or T_1 it is a one-to-one map. Using these facts, we can find a unique subinterval $I_{t_0 t_1}(r) \subset I_{t_0}(r)$ formed by points with forward orbits inside H_r and itineraries starting as $(t_0, t_1, \ldots)$. Inductively, for each $n > 0$, $I_{t_0 t_1 \ldots t_n}(r)$ is the unique subinterval of $I_{t_0 t_1 \ldots t_{n-1}}(r)$ formed by points with forward orbits inside H_r and itineraries starting with $(t_0, t_1, \ldots, t_n, \ldots)$. Clearly,

$$I_{t_0 t_1, \ldots t_n}(r) \subset I_{t_0 t_1 \ldots t_{n-1}}(r) \subset \ldots \subset I_{t_0 t_1}(r) \subset I_{t_0}(r).$$

Due to the expansivity of f_a in H_R we obtain $\cap_{n \geq 0} I_{t_0 t_1, \ldots t_n}(r)$ is a unique point. Let $q = q(r)$ be that point. By construction, q has itinerary t and $f^k(q) \to \infty$ as $k \to \infty$, so q must belong to the unique curve ω_t described in (c). The above arguments imply that ω_t intersects $\mathrm{Re}(z) = r$ at a unique point, thus $q(r)$ gives a parametrization of ω_t on the interval $[r, \infty)$.

In fact, ω_t can be shown to be Lipschitz. Since the upper and lower boundaries of H_R are asympotically bounded by $\mathrm{Im}(z) = \pm 2\pi$, we have $|\mathrm{Im}(f_a^j(z))| < 2\pi$ for all $j \geq 0$. If $z \in H_R$ has an orbit entirely contained in H_R one can check that $\mathrm{Re}(f_a(z)) > c\mathrm{Re}(z)e^{\mathrm{Re}(z)} > cR$ with $c > 1$. Using induction it is easy to see that

$$\mathrm{Re}(f_a^k(z)) > c\mathrm{Re}(f_a^{k-1}(z)) \exp\left(\mathrm{Re}(f_a^{k-1}(z))\right) > c^k R^k.$$

Let $\theta_k(z) := \arg(f_a^k(z))$. Then, given $\epsilon > 0$ small, for each k $|\tan\theta_k(z)| \leq 2\pi(cR)^{-k} < \epsilon$ for R sufficiently large. In particular, we may adjust R so that $\sum_{k \geq 0} |\theta_k(z)| < \epsilon$ whenever $z \in \omega_t(R)$.

Finally, let $\Gamma_\pm^k$ be the upper and lower horizontal boundaries of $f_a^{-k}(T_{t_k} \cap H_R)$, where inverses are chosen according to the itinerary t. Applying the above estimates we obtain that $\Gamma_\pm^k$ can be parametrized by $t \mapsto (t, \mu_\pm^k(t))$ for $t \geq R$ and $\mu_\pm^k$ analytic. As vertical widths of $f_a^{-k}(T_{t_k} \cap H_R)$ tend to zero, then $\Gamma_\pm^k \to \omega_t(R)$ as $k \to \infty$. And since $|(\mu_\pm^k)'(t)| < \epsilon$, we conclude that $\omega_t(R)$ is the graph of a Lipschitz function with Lipschitz constant ϵ.

Finally to see statement (e) we observe that there are no endpoints associated to bounded itineraries in H_R, implying that each

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w_t is a quasiconformal copy of the tail of some hair in (a). On one hand, the only points in $T_0 \cup T_1$ with bounded orbit belong to $\partial A^*(-a)$, so they are not in H_R. On the other hand if there were and endpoint *of a hair* with orbit escaping to infinity in H_R, it should have an itinerary, say, t. But from statement (d) there is a (unique) curve w_t going from $\mathrm{Re}(z) = R$ to infinity with such itinerary t, a contradiction.

$\square$

Definition 2.10. *Given any A-sequence t, each curve described in Proposition 2.9 will be denote by $\omega_t = \omega_t(R)$ and it will be called the tail with itinerary t contained in the half plane $\mathrm{Re}(z) \geq R$.*

The component of ω_t that cuts across

$$F_R = \{ z \in H_R \mid |z| < f_a(R + i\zeta_1(R)) \}$$

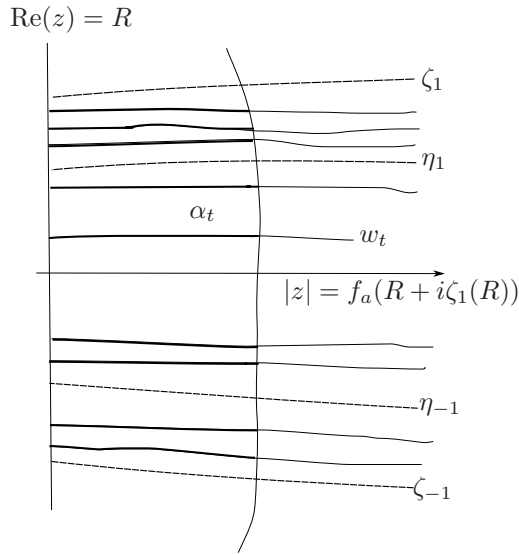*is called the* base *of the tail, and it will be denote by $\alpha_t = \alpha_t(R)$.*

See Figure 5.

Figure 5: The set H_R, the tail ω_t and its base α_t for some $t \in \Sigma_A$. Bases are depicted as bold lines.

