

Conductor Sobolev-Type Estimates and Isocapacitary Inequalities

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ABSTRACT. In this paper we present an integral inequality connecting a function space (quasi-)norm of the gradient of a function to an integral of the corresponding capacity of the conductor between two level surfaces of the function, which extends the estimates obtained by V. Maz'ya and S. Costea, and sharp capacity inequalities due to V. Maz'ya in the case of the Sobolev norm. The inequality, obtained under appropriate convexity conditions on the function space, gives a characterization of Sobolev-type inequalities involving two measures, necessary and sufficient conditions for Sobolev isocapacitary-type inequalities, and self-improvements for integrability of Lipschitz functions.

1. INTRODUCTION

If $\text{Lip}_0(\Omega)$ is the class of all Lipschitz functions with compact support in a domain $\Omega \subset \mathbb{R}^n$, Wiener's capacity of a compact subset K of Ω ,

$$\text{Cap}(K, \Omega) = \inf_{0 \leq f \leq 1, f=1 \text{ on } K} \|\nabla f\|_2^2 \quad (f \in \text{Lip}_0(\Omega)),$$

extended in the obvious way for any $p \geq 1$ as the p -capacity

$$\text{Cap}_p(K, \Omega) = \inf_{0 \leq f \leq 1, f=1 \text{ on } K} \|\nabla f\|_p^p \quad (f \in \text{Lip}_0(\Omega)),$$

was used in [M05] to obtain the Sobolev inequality

$$(1.1) \quad \int_0^\infty \text{Cap}_p(\bar{M}_{at}, M_t) d(t^p) \leq c(a, p) \|\nabla f\|_p^p,$$

where M_t is the level set $\{x \in \Omega : |f(x)| > t\}$ ($t > 0$).

This “conductor inequality” is a powerful tool with applications to Sobolev-type imbedding theorems, which for $p > 1$ plays the same role as the co-area formula for $p = 1$.

With its variants, (1.1) has many applications to very different areas, such as to Sobolev inequalities on domains of \mathbb{R}^n and on metric spaces, to linear and non-linear partial differential equations, to calculus of variations, to Markov processes, etc. (See, e.g., [AH], [AP], [Ci], [Da], [DKX], [Han], [K84], [MM], [MM1], [MM2], [M85], [M11], [M05], [M06], [MN], [MP], [Ra], [V99], and the references therein).

An interesting extension based on the Lorentz space $L^{p,q}(\Omega)$ ($1 < p < \infty$, $1 \leq q \leq \infty$) has been recently obtained in [CoMa], showing that

$$\int_0^\infty \text{Cap}_{p,q}(\bar{M}_{at}, M_t) d(t^p) \leq c(a, p, q) \|\nabla f\|_{L^{p,q}(\Omega)}^p \quad (1 \leq q \leq p)$$

and

$$\int_0^\infty \text{Cap}_{p,q}(\bar{M}_{at}, M_t)^{q/p} d(t^q) \leq c(a, p, q) \|\nabla f\|_{L^{p,q}(\Omega)}^q \quad (p < q < \infty),$$

where now

$$\text{Cap}_{p,q}(K, \Omega) = \inf_{0 \leq f \leq 1, f=1 \text{ on } K} \|\nabla f\|_{L^{p,q}(\Omega)}^p \quad (f \in \text{Lip}_0(\Omega)).$$

Our aim in this paper is to extend these capacity estimates when a general function space X substitutes $L^p(\Omega)$ or $L^{p,q}(\Omega)$ in the definition of Cap_p and $\text{Cap}_{p,q}$.

It could seem that for improvements of integrability only truncations methods are needed. In [KO] it appears that inequalities of Sobolev-Poincaré type are improved to Lorentz-type scales thanks to stability under truncations, but there and also in [CoMa], p -convexity is implicitly used, since the proofs are based on the inequalities

$$\begin{aligned} \|f\|_{L^{p,q}(\Omega,\mu)}^p + \|\mathcal{G}\|_{L^{p,q}(\Omega,\mu)}^p &\leq \|f + \mathcal{G}\|_{L^{p,q}(\Omega,\mu)}^p \quad (1 \leq q \leq p), \\ \|f\|_{L^{p,q}(\Omega,\mu)}^q + \|\mathcal{G}\|_{L^{p,q}(\Omega,\mu)}^q &\leq \|f + \mathcal{G}\|_{L^{p,q}(\Omega,\mu)}^q \quad (1 < p < q) \end{aligned}$$

of the Lorentz (quasi-)norms, for disjointly supported functions. With use of the fact that the constant in the right-hand side of the inequalities is 1, they can be extended to an arbitrary set of disjoint functions, and $L^{p,q}$ satisfies lower estimates with constant 1 (see Section 2).

A perusal in the proofs also shows that the limitation of the usual techniques is that they allow us to cover only a certain particular kind of spaces because of the lower p -estimates with constant 1, and it does not apply to a wider class of spaces.