We now describe a pullback construction to extend the tail ω_t into a longer curve. Recall $g_a^{j_i} = f_a^{-1}|T_{j_i}, j_i \in \{0_1, 0_2, 1_1, 1_2\}$ and consider the shift map $\sigma : \Sigma_A \to \Sigma_A$ acting on the space of A-sequences. Let $t = (t_0, t_1, \dots) \in \Sigma_A$. By the conjugacy of $f_a|\bigcup_{t \in \Sigma_A} \omega_t$ with $\sigma|\Sigma_A$,

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it follows that $\omega_{\sigma(t)}$ properly contains $f_a(\omega_t)$ since this curve lies in $H_R \setminus F_R$ (see Proposition 2.9). Hence $f_a(\omega_t)$ is a curve that misses the base $\alpha_{\sigma(t)}$. Consequently, $g_a^{t_0}(\omega_{\sigma(t)})$ is a continuous curve that lies in T_{t_0} and extends ω_t to the left of $\mathrm{Re}(z) = R$. Clearly, any point in the extended curve $g_a^{t_0}(\omega_{\sigma(t)})$ has itinerary t. Inductively, consider

$$g_a^{t_0} \circ \ldots \circ g_a^{t_{n-1}} \left(\omega_{\sigma^n(t)} \right).$$

This pullback iteration is always defined as long as the extended curve does not meet $z = 0$, which is impossible since $\mathbb{R}$ is forward invariant. We thus obtain a curve of points with itinerary t, and each pullback iteration extends its predecessor.

Definition 2.11. *Let*

$$\gamma(t) = \bigcup_{n=0}^{\infty} g_a^{t_0} \circ \ldots \circ g_a^{t_{n-1}} \left(\omega_{\sigma^n(t)} \right). \tag{2}$$

We call $\gamma(t)$ the hair *associated to t.*

Note that the pullback process described before may or may not produce an endpoint. In Section 4 we show that in some cases, $\gamma(t)$ is a hair with an endpoint in $\partial \mathcal{A}^*(-a)$, and in some other cases $\gamma(t)$ is a non-landing hair and accumulates everywhere upon itself. By the following theorem found in [4], we will conclude that $\overline{\gamma(t)}$ is an indecomposable continuum.

Theorem 2.12 (Curry, 1991)**.** *Suppose that X is a one-dimensional nonseparating plane continuum which is the closure of a ray that limits on itself. Then X is either an indecomposable continuum or the union of two indecomposable continua.*

Remark 2.13. *A* ray *is defined as the image of $[0, +\infty)$ under a continuous, one to one map. Giving any positive number α, the image of $[\alpha, +\infty)$ under the same map is known as a* final segment *of the ray. Then, the ray* limits on itself *if it is contained in the closure of any final segment of itself.*

3 Targets in H_R

The main result in this section will be Theorem 3.6, where we construct B-sequences so their associated A-sequences produce hairs

that accumulate everywhere on themselves.

We now set up *targets* around the nth *image of the base* α_t of each tail ω_t. The construction is very similar to the one presented in Devaney and Jarque [8] and Devaney, Jarque and Moreno Rocha in [9], although the existence of a critical point in our present case requires some modifications.

Let t be a given A-sequence. We first enlarge the base α_t inductively along ω_t. Set $\alpha_{t,0} = \alpha_t$ and consider the two bases $\alpha_{\sigma^{-1}(t)}$. The set $f_a(\alpha_{\sigma^{-1}(t)})$ (taking the two possible bases) is a subset of ω_t. Thus, the set

$$\alpha_{t,1} = \alpha_{t,0} \cup f_a(\alpha_{\sigma^{-1}(t)}),$$

is an extension of $\alpha_{t,0}$ along ω_t. Inductively, define the nth image of the base α_t as

$$\alpha_{t,n} = \alpha_{t,n-1} \cup f_a^n(\alpha_{\sigma^{-n}(t)}).$$

It is easy to verify that $\{\alpha_{t,n}\}_{n \in \mathbb{N}}$ is a sequence of curves satisfying the following three conditions:

  (i) $\alpha_{t,0} = \alpha_t$,

 (ii) $\alpha_{t,n} \subset \alpha_{t,n+1}$, and

(iii) $\displaystyle\bigcup_{n \geq 0} \alpha_{t,n} = \omega_t$.

In order to define a target around each $\alpha_{t,n}$, consider $\xi, \eta \in \mathbb{R}^+$ and let

$$V(\xi, \eta) = \{z \in H_R \mid \xi - 1 < \operatorname{Re}(z) < \eta + 1\}.$$

By definition $V(\xi, \eta)$ is a rectangular region bounded above and below by components of ζ_{-1} and ζ_1, respectively.

Lemma 3.1. *Let $R > 0$ be large enough. For all $n \geq 0$ there exist positive real numbers ξ_n and η_n such that the following statements hold.*

  (a) *For every $t \in \Sigma_A$, the nth iterate of α_t belongs to the interior of $V(\xi_n, \eta_n)$.*

  (b) *For every $\ell \geq 0$, $V(\xi_{n+1}, \eta_{n+\ell+1})$ is compactly contained inside $f_a(V(\xi_n, \eta_{n+\ell}))$.*

Proof. Set $\eta_0 = f_a(R + i\zeta_1(R))$, $\eta_{n+1} = f_a(\eta_n + i\zeta_1(\eta_n))$ and $\xi_n = f_a^n(R)$ for $n \geq 0$. It remains to verify that these values satisfy

statements (a) and (b). Observe that for R large enough, the image of every vertical segment $L[R]$ cuts across $T_0 \cup T_1$ in two almost vertical lines: one *near* $\operatorname{Re}(z) = f_a(R)$ and the other *near* $\operatorname{Re}(z) = f_a(R + i\zeta_1(R))$ (see Lemma 2.3).