However, by means of new techniques, we will see that an extension is possible in the setting of (quasi-)Banach spaces with lower p -estimates, independently of

the value of the constant. Our results can be applied to many examples, which include Lebesgue spaces, Lorentz spaces, classical Lorentz spaces, Orlicz spaces, and mixed norm spaces.

The organization of the article is as follows. Since certain convexity conditions on the space are needed, in Section 2 we recall some basic definitions and known results concerning these concepts and present the most classical examples of spaces, not necessarily rearrangement invariant, satisfying these kinds of properties, and we include some facts concerning capacities and submeasures that we will require in the development of our results.

In Section 3, using a result due to Kalton and Montgomery-Smith on submeasures satisfying an upper p -estimate, we prove our main results.

In Section 4 we characterize Sobolev-type inequalities in the setting of rearrangement invariant (r.i.) spaces. Under appropriate conditions on the space X (see Theorem 4.2) and for any $0 < p < \infty$, we show the equivalence of the following properties:

- (i) For every compact set K on Ω , $\varphi_Y(\mu(K)) \lesssim \text{Cap}_X(K)$,
- (ii) $\|f\|_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|_X$ ($f \in \text{Lip}_0(\Omega)$),
- (iii) $\|f\|_{\Lambda^{1,\infty}(Y)} \lesssim \|\nabla f\|_X$ ($f \in \text{Lip}_0(\Omega)$),

where φ_Y denotes the fundamental function of Y and $\Lambda^{p,q}(Y)$ ($0 < q \leq \infty$) a Lorentz space, defined in Section 4. Moreover, under the appropriate conditions on Y , we show that

$$\|f\|_{\Lambda^{1,\infty}(Y)} \lesssim \|\nabla f\|_X \Leftrightarrow \|f\|_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|_X \Leftrightarrow \|f\|_Y \lesssim \|\nabla f\|_X.$$

In the particular case when $X = L^p$, $p \in (1, n)$, and $Y = L^s$ with $s = np/(n - p)$, we recover the well-known self-improvement of integrability of Lipschitz functions

$$\|f\|_{L^{s,p}} = \|f\|_{\Lambda^{1,p}(L^s)} \lesssim \|\nabla f\|_{L^p}.$$

In Section 5 we derive necessary and sufficient conditions for Sobolev-type inequalities in r.i. spaces involving two measures, recovering results obtained in [CoMa], [M05] and [M06] for Lorentz spaces.

Finally, in Section 6, we include some connections with the theory of the capacity function spaces studied in [Ce], [CMS], and [CMS1].

As usual, the symbol $f \lesssim g$ means that there exists a universal constant $c > 0$ (independent of all parameters involved) such that $f \leq cg$, and $f \simeq g$ means that $f \lesssim g \lesssim f$.

2. PRELIMINARIES

2.1. Function spaces. Let (Ω, μ) be a measure space and $L^0(\Omega)$ the vector space of all (equivalence classes of) measurable real functions on Ω . We shall say that X is a *quasi-Banach function space* if it is a quasi-Banach linear subspace of $L^0(\Omega)$ with the following properties:

(i) (Lattice property) If $g \in X$ and $f \in L^0(\Omega)$ such that $|f| \leq |g|$, then $f \in X$ and $\|f\|_X \leq \|g\|_X$.

(ii) (Fatou property) If $0 \leq f_n \uparrow f$ almost everywhere, then $\|f_n\|_X \leq \|f\|_X$.

For $0 < p < \infty$, recall that X is said to satisfy an upper p -estimate or a lower p -estimate if there exists a constant M so that, for all $n \in \mathbb{N}$ and for any choice of disjointly supported elements $\{f_i\}_{i=1}^n \subset X$,

$$(2.1) \quad \left\| \left(\sum_{i=1}^n |f_i|^p \right)^{1/p} \right\| \leq M \left(\sum_{i=1}^n \|f_i\|^p \right)^{1/p}$$

or

$$(2.2) \quad \left(\sum_{i=1}^n \|f_i\|^p \right)^{1/p} \leq M \left\| \left(\sum_{i=1}^n |f_i|^p \right)^{1/p} \right\|,$$

respectively. The smaller constant M is called the upper p -estimate constant or the lower p -estimate constant, and it will be denoted by $M^{(p)}(X)$ or $M_{(p)}(X)$, respectively.

2.2. Some examples. For the sake of the reader's convenience, let us present some examples of spaces satisfying these kinds of properties.

As usual, if $f \in L^0(\Omega)$, f^* will denote the non-increasing rearrangement of f defined by

$$f^*(t) = \inf \{ \lambda > 0 : \mu \{ x \in \Omega : |f(x)| > \lambda \} \leq t \},$$

and $f^{**}(t) := t^{-1} \int_0^t f^*(s) ds$ the average function.

Recall that a quasi-Banach function space X on Ω is said to be rearrangement invariant (r.i.) if $f \in X$, $g \in L^0(\Omega)$ and $g^* \leq f^*$ imply $g \in X$ and $\|g\|_X \leq \|f\|_X$.