Statement (a) follows directly from the definition of ξ_n and η_n, since at each step we choose ξ_n and η_n to be respectively the smallest and largest possible values of $\operatorname{Re}(f_a^n(\alpha_t))$ for all $t \in \Sigma_A$, and moreover it is easy to check that $\xi_n < \eta_{n-1}$.

We prove statement (b) when $\ell = 0$. The case $\ell > 0$ follows similarly. The proof of the statement proceeds in two steps. The first one is to verify the inequality

$$f_a\left(\xi_n - 1 + i\zeta_1\left(\xi_n - 1\right)\right) < \xi_{n+1} - 1\,.$$

From the definition of f_a we obtain that

$$
\begin{aligned}
f_a\left(\xi_n - 1 + i\zeta_1\left(\xi_n - 1\right)\right) \;&<\; ae^{a+\xi_n-1}\sqrt{(\xi_n - 1 - (1-a))^2 + 4\pi^2} \\
&<\; ae^{a+\xi_n}(\xi_n - (1-a)) - 1 \\
&=\; \xi_{n+1} - 1,
\end{aligned}
$$

where the second inequality is satified if R is large enough. The second step is to verify the inequality

$$f_a(\eta_n + 1) > \eta_{n+1} + 1 = f_a(\eta_n + i\zeta_1(\eta_n)) + 1.$$

By evaluating both sides we obtain

$$ae^{a+\eta_n+1}(\eta_n + a) > ae^{a+\eta_n}\sqrt{(\eta_n + a - 1)^2 + (\zeta_1(\eta_n))^2} + 1.$$

Consequently the image of $V(\xi_n, \eta_n)$ contains $\overline{V(\xi_{n+1}, \eta_{n+1})}$ as desired. $\qquad\square$

Definition 3.2. *The set $V(\xi_n, \eta_n)$ will be called the nth target of f_a.*

Targets provide a useful tool in the proof of Theorem 3.6. Before getting into the details, we briefly explain why targets are so important in our construction. For large values of n, an nth target corresponds to a rectangular region with arbitrarily large real part and imaginary part bounded, in absolute value, by 2π. Then, we may pullback the nth target using suitable branches of f_a^{-1} to obtain two sequences of nested subsets inside $V(\xi_0, \eta_0)$. We intend to prove each nested sequence contains not only a base $\alpha_{t^i}, i = 1, 2$, but there

are also other components of $\gamma(t^i)$ accumulating into the base.

As observed before, each target $V(\xi_n, \eta_{n+\ell})$ intersects the fundamental domains T_0 and T_1 for any $n \in \mathbb{Z}^+$. Denote by $W_{n,\ell}^{0_i}$ and $W_{n,\ell}^{1_i}$ the domains given by $V(\xi_n, \eta_{n+\ell}) \cap T_{0_i}$ and $V(\xi_n, \eta_{n+\ell}) \cap T_{1_i}$, respectively. The next step is to show that, by considering appropriate preimages of these W-sets, we obtain a nested sequence of neighborhoods around two particular bases of tails.

In what follows, we will work solely with B-sequences and its respective A-sequences. We may also assume that every B-sequence has infinitely many 1's to avoid taking a preimage of the positive real line. The next result is a suitable restatement of Lemma 4.3 in [9] following our notation.

Lemma 3.3. *Let $\ell \geq 0$ and let $t = (\tau_0, 1, \tau_1, 1, \tau_2, 1, \ldots)$ be a B-sequence, where each $\tau_k = \left(\tau_k^1, \ldots, \tau_k^{n_k}\right)$ denotes a finite block of binary symbols $\{0, 1\}$ of length n_k. Let $m_j = j + 1 + n_0 + n_1 + \ldots + n_j$, $j \geq 0$. Then for each m_j the sets*

$$g_a^{\tau_0^1} \circ \cdots \circ g_a^{\tau_j^{n_j}} \left(W_{m_j, \ell}^{1_i}\right), \ i = 1, 2, \tag{3}$$

form two nested sequences of subsets of $V(\xi_0, \eta_\ell)$. Moreover, if we denote by t^1 and t^2 the two allowable A-sequences satisfying $\pi\left(t^1\right) = \pi\left(t^2\right) = t$, then $\alpha_{t^1, \ell}$ and $\alpha_{t^2, \ell}$ are contained each in a nested sequence of subsets of $V(\xi_0, \eta_\ell)$ given by (3).