A function $F : (0, \infty) \rightarrow (0, \infty)$ is called quasi-increasing (respectively, quasi-decreasing) if $F(s) \leq F(t)$ (respectively, $F(t) \leq F(s)$) for any $0 < s < t$. Moreover, F is said to be quasi-superadditive if there exists a constant $d > 0$ such that $F(x) + F(y) \leq dF(x + y)$ for all $0 < x, y < \infty$, and it is said to be superadditive when $d = 1$.

Example 2.1. The space X is said to be p -concave (respectively, p -convex) if (2.2) (respectively, (2.1)) holds for arbitrary functions.

Since $\sum_{i=1}^n |g_i|^p = |\sum_{i=1}^n g_i|^p$ when $\{g_i\}_{i=1}^n \subset X$ are disjointly supported, if X is p -concave, then X satisfies a lower p -estimate.

Example 2.2 (Lorentz spaces). Suppose $0 < p < \infty$, and let w be a weight on $(0, \infty)$ satisfying the Δ_2 -condition $\int_0^{2t} w(s) ds \lesssim \int_0^t w(s) ds$, so that the classical Lorentz space

$$\Lambda^p(w) = \left\{ f \in L^0(\Omega) : \|f\|_{\Lambda^p(w)} := \left(\int_0^\infty f^*(x)^p w(x) dx \right)^{1/p} < \infty \right\}$$

is a quasi-Banach space (see, e.g., [CRS, Section 2.2]).

It is well known that $\Lambda^p(w)$ is p -convex with constant 1 when w is decreasing and p -concave with constant 1 when w is increasing (see [KM]).

If w is decreasing, by [KP, Theorem 1], for $r > p$ and $p/r + 1/s = 1$, the r -concavity constant of $\Lambda^p(w)$ is

$$\sup_{t>0} \left[\frac{\left(\frac{1}{t} \int_0^t w^s\right)^{1/s}}{\frac{1}{t} \int_0^t w} \right]^{1/p}.$$

Moreover, if $0 < \int_0^x w(t) dt < \infty$ and $\int_x^\infty t^{-p} w(t) dt < \infty$, then, for $p \leq r < \infty$, $\Lambda^p(w)$ satisfies a lower r -estimate if and only if

$$t^{-p/r} \int_0^t w(s) ds$$

is quasi-increasing.

These spaces generalize many known spaces in the literature. For instance, if $w(t) := t^{p/q-1}(1 + \log(t))^{\lambda p}$, then we obtain the Lorentz-Zygmund space, that is, $\Lambda^p(w) = L^{q,p}(\text{Log}L)^\lambda(\Omega)$ (see, e.g., [BR]).

More generally, a positive function b is said to be slowly varying on $(1, \infty)$ (in the sense of Karamata) if for each $\varepsilon > 0$, $t^\varepsilon b(t)$ is quasi-increasing and $t^{-\varepsilon} b(t)$ is quasi-decreasing. For example,

$$b(t) = \exp(\sqrt{\log t}) \quad \text{and} \quad b(t) = (e + \log t)^\alpha (\log(e + \log t))^\beta,$$

with $\alpha, \beta \in \mathbb{R}$, are slowly varying.

If $w(t) = t^{q/p-1} b(1/t)^q$ on $(0 < t < 1)$ with b slowly varying, then $\Lambda^q(w)$ is the Lorentz-Karamata space $L_{p,q,b}(\Omega)$ (see, e.g., [Nev]).

Example 2.3 ($\Gamma^p(w)$). Suppose that the weight w satisfies the nondegeneracy conditions $\int_0^1 s^{-p} w(s) ds = \int_1^\infty w(s) ds = \infty$.

If $t^{-p} \int_0^t w(s) ds \lesssim \int_t^\infty s^{-p} w(s) ds$ and $1 < p \leq r < \infty$, then

$$\Gamma^p(w) = \left\{ f \in L^0(\Omega) : \|f\|_{\Gamma^p(w)} := \left(\int_0^\infty f^{**}(x)^p w(x) dx \right)^{1/p} < \infty \right\}$$

satisfies a lower r -estimate if and only if $t^{p(1-1/r)} \int_t^\infty s^{-p} w(s) ds$ is quasi-increasing.

For $0 < p \leq 1$ and $r \geq 1$, or for $1 < p < r < \infty$, $\Gamma^p(w)$ is r -concave if and only if

$$t^{p(1-1/r)-\varepsilon} \int_t^\infty s^{-p} w(s) ds$$

is quasi-increasing for some $\varepsilon > 0$.

For details see [KM1].