Proof. Due to Lemma 3.1, for each m_j and $i = 1, 2$, the sets $W_{m_j, \ell}^{1_i} \subset V(\xi_{m_j}, \eta_{m_j + \ell})$, can be pulled back following $(\tau_0, 1, \tau_1, 1, \ldots, \tau_j)$. These preimages yield two nested sequences of subsets in $V(\xi_0, \eta_\ell)$, one sequence corresponds to preimages of $W_{m_j, \ell}^{1_1}$ and the other to preimages of $W_{m_j, \ell}^{1_2}$. Notice there is a unique way of pulling back each $W_{m_j, \ell}^{1_i}$, $i = 1, 2$ following the B-sequence t, since $f_a | T_k$, $k = 0, 1$ are one-to-one maps. Also, at each step of the construction, the points belonging to these nested subsets are points of $V(\xi_0, \eta_\ell)$ with B-itinerary $t = (\tau_0, 1, \tau_1, 1, \ldots, \tau_j, \ldots)$. So, $\alpha_{t^1, \ell}$ and $\alpha_{t^2, \ell}$ must be inside all of them. $\qquad \square$

The next step in the construction is to show that in each of the nested subsets of $V(\xi_0, \eta_\ell)$ given by the previous lemma we will have not only the bases $\alpha_{t^1, \ell}$ and $\alpha_{t^2, \ell}$, but also other components of $\gamma(t)$. We split this step into two lemmas. The first lemma shows that for

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suitable choices of A-sequences the extended tails cut twice across the line $\text{Re}(z) = -\mu$ for $\mu > 0$ arbitrarily large.

A finite block of 0's (respectively 0_1's or 0_2's) of length k will be denoted by 0^k (respectively 0_1^k or 0_2^k).

Lemma 3.4. *Let t be any B-sequence, and let $t^1, t^2 \in \Sigma_A$ so that $\pi(t^1) = \pi(t^2) = t$. Assume also that $0_1 t^1$ and $0_2 t^2$ are allowable. Given any $\mu > 0$, there exists $K > 0$ such that for all $k > K$, there exist two continuous curves, denoted by $\tilde{\omega}_{1_2 0_1^k t^1}$ and $\tilde{\omega}_{1_1 0_2^k t^2}$, that extend to infinity to the right and satisfy:*

   (a) $\tilde{\omega}_{1_2 0_1^k t^1}$ and $\tilde{\omega}_{1_1 0_2^k t^2}$ are enlargements (to the left) of the tails with itineraries $1_2 0_1^k t^1$ and $1_1 0_2^k t^2$, respectively; and

   (b) $\tilde{\omega}_{1_2 0_1^k t^1}$ and $\tilde{\omega}_{1_1 0_2^k t^2}$ cuts twice across $\text{Re}\, z = -\mu$.

Proof. Consider any A-sequence of the form $0_1^m t^1$ with $m > 0$ large enough so the unique tail $\omega_{0_1^m t^1}$ is parametrized by $z = (x, h(x))$, $x > R$ (see Proposition 2.9). Lemma 2.2 and its proof imply that $\omega_{0_1^m t^1}$ is ε-close to $z = (x, 0)$, $x > R$ (that is, $|h(x)| < \varepsilon$ for all $x > R$).

By pulling back $\omega_{0_1^m t^1}$ using g_a^0, we obtain a new curve which is precisely the extended tail $\tilde{\omega}_{0_1^{m+1} t^1}$. Moreover, since the positive real line is repelling, we observe that $\tilde{\omega}_{0_1^{m+1} t^1}$ is also ε-close to $z = (x, 0)$ with $x > g_a^0(R)$. Successive pullbacks via g_a^0 allow us to find a first positive integer r such that the extended tail $\tilde{\omega}_{0_1^{m+r} t^1}$ will be ε-close to $z = (x, 0)$, $x > p_a$ where p_a is the real fixed point in the boundary of $\mathcal{A}^*(-a)$.

At this stage we pullback again $\tilde{\omega}_{0_1^{m+r} t^1}$ using g_a^1. The obtained curve must coincide with the extended tail $\tilde{\omega}_{1_2 0_1^{m+r} t^1}$. By construction it is a curve extending to infinity to the right, extends far into the left half plane (since $\tilde{\omega}_{0_1^{m+r} t^1}$ is close to $z = 0$) and lies close to q_a (the preimage of p_a in $\partial \mathcal{A}^*(-a)$) since $\tilde{\omega}_{0_1^{m+r} t^1}$ gets arbitrarily close to $z = p_a$. For a given μ, if we choose m large enough, this pullback construction guarantees that $\tilde{\omega}_{1_2 0_1^{m+r} t^1}$ cuts across $\text{Re}(z) = -\mu$ twice, as desired.

Analogously, if we start the construction with $0_2^m t^2$, $m > 0$, we get a similar result for the extended tail $\tilde{\omega}_{1_1 0_2^{m+r} t^2}$. $\qquad\square$

Proposition 3.5. *Fix $\ell \geq 0$. Let t be any B-sequence with t^1 and t^2 its associated A-sequences. Let τ be any finite block of binary symbols*

of length n and denote by $s = \tau 110^k t$, and by s^1, s^2 its associated A-sequences. Then there exists $K > 0$ such that for all $k > K$, the following statements hold.

(a) The forward image of $W_{n,\ell}^{1_1}$ cuts three times (twice far to the left and once far to the right) across the extended tail $\tilde{\omega}_{1_2 0_1^k t^1}$. In other words, the hair $\gamma(1_1 1_2 0_1^k t^1)$ cuts three times across $W_{n,\ell}^{1_1}$.

(b) The hair $\gamma(s^1)$ cuts three times across

$$g_a^{\tau_0} \circ \cdots \circ g_a^{\tau_{n-1}} \left(W_{n,\ell}^{1_1} \right) .$$

Moreover one of the connected components of $\gamma(s^1)$ in the above expression contains $\alpha_{s^1, l}$.