Example 2.4 (Orlicz spaces). Let ϕ be a Young function, and consider the Luxemburg norm defined by

$$\|f\|_\phi := \inf \left\{ \varepsilon > 0 : \int_0^{\mu(\Omega)} \phi \left(\frac{|f(t)|}{\varepsilon} \right) dt \leq 1 \right\}.$$

- For $0 < q < \infty$, $L_\phi(\Omega)$ satisfies a lower q -estimate if and only if $\phi(\lambda u) \lesssim \lambda^q \phi(u)$ for all $\lambda \geq 1$ and all u .
- Suppose $\mu(\Omega) < \infty$ and $1 < p \leq 2 \leq q < r < \infty$. If $\phi(\lambda u) \lesssim \lambda^q \phi(u)$ for all $\lambda \geq 1$ and $u \geq u_0 \geq 0$, then $L_\phi(\Omega)$ is r -concave.
- If $\mu(\Omega) = \infty$, then the above inequalities need to be satisfied for all $u \geq 0$.

For details see [K1] and [K2].

Function spaces that are not rearrangement invariant may also be considered:

Example 2.5 (Mixed norm L^p spaces). The space $L^q(\Omega_2)[L^p(\Omega_1)]$ for $1 \leq p, q \leq \infty$, defined by the condition

$$\|f\| := \left(\int \left(\int |f(x, y)|^p d\mu_1(x) \right)^{q/p} d\mu_2(y) \right)^{1/q} < \infty,$$

satisfies a lower pq -estimate with constant 1.

Indeed, if f and g are two disjointly supported functions, it follows from [BP, Theorem 1] that $\|f + g\|^{pq} \geq \|f\|^{pq} + \|g\|^{pq}$.

Similarly, in the case $L^{p_n}(\mu_n)[\dots[L^{p_1}(\mu_1)]]$ of n parameters, we have a lower $p_1 \cdots p_n$ -estimate with constant 1.

Example 2.6 (Mixed norm weighted Lorentz spaces). Suppose $1 \leq p, q < \infty$, and, for a measurable function f on $\Omega = \Omega_1 \times \Omega_2$, let $f_y^*(x, t)$ denote the decreasing rearrangement of f with respect to the second variable y , when the first variable x is fixed (see [BK]).

Let u and v be weights on Ω_1 and Ω_2 , u such that $U(x) := \int_0^x u(t) dt$ is quasi-superadditive. Then the space $\Lambda^q(v)[\Lambda^p(u)]$ defined by the condition

$$\|f\|_{\Lambda^q(v)[\Lambda^p(u)]} := \left(\int_0^\infty \left[\left(\int_0^\infty (f_y^*(\cdot, t))^p u(t) dt \right)^* (s) \right]^{q/p} v(s) ds \right)^{1/q} < \infty$$

also satisfies a lower pq -estimate.

Let $0 \leq a \leq 1$. An application of Hölder's inequality gives

$$(|x|^p + |y|^p)^{1/p} \geq a^{1-1/p} x + (1-a)^{1-1/p} y \quad (1 \leq p < \infty).$$

It follows that, since $\Lambda^p(u)$ satisfies a lower p -estimate (see [CS, Lemma 3.2]), if $f, g \in \Lambda^p(u)$ are disjointly supported, then

$$\begin{aligned} M_{(p)}(\Lambda^p(u))\|f + g\|_{\Lambda^p(u)} &\geq (\|f\|_{\Lambda^p(u)}^p + \|g\|_{\Lambda^p(u)}^p)^{1/p} \\ &\geq a^{1-1/p} \|f\|_{\Lambda^p(u)} + (1-a)^{1-1/p} \|g\|_{\Lambda^p(u)}, \end{aligned}$$

if $0 \leq a \leq 1$.

Let now $f, g \in \Lambda^q(v)[\Lambda^p(u)]$ be disjointly supported. Then,

$$\begin{aligned} &M_{(p)}(\Lambda^p(u))\|f + g\|_{\Lambda^q(v)[\Lambda^p(u)]} \\ &= M_{(p)}(\Lambda^p(u))\left(\int_0^\infty \left[\left(\int_0^\infty (f + g)_y^*(\cdot, t)^p u(t) dt\right)^*(s)\right]^{q/p} v(s) ds\right)^{1/q} \\ &\geq \left(\int_0^\infty [a^{1-1/p}\|f_y(s)\|_{\Lambda^p(u)}^q + (1-a)^{1-1/p}\|g_y(s)\|_{\Lambda^p(u)}^q] v(s) ds\right)^{1/q} \\ &\geq a^{1-1/(pq)}\|f\|_{\Lambda^q(v)[\Lambda^p(u)]} + (1-a)^{1-1/(pq)}\|g\|_{\Lambda^q(v)[\Lambda^p(u)]}. \end{aligned}$$

Finally, choosing

$$a = \frac{\|f\|_{\Lambda^q(v)[\Lambda^p(u)]}^{pq}}{\|f\|_{\Lambda^q(v)[\Lambda^p(u)]}^{pq} + \|g\|_{\Lambda^q(v)[\Lambda^p(u)]}^{pq}},$$

we obtain

$$M_{(p)}(\Lambda^p(u))\|f + g\|_{\Lambda^q(v)[\Lambda^p(u)]} \geq (\|f\|_{\Lambda^q(v)[\Lambda^p(u)]}^{pq} + \|g\|_{\Lambda^q(v)[\Lambda^p(u)]}^{pq})^{1/(pq)}.$$

Observe that, if U is superadditive, then $M_{(p)}(\Lambda^p(u)) = 1$.