Analogous results corresponding for s^2 also apply.

Proof. The set $f_a \left(W_{n,\ell}^{1_1} \right)$ is a large semi-annulus in the upper half plane and intersects $T_0 \cup T_1$ far to the right and far to the left. From Lemma 3.4 we can choose K such that for all $k > K$, the extended tail $\tilde{\omega}_{1_2 0_1^k t^1}$ cuts across the semi-annulus $f_a \left(W_{n,\ell}^{1_1} \right)$ three times: twice far to the left and one far to the right. Consequently there must be three components of $\gamma(1_1 1_2 0_1^k t^1)$ in $W_{n,\ell}^{1_1}$ that map into three components of the extended tail $\tilde{\omega}_{1_2 0_1^k t^1}$ in $f_a \left(W_{n,\ell}^{1_1} \right)$.

By taking suitable pullbacks of $W_{n,\ell}^{1_1}$ that follow the string of symbols $\tau = (\tau_0, \ldots, \tau_{n-1})$ and applying Lemma 3.3, we can now conclude statement (b) of the present proposition. $\square$

Theorem 3.6. *Let τ be a finite block of binary symbols of length n. There exists an increasing sequence of integers k_j, $j \geq 1$ so for the B-sequence*

$$\mathfrak{T} = \tau 110^{k_1} 110^{k_2} 110^{k_3} \ldots, \tag{4}$$

its associated A-sequences $\mathfrak{T}^1$ and $\mathfrak{T}^2$ determine two distinct hairs $\gamma(\mathfrak{T}^1)$ and $\gamma(\mathfrak{T}^2)$ that limit upon themselves, thus becoming non-landing hairs.

Proof. Let $\ell > 0$, $p_0 = n + 1$ and for each $i \geq 1$, set

$$p_i = n + 1 + \sum_{j=1}^{i} k_j + 2i.$$

The symbol at the p_i^{th} position in (yet to be constructed) $\mathfrak{T}$ is equal to 1. In general, the symbol at position p_i in the A-sequence $\mathfrak{T}^j$, $j = 1, 2$

can be either 1_1 or 1_2, depending on the previous entry in $\mathfrak{T}^j$. In what follows we assume that all symbols at the p_i^{th} position in $\mathfrak{T}^1$ are equal to 1_1 (hence, all symbols at position p_i^{th} in $\mathfrak{T}^2$ are 1_2). It will become clear from the proof that this assumption is without lost of generality. Moreover, we shall prove only that $\gamma(\mathfrak{T}^1)$ limits upon itself since the case of $\gamma(\mathfrak{T}^2)$ follows in the same way.

Let s be any B-sequence not ending in all 0's, and let s^1 and s^2 be its associated A-sequences. For each $j \geq 1$ we aim to construct inductively B-sequences of the form

$$u_j \;\; = \;\; \tau 110^{k_1}110^{k_2}\ldots 110^{k_j}\sigma^{p_j}(s), \tag{5}$$

so that, by carefully selecting longer blocks of zeros, the u_j will converge to the B-sequence $\mathfrak{T}$ with the desired properties. Note that for all j, u_j will be a concatenation of the first p_j^{th} symbols in $\mathfrak{T} = (\mathfrak{t}_0, \mathfrak{t}_1, \ldots)$ with $\sigma^{p_j}(s)$. In other words,

$$u_j \;\; = \;\; (\mathfrak{t}_0, \mathfrak{t}_1, \ldots, \mathfrak{t}_{p_j-1}, s_{p_j}, s_{p_j+1}, \ldots). \tag{6}$$

As before, u_j^1 and u_j^2 will denote the A-sequences that project into u_j.

We first show how to define u_1. Applying Proposition 3.5(a), we can choose an integer $k_1 > 0$ so the forward image of $W^{1_1}_{p_0-1,\ell}$ is cut across by the extended tail $\tilde{\omega}_{1_2 0_1^{k_1}\sigma^{p_1}(s^1)}$ three times (twice far to the left and once far to the right). Thus the hair $\gamma(1_1 1_2 0_1^{k_1}\sigma^{p_1}(s^1))$ cuts three times across $W^{1_1}_{p_0-1,\ell}$. For the given block τ, denote by τ_1 its corresponding A-block that makes $u_1^1 = \tau_1 1_1 1_2 0_1^{k_1}\sigma^{p_1}(s^1)$ an allowable A-sequence. Thus Proposition 3.5(b) implies that

$$g_a^{\mathfrak{t}_0} \circ \cdots \circ g_a^{\mathfrak{t}_{p_0-2}}\left(W^{1_1}_{p_0-1,\ell}\right)$$

is a subset of $V(\xi_0, \eta_\ell)$ that contains $\alpha_{u_1^1,\ell}$ and (at least) two other components of the hair $\gamma(u_1^1)$.