Remark 2.7. In Example 2.2, if w is decreasing (respectively, increasing), then for all $q > p$, $\Lambda^p(w)$ is q -concave with constant 1 (respectively, q -convex with constant 1 for all $0 < q < p$) if and only if $\Lambda^p(w)$ is isometric to L^p . See [KP, Corollary 4].

For a Lorentz-Karamata space, we have (see [EP])

$$\begin{aligned} \|f\|_{p,q,b} &= \left(\int_0^\infty [t^{1/p-1/q} f^*(t)b(t)]^q dt\right)^{1/q} \\ &= \left(\int_0^\infty f^*(t)^q t^{q/p-1} b(t)^q dt\right)^{1/q} = \|f\|_{\Lambda^q(w)}, \end{aligned}$$

where w is the weight defined as $w(s) := s^{q/p-1}b(s)^q$, $s > 0$. For $q > p$, $W(x) = \int_0^x w(s) ds$ is quasi-superadditive, and, by [CS, Lemma 3.2], $\Lambda^p(w)$ satisfies a lower q -estimate. For $q \leq p$, by [KM, Theorem 6], $\Lambda^p(w)$ is not q -concave.

A function ϕ is said to satisfy (RC) if

$$\frac{\phi(au)}{\phi(u)} + \frac{\phi((1-a)v)}{\phi(v)} \geq 1 \quad \text{for all } u, v > 0 \text{ and } 0 < a < 1.$$

Assume that ϕ is an Orlicz function, that is, ϕ is strictly increasing and continuous with $\lim_{u \rightarrow \infty} \phi(u) = \infty$, $\phi(0) = 0$ and $\phi(1) = 1$.

When $\mu(\Omega) < \infty$, if $\phi(u^{1/p})$ satisfies (RC), by [HKT, Corollary 3.3], $L_\phi(\Omega)$ satisfies a lower p -estimate with constant 1.

If $\mu(\Omega) = \infty$, then $\phi(u^{1/p})$ satisfies (RC) if and only if $L_\phi(\Omega)$ satisfies a lower p -estimate with constant 1.

2.3. Capacities. Let Ω be a domain of \mathbb{R}^n endowed with the Lebesgue measure m_n , $\text{Lip}(\Omega)$ be the class of Lipschitz functions on Ω , and $\text{Lip}_0(\Omega) = \{u \in \text{Lip}(\Omega) : \sup u \text{ compact in } \Omega\}$.

From now on, $X = X(\Omega)$ denotes a quasi-Banach function space on Ω .

Given a compact set $K \subset \Omega$ and an open set $G \subset \Omega$ containing K , we call the couple (K, G) a conductor and denote

$$W(K, G) := \{u \in \text{Lip}_0(G) : u = 1 \text{ on a neighbourhood of } K, 0 \leq u \leq 1\}.$$

Each conductor has an X -capacity defined by

$$\text{Cap}_X(K, G) := \inf\{\|\nabla u\|_X : u \in W(K, G)\}$$

that for $X = L^{p,q}$ recovers the capacity $\text{Cap}_X = \text{Cap}_{p,q}^{1/p}$ from [Co]. From the definition (see [M85], [M05] and [Co]) we have the following statements:

- If $K_1 \subset K_2$ are compact sets in G , $\text{Cap}_X(K_1, G) \leq \text{Cap}_X(K_2, G)$.
- If $\Omega_1 \subset \Omega_2$ are open and K is a compact subset of Ω_1 , then

$$\text{Cap}_X(K, \Omega_2) \leq \text{Cap}_X(K, \Omega_1).$$

- If $\{K_i\}$ is a decreasing sequence of compact subsets of G with $K := \bigcap_{i=1}^\infty K_i$, then

$$\text{Cap}_X(K, G) = \lim_{i \rightarrow \infty} \text{Cap}_X(K_i, G).$$

- If $\{\Omega_i\}$ is an increasing sequence of open subsets of Ω with $\Omega := \bigcup_{i=1}^\infty \Omega_i$ and K is a compact subset of Ω_1 , then

$$\text{Cap}_X(K, \Omega) = \lim_{i \rightarrow \infty} \text{Cap}_X(K, \Omega_i).$$

We will write $\text{Cap}_X(\cdot) = \text{Cap}_X(\cdot, \Omega)$ if Ω has been fixed.

2.4. Submeasures. If \mathcal{A} is an algebra of subsets on Ω , a set-function $\phi : \mathcal{A} \rightarrow \mathbb{R}$ is said to be monotone if it satisfies $\phi(\emptyset) = 0$ and $\phi(A) \leq \phi(B)$ whenever $A \subset B$, and ϕ is said to be normalized when $\phi(\Omega) = 1$. A monotone set-function ϕ is a submeasure if

$$\phi(A \cup B) \leq \phi(A) + \phi(B)$$

whenever $A, B \in \mathcal{A}$ are disjoint, and ϕ is a supermeasure if

$$\phi(A \cup B) \geq \phi(A) + \phi(B)$$

whenever $A, B \in \mathcal{A}$ are disjoint.