For $j > 1$ we apply again Proposition 3.5(a) to the sequence

$$\tilde{\tau} 110^{k_j}\sigma^{p_j}(s),$$

where $\tilde{\tau} = \tau 110^{k_1}110^{k_2}\ldots 110^{k_{j-1}}$ and with an integer $k_j > 0$ large enough so that the forward image of $W^{1_1}_{p_j-1,\ell}$ is cut across by the extended tail $\tilde{\omega}_{1_2 0_1^{k_j}\sigma^{p_j}(s^1)}$ three times (twice far to the left and once far to the right). Thus the hair $\gamma(1_1 1_2 0_1^{k_j}\sigma^{p_j}(s^1))$ cuts three times

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across $W^{1_1}_{p_j-1,\ell}$. Pulling back $W^{1_1}_{p_j-1,\ell}$ through the finite block of binary symbols given by $\tilde{\tau}$, we get from Proposition 3.5(b) that

$$g_a^{t_0} \circ \cdots \circ g_a^{t_{p_j-2}} \left(W^{1_1}_{p_j-1,\ell} \right)$$

is a subset of $V(\xi_0, \eta_\ell)$ that contains $\alpha_{u_j^1,\ell}$ and (at least) two other components of the hair $\gamma(u_j^1)$.

As j tends to infinity, the sequence u_j converges to the desired sequence $\mathfrak{T}$. Indeed, from Proposition 3.5(b), the sets

$$g_a^{t_0} \circ \cdots \circ g_a^{t_{p_j-2}} \left(W^{1_1}_{p_j-1,\ell} \right)$$

form a nested sequence of subsets of $V(\xi_0, \eta_\ell)$ that contains $\alpha_{u_j,\ell}$ and two further components of the hair $\gamma(u_j)$. Since $\ell \geq 0$ was selected in an arbitrary manner, as $j \to \infty$ we obtain $V(\xi_0, \infty)$ contains $\omega_{\mathfrak{T}^1}$ and infinitely many distinct components of the hair $\gamma(\mathfrak{T}^1)$ accumulating on it. To see this, let $z \in \omega_{\mathfrak{T}^1}$ be given and select $\ell_z > 0$ large enough so z lies in $\alpha_{\mathfrak{T}^1,\ell_z}$. As before, this base lies in the target $V(\xi_0, \eta_{\ell_z})$. Since $\mathfrak{T}$ has been already constructed, we may now apply Proposition 3.5 to $\gamma(\mathfrak{T}^1)$ itself. There exists an integer $N > 0$ sufficiently large so for all $n > N$, $\gamma(\sigma^{p_n+1}(\mathfrak{T}^1))$ cuts across $W^{1_1}_{p_n-1,\ell_z}$ in (at least) three components, say A_1^n, A_2^n and A_3^n, with one of them being the component of the tail $\omega_{\sigma^{p_n+1}(\mathfrak{T}^1)}$.

As n increases, the diameter of $g_a^{t_0} \circ \ldots \circ g_a^{t_{p_n}}(W^{1_1}_{p_n-1,\ell_z})$ decreases, nonetheless, Proposition 3.5 implies that for each $j = 1, 2, 3$,

$$g_a^{t_0} \circ \ldots \circ g_a^{t_{p_n}}(A_j^n) \text{ cuts across } g_a^{t_0} \circ \ldots \circ g_a^{t_{p_n}}(W^{1_1}_{p_n-1,\ell_z}).$$

Since $g_a^{t_0} \circ \ldots \circ g_a^{t_{p_n}}(A_j^n) = \alpha_{\mathfrak{T}^1,\ell_z}$ for some j, the pullback images of the remaining A_j^n's accumulate lenghtwise over $\alpha_{\mathfrak{T}^1,\ell_z}$ (and thus along z) for each $n > N$.

To see that $\gamma(\mathfrak{T}^1)$ accumulates on each of its points and not only on its tail portion, we may perform the same construction for the sequences

$$110^{k_i}110^{k_{i+1}} \ldots,$$

for $i \geq 1$. Then we may pullback the corresponding hairs and their accumulations by the appropriate inverse branches of f_a to show that $\gamma(\mathfrak{T}^1)$ must accumulate on any point in the hair $\gamma(\mathfrak{T}^1)$. $\square$

4   Geometry of hairs

This last section will focus on the geometry of sets $\overline{\gamma(t)}$ when t is a B-sequence that is either periodic or a sequence as constructed in Theorem 3.6. We show that in the former case, the closure of the hair has a landing point in $\partial\mathcal{A}^*(-a)$ while, in the latter case, $\overline{\gamma(t^i)}$ is an indecomposable continuum for each $i = 1, 2$.

Proposition 4.1. *Let t be a periodic B-sequence and t^1, t^2 their periodic associated A-sequences. Then, the hairs $\gamma(t^1)$ and $\gamma(t^2)$ land at two (repelling) periodic points $p_1, p_2 \in \partial\mathcal{A}^*(-a)$.*

Proof. Let $t = \overline{t_0 \ldots t_{n-1}}$, so every block of 0's in t has bounded length. We work only with t^1 since the other case follows similarly.

There exists a value $\delta > 0$ and a closed ball $\overline{B_\delta(0)}$ so the orbit of $\overline{\gamma(t^1)}$ stays always outside $\overline{B_\delta(0)}$. Analogously, we can find a real value m so the orbit of $\overline{\gamma(t^1)}$ does not intersect the half plane $\operatorname{Re}(z) < m < 0$.

Let $R > 0$ as in Proposition 2.9. Then, there exists a value $M \geq R$ for which the intersection of the tail ω_{t^1} with the line $\operatorname{Re}(z) = M$ is a single point. Since the orbit of the origin escapes along the positive real line, there exists an integer $N = N(M) > 0$ for which $f_a^{N-1}(0) < M \leq f_a^N(0)$. Select $0 < \epsilon \leq \delta$ small enough so for $B_\epsilon = \overline{B_\varepsilon(0)}$ and $f_a(B_\epsilon), \ldots, f_a^N(B_\epsilon)$, they are compact domains contained in T_0. Finally, select $m' \leq m < 0$ so the left half plane $\operatorname{Re}(z) \leq m'$ maps completely inside B_ϵ.

With these constants we can define an open, connected, simply connected region E as in the proof of Theorem 2.6. Then, the pullback process that defines the hair $\gamma(t^1)$ together with the contracting map $\Psi_\ell = \psi_{t_0^1}^{-1} \circ (g_a^{t_0^1} \circ \cdots \circ g_a^{t_{n-1}^1})^\ell \circ \psi_{t_0^1}$ (where $(g_a^{t_0^1} \circ \cdots \circ g_a^{t_{n-1}^1})^\ell$ denotes the ℓ-fold composition with itself), shows that $\overline{\gamma(t^1)} \setminus \gamma(t^1)$ is a unique periodic point in E with itinerary t^1. By Theorem 2.6, this periodic point lies in $\partial A^*(-a)$. $\square$

Theorem 4.2. *Let $\mathfrak{T}$ be a B-sequence as in Theorem 3.6, $\mathfrak{T}^1, \mathfrak{T}^2$ be its associated A-sequences that project onto $\mathfrak{T}$ and $\gamma(\mathfrak{T}^1), \gamma(\mathfrak{T}^2)$ their associated hairs. Then, the closure of each of these hairs is an indecomposable continuum.*

Proof. As before, we restrict the proof to the A-sequence $\mathfrak{T}^1 = \tau_1 1_1 1_2 0_1^{k_1} 1_1 1_2 0_1^{k_2} \ldots$. Let Γ^1 be the closure of the hair $\gamma(\mathfrak{T}^1)$. In order to apply Theorem 2.12, we must verify first that Γ^1 does not

separate the plane, as from Theorem 3.6 $\gamma(\mathfrak{T}^1)$ is a curve that accumulates upon itself.

First observe that Γ^1 is a set with bounded negative real part. Indeed, the first block of 0_i's in $\mathfrak{T}^1$ has finite length and this implies $\gamma(\mathfrak{T}^1)$ (and thus Γ^1) lies to the right of the line $\operatorname{Re}(z) = m$ for some $m < 0$. For simplicity, denote by $k_0 \in \{0_1, 0_2, 1_1, 1_2\}$ the first entry in τ_1, so Γ^1 lies inside the set T_{k_0} and to the right of the line $\operatorname{Re}(z) = m$. Hence

$$\mathbb{C} \setminus (T_{k_0} \cap \{z \mid \operatorname{Re}(z) > m\}) \subset \mathbb{C} \setminus \Gamma^1,$$

and since the bounderies of T_0 and T_1 are the graphs of strictly monotonic functions, the set in the left hand side is in fact a single component with unbounded imaginary part. This implies the existence of a single complementary component $U \subset \mathbb{C} \setminus \Gamma^1$ with unbounded imaginary part, while all other complementary components must have bounded imaginary parts. Also, note that $-a \in U$ and since $-a$ is a fixed point then $-a \in f_a^k(U)$ for all $k > 0$.

Let $V \neq \emptyset$ be a component in $\mathbb{C} \setminus \Gamma^1$ with bounded imaginary part. Firstly, we assume that $V \cap \mathcal{J}(f_a) \neq \emptyset$. Then, by Montel's Theorem, there exists $N > 0$ such that $-a \notin f^k(V)$ for $0 \leq k < N$ while $-a \in f_a^N(V)$, since $-a$ is not an exceptional value (indeed, $-a$ has infinitely many preimages). But this implies that $f_a^{N-1}(V)$ is a component with bounded imaginary part inside $T_0 \cup T_1$ that must contain a point in $f_a^{-1}(-a)$, a contradiction since in $T_0 \cup T_1$ there are no preimages of $-a$ different from $-a$ itself. Secondly, we assume that V does not contain points in $\mathcal{J}(f_a)$ and thus is a Fatou component. Since the Fatou set coincides with the basin of attraction of $-a$, there exists an integer $N > 0$ so that $f_a^N(U) = \mathcal{A}^*(-a)$ and thus $\partial\mathcal{A}^*(-a) \subset \partial U$, which is a contradiction with Proposition 2.5(b), as only one point in $\partial\mathcal{A}^*(-a)$ has itinerary $\mathfrak{T}^1$.

We conclude that Γ^1 does not separate the plane and by Theorem 2.12, it is either an indecomposable continuum or the union of two indecomposable continua. Nevertheless, $\gamma(\mathfrak{T}^1)$ has a unique tail extending to infinity so it cannot be the union of two indecomposable continua. $\qquad\square$

Remark 4.3. *The above two results describe an interesting relationship between the combinatorics of an itinerary s and the landing properties of the curve $\gamma(s)$. Indeed, if s is an A-sequence whose blocks of 0_i's have bounded length, the same arguments as in Proposition 4.1 can be applied to show that the accumulation set of the hair is no other than the unique point $p(s) \in \partial\mathcal{A}^*(-a)$ that follows*

the given itinerary. The key step is to ensure that pullbacks of the tail are always bounded away from the postcritical and asymptotical orbits (see, for instance, Proposition 3.6 in [14]), and this is always the case for A-sequences whose blocks of 0_i's have bounded length.