For any $0 < p < \infty$, we say that a monotone set-function ϕ satisfies an upper p -estimate if ϕ^p is a submeasure, and a lower p -estimate if ϕ^p is a supermeasure.

In the proof of our main result, Theorem 3.1, we shall use [KM0, Theorem 2.2], where it is shown that if $0 < p < 1$ and φ is a normalized supermeasure which satisfies an upper p -estimate, then there exists a measure μ on Ω such that $\varphi \leq \mu$ and $\mu(\Omega) \leq K_p$, where

$$K_p = \frac{2}{(2^p - 1)^{1/p}} - 1.$$

For a more complete treatment, see [KM0] and the references quoted therein.

3. SOBOLEV CAPACITARY INEQUALITIES

In this section we will present our extensions of [CoMa, Theorem 4.2] to an arbitrary parameter p , $0 < p < \infty$, and X a Banach or quasi-Banach function space which satisfies a lower p -estimate.

Theorem 3.1. *Suppose $0 < p < \infty$, and let $\alpha > 1$ be a constant. If X is a Banach function space that satisfies a lower p -estimate, then*

$$(3.1) \quad \int_0^\infty t^p \text{Cap}_X(\{\overline{|f|} > \alpha t\}, \{|f| > t\})^p \frac{dt}{t} \leq c \|\nabla f\|_X^p \quad (f \in \text{Lip}_0(\Omega)),$$

where c is a constant that depends on α , p and $M_{(p)}(X)$.

In particular,

$$(3.2) \quad \int_0^\infty t^p \text{Cap}_X(\{|f| \geq t\})^p \frac{dt}{t} \leq 2^p c \|\nabla f\|_X^p \quad (f \in \text{Lip}_0(\Omega)),$$

where c depends on p and $M_{(p)}(X)$.

Proof. Without loss of generality, assume that $\|\nabla f\|_X < \infty$, and that $f \geq 0$, since $|\nabla|f|| \leq |\nabla f|$.

Since X is a Banach function space, the set-function

$$\phi(A) := \frac{\|\nabla f|_{X_A}\|_X}{\|\nabla f\|_X} \quad (A \in \mathcal{B}(\Omega))$$

is a submeasure. Moreover, using that X satisfies a lower p -estimate, we conclude that if A_1, \dots, A_n are disjoint, then

$$(3.3) \quad \phi(A_1 \cup \dots \cup A_n) \geq \frac{1}{M_{(p)}(X)} (\phi^p(A_1) + \dots + \phi^p(A_n))^{1/p}.$$

Let us consider the set-function ψ , defined by

$$(3.4) \quad \psi(A) := \sup \left\{ \sum_{i=1}^n \phi^p(A_i) \right\},$$

the supremum being taken over all finite partitions (A_1, \dots, A_n) of A .

It follows from (3.3) and (3.4) that

$$(3.5) \quad \frac{\psi}{(M_{(p)}(X))^p} \leq \phi^p \leq \psi,$$

and we claim that ψ is a supermeasure satisfying an upper $\min(p, 1/p)$ -estimate. Indeed, given any $\varepsilon > 0$ and two disjoint sets A and B , choose finite partitions $A = \bigcup_{i=1}^{n_a} A_i$, $B = \bigcup_{j=1}^{n_b} B_j$ such that

$$\psi(A)(1 - \varepsilon) \leq \sum_{i=1}^{n_a} \phi^p(A_i) \quad \text{and} \quad \psi(B)(1 - \varepsilon) \leq \sum_{j=1}^{n_b} \phi^p(B_j).$$

Then $\{D_k\}_{k=1}^{n_a+n_b} = \{A_k\}_{k=1}^{n_a} \cup \{B_k\}_{k=1}^{n_b}$ is a partition of $A \cup B$ which satisfies

$$\begin{aligned} \psi(A)(1 - \varepsilon) + \psi(B)(1 - \varepsilon) &\leq \sum_{i=1}^{n_a} \phi^p(A_i) + \sum_{j=1}^{n_b} \phi^p(B_j) \\ &\leq \sum_{k=1}^{n_a+n_b} \phi^p(D_k) \leq \psi(A \cup B), \end{aligned}$$

and ψ is a supermeasure.

Let $r = \min(p, 1/p)$. Recall that ψ satisfies an upper r -estimate if ψ^r is a submeasure.