In contrast, whenever $s = \mathfrak{T}$ is a B-sequence as in Theorem 3.6, then the corresponding point $p(\mathfrak{T}^1)$ in $\partial \mathcal{A}^*(-a)$ is an accumulation point of $\gamma(\mathfrak{T}^1)$, nevertheless, is not its endpoint.

Theorem 4.4. *Consider $\mathfrak{T}$ and $\gamma(\mathfrak{T}^1)$ as in Theorem 3.6, set $\Gamma^1 = \overline{\gamma(\mathfrak{T}^1)}$. Then, there exists a unique point $p_1 \in \partial \mathcal{A}^*(-a)$ such that $p_1 \in \Gamma^1$ and p_1 has itinerary $\mathfrak{T}^1$.*

Proof. Recall that $\mathfrak{T} = \tau 110^{k_1} 110^{k_2} \ldots$. For each $n \geq 1$ define two preperiodic sequences given by

$$
\begin{aligned}
s_n &= \tau \overline{110^{k_1} \ldots 110^{k_n} 0}, \\
r_n &= \tau \overline{110^{k_1} \ldots 110^{k_n} 1}.
\end{aligned}
$$

Clearly, their associated A-sequences satisfy $s_n^1 < \mathfrak{T}^1 < r_n^1$ for all $n \geq 1$ with respect to the distance induced by the usual order in Σ_B. Moreover, if $p(s_n^1)$ and $q(r_n^1)$ denote endpoints of the corresponding preperiodic hairs associated to s_n^1 and r_n^1, then it follows by Proposition 4.1 that $p(s_n^1), q(r_n^1) \in \partial \mathcal{A}^*(-a)$. Let $p(\mathfrak{T}^1)$ be the unique point in $\partial \mathcal{A}^*(-a)$ following the itinerary $\mathfrak{T}^1$ under the action of $f_a | \partial \mathcal{A}^*(-a)$.

In the Euclidean distance restricted to $\partial \mathcal{A}^*(-a)$ we obtain $|p(s_n^1) - q(r_n^1)| \to 0$ as $n \to +\infty$ and clearly $p(s_n^1) < p(\mathfrak{T}^1) < q(r_n^1)$ with the order inherited by S^1 under the continuous extension of the Böttcher mapping $\Phi_a : \mathbb{D} \to \mathcal{A}^*(-a)$.

For each n, $\gamma(\mathfrak{T}^1)$ belongs to the region bounded above and below by $\gamma(r_n^1)$ and $\gamma(s_n^1)$, and the arc $[p(s_n^1), q(r_n^1)] \subset \partial \mathcal{A}^*(-a)$ containing $p(\mathfrak{T}^1)$. Thus, if $\gamma(\mathfrak{T}^1)$ accumulates on $\partial \mathcal{A}^*(-a)$, then it must accumulate at the point $p(\mathfrak{T}^1)$.

In order to show that $\overline{\gamma(\mathfrak{T}^1)}$ accumulates on the boundary of $\mathcal{A}^*(-a)$, we employ the polynomial-like construction and the symbolics of $\mathfrak{T}^1$. Let $m > 0$ and consider the hair associated to the itinerary

$$
t_m = 0_{k_m} 110_{k_{m+1}} 11 \ldots.
$$

Note that t_m is the image of $\mathfrak{T}^1$ under some iterates of the shift $\sigma | \Sigma_A$. By Proposition 3.5, the hair $\gamma(t_m)$ is clearly close to the origin and also the hair $\gamma(1 t_m)$ intersects a left half plane. Hence, a portion

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of $\gamma(1t_m)$ must intersect V_a and its preimages under $f_a|V_a$ following $\tau 110^{k_1} \dots 0^{k_{m-1}}1$ must accumulate in $\partial \mathcal{A}^*(-a)$ since by Proposition 2.4,

$$\bigcap_{n \geq 0} f_a^{-n}(V_a) = \partial \mathcal{A}^*(-a),$$

and $m > 0$ has been taken arbitrarily large. $\qquad \square$

We end this section by briefly discussing the set of points in the Julia set that follow non-binary sequences. Note first that Proposition 2.9 can be easily modified to show the existence of tails with itineraries corresponding to the partition $\cup_{j \in \mathbb{Z}} T_j$. If $s = (s_0, s_1, \dots)$ is a sequence that contains infinitely many non-binary symbols and $T_{s_j} \subset f_a(T_{s_{j+1}})$ for all $j \geq 0$. Let ω_s be the tail associated to s that lies in the right half plane $\text{Re}(z) > R$. If in addition s has (if any) blocks of 0's with bounded length, then the pullbacks of ω_s are always bounded away from the postcritical and asymptotic orbits, hence $\gamma(s)$ is a landing hair with an endpoint $p(s)$. By Proposition 2.5, $p(s)$ (and in fact $\overline{\gamma(s)}$) do not lie in the boundary of any Fatou component, as otherwise, s will have to end in a binary sequence. This establishes

Corollary 4.5. *If $s = (s_0, s_1, \dots) \in \mathbb{Z}^{\mathbb{N}}$ is realizable by f_a, contains infinitely many non-binary symbols and its blocks of 0's have bounded lenght, then $\gamma(s)$ is a landing hair and a buried component of $\mathcal{J}(f_a)$.*

Acknowledgments

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