Suppose first $p \geq 1$, that is, $r = 1/p$, and let A, B be disjoint sets. If (C_1, \dots, C_n) is a partition of $A \cup B$, then, since ϕ is a submeasure,

$$\begin{aligned} \left(\sum \phi^p(C_i) \right)^{1/p} &= \left(\sum_i \phi^p((C_i \cap A) \cup (C_i \cap B)) \right)^{1/p} \\ &\leq \left(\sum_i (\phi(C_i \cap A) + \phi(C_i \cap B))^p \right)^{1/p} \\ &= \|\{\phi(C_i \cap A) + \phi(C_i \cap B)\}_{i=1}^n\|_{\ell^p} \\ &\leq \|\{\phi(C_i \cap A)\}_{i=1}^n\|_{\ell^p} + \|\{\phi(C_i \cap B)\}_{i=1}^n\|_{\ell^p} \\ &= \left(\sum \phi^p(C_i \cap A) \right)^{1/p} + \left(\sum \phi^p(C_i \cap B) \right)^{1/p} \\ &\leq \psi(A)^{1/p} + \psi(B)^{1/p}. \end{aligned}$$

Therefore, taking the supremum over all partitions, we obtain that

$$\psi(A \cup B) = \sup \sum \phi^p(C_i) \leq (\psi(A)^{1/p} + \psi(B)^{1/p})^p,$$

and $\psi^{1/p}$ is a submeasure.

If $p < 1$ and (C_1, \dots, C_n) is a partition of $A \cup B$, then, since ϕ is a submeasure, using that

$$(x + y)^p \leq x^p + y^p \quad (x, y \geq 0),$$

we have that

$$\begin{aligned} \left(\sum \phi^p(C_i)\right)^p &\leq \left(\sum_i (\phi(C_i \cap A) + \phi(C_i \cap B))^p\right)^p \\ &\leq \left(\sum \phi^p(C_i \cap A)\right)^p + \left(\sum \phi^p(C_i \cap B)\right)^p \\ &\leq \psi(A)^p + \psi(B)^p. \end{aligned}$$

Therefore, taking the supremum over all partitions, we obtain that

$$\psi(A \cup B) = \sup \sum \phi^p(C_i) \leq (\psi(A)^p + \psi(B)^p)^{1/p},$$

and ψ^p is a submeasure.

We normalize ψ and define

$$\varphi(A) := \frac{\psi(A)}{\psi(\Omega)},$$

a normalized supermeasure which satisfies an upper r -estimate. Thus, by [KM0, Theorem 2.2], there is a measure μ on Ω such that

$$(3.6) \quad \varphi \leq \mu \quad \text{and} \quad \mu(\Omega) \leq K_r.$$

Now, if $M_t := \{|f| > t\} = \{f > t\}$, the function $\gamma(t) := \mu(M_t)$ is decreasing on $(0, \infty)$ and the limits $\gamma(0)$ and $\gamma(\infty)$ exist, so that

$$\int_0^\infty (\gamma(t) - \gamma(at)) \frac{dt}{t} = \lim_{\varepsilon \rightarrow 0, N \rightarrow \infty} \int_\varepsilon^N (\gamma(t) - \gamma(at)) \frac{dt}{t}$$

as an improper integral.

We have

$$\mu(M_t) = \mu(M_t \setminus M_{at}) + \mu(M_{at})$$

and therefore

$$\int_0^\infty \mu(M_t \setminus M_{at}) \frac{dt}{t} = \int_0^\infty (\mu(M_t) - \mu(M_{at})) \frac{dt}{t} = \mu(M_0) \log a.$$

By (3.5) and (3.6) we obtain

$$\begin{aligned} \int_0^\infty \mu(M_t \setminus M_{at}) \frac{dt}{t} &\geq \int_0^\infty \varphi(M_t \setminus M_{at}) \frac{dt}{t} = \int_0^\infty \frac{\psi(M_t \setminus M_{at})}{\psi(\Omega)} \frac{dt}{t} \\ &\geq \frac{1}{\psi(\Omega)} \int_0^\infty \phi^p(M_t \setminus M_{at}) \frac{dt}{t} \\ &= \frac{1}{\psi(\Omega)} \int_0^\infty \frac{\|\nabla f|_{X_{M_t \setminus M_{at}}}\|_X^p}{\|\nabla f\|_X^p} \frac{dt}{t}, \end{aligned}$$

so that

$$(3.7) \quad \begin{aligned} \int_0^\infty \|\nabla f|_{X_{M_t \setminus M_{at}}}\|_X^p \frac{dt}{t} &\leq \psi(\Omega) \mu(M_0) \log a \|\nabla f\|_X^p \\ &\leq K_r M_{(p)}(X)^p \log a \|\nabla f\|_X^p. \end{aligned}$$

Consider now

$$\Lambda_t(f) = \min \left\{ \frac{(|f| - t)_+}{(a - 1)t}, 1 \right\}.$$

Since $f \in \text{Lip}_0(\Omega)$, an easy computation shows that

$$|\nabla \Lambda_t(f)| = \frac{1}{(a - 1)t} |\nabla f|_{X_{M_t \setminus M_{at}}}$$

and obviously

$$\|\nabla f|_{X_{M_t \setminus M_{at}}}\|_X^p = (a - 1)^p t^p \|\nabla \Lambda_t(f)\|_X^p.$$

Moreover, since $\Lambda_t(f) \in W(\bar{M}_{at}, M_t)$,

$$\|\nabla f|_{X_{M_t \setminus M_{at}}}\|_X^p \geq (a - 1)^p t^p \text{Cap}_X(\bar{M}_{at}, M_t)^p,$$

and the proof of (3.1) with

$$c := c(a, p, M_{(p)}(X)) = \frac{M_{(p)}(X)^p K_r \log a}{(a - 1)^p}$$

ends by inserting the last estimate in the left-hand side of (3.7).

If $p = 1$, then X satisfies a lower 1-estimate, and it follows from [LZ, Proposition 1.f.7] that X is q -concave for all $q > 1$. Therefore, X can be equivalently renormed so that, with the new norm, it satisfies a lower q -estimate with constant 1. Hence, the result follows with similar arguments to those in [CoMa].

The capacity inequality (3.2) follows from using (3.1) with $a = 2$ and $\text{Cap}_X(\bar{M}_{at}) \leq \text{Cap}_X(\bar{M}_{at}, M_t)$. In this case

$$2^p c = 2^p c(2, p, M_{(p)}(X)) = M_{(p)}(X)^p K_r 2^p \log 2. \quad \square$$

Theorem 3.1 can be extended to the setting of quasi-Banach spaces by using Aoki-Rolewicz's Theorem (see, e.g., [BL, Section 3.10]):

Theorem 3.2. *Suppose $0 < p < \infty$, and let $a > 1$ be a constant. If X is a quasi-Banach function space which satisfies a lower p -estimate, then*

$$\int_0^\infty t^p \operatorname{Cap}_X(\overline{\{|f| > at\}}, \{f > t\})^p \frac{dt}{t} \leq c_1 \|\nabla f\|_X^p \quad (f \in \operatorname{Lip}_0(\Omega)),$$

where the constant c_1 depends on a , p , $M_{(p)}(X)$ and on the quasi-subadditivity constant c of the quasi-norm in X .

In particular,

$$\int_0^\infty t^p \operatorname{Cap}_X(\{|f| \geq t\})^p \frac{dt}{t} \leq 2^p c_1 \|\nabla f\|_X^p \quad (f \in \operatorname{Lip}_0(\Omega)),$$

with c_1 depending on p , $M_{(p)}(X)$ and c .

Proof. The proof of Theorem 3.1 can be adapted to this case as follows.

By Aoki-Rolewicz's Theorem, if ϱ is defined as $(2c)^\varrho = 2$, there is a 1-seminorm $\|\cdot\|^*$ such that, for all $f \in X$,

$$\|f\|^* \leq \|f\|_X^\varrho \leq 2\|f\|^*.$$

Endowed with this 1-seminorm, X satisfies a lower p/ϱ -estimate, since if f_1, \dots, f_n are disjointly supported functions in X , then

$$\begin{aligned} \left(\sum_{i=1}^n (\|f_i\|^*)^{p/\varrho}\right)^{\varrho/p} &\leq \left(\sum_{i=1}^n \|f_i\|_X^p\right)^{\varrho/p} \leq M_{(p)}(X)^\varrho \left\|\sum_{i=1}^n |f_i|\right\|_X^\varrho \\ &\leq 2M_{(p)}(X)^\varrho \left\|\sum_{i=1}^n |f_i|\right\|^*. \end{aligned}$$

Now consider

$$\psi(A) = \sup \left\{ \sum_{i=1}^n \phi^{p/\varrho}(A_i) \right\} \quad \text{with} \quad \phi(A) = \frac{\|\nabla f|_{X_A}\|^*}{\|\nabla f\|^*}.$$

With the same arguments as in Theorem 3.1, it can be shown that ψ is a supermeasure that satisfies an upper r -estimate, and the proof ends in the same way, now with

$$c_1 := c_1(a, p, c, M_{(p)}(X)) = \frac{2^{2p/\varrho} K_r \log a M_{(p)}(X)^p}{(a-1)^p},$$

for ϱ such that $(2c)^\varrho = 2$ and $r = \min(p/\varrho, \varrho/p)$. □

4. ISOCAPACITARY INEQUALITIES AND SOBOLEV-TYPE ESTIMATES

Let μ be a Borel measure on Ω and X be a quasi-Banach r.i. space on Ω . Recall that the distribution function of f is defined as

$$\mu_f(\lambda) := \mu\{x \in \Omega : |f(x)| > \lambda\} \quad (\lambda \geq 0),$$

and the fundamental function of X (see [BS] and [BrK]) is defined by

$$\varphi_X(t) = \|\chi_A\|_X \quad (t = \mu(A)).$$

Given $0 < p < \infty$ and $0 < q \leq \infty$, the Lorentz space $\Lambda^{p,q}(X)$ associated to X is defined as

$$\left\{ f \in L^0(\Omega) : \|f\|_{\Lambda^{p,q}(X)} = \left(\int_0^\infty p t^{q-1} (\varphi_X(\mu_f(t)))^{q/p} dt \right)^{1/q} < \infty \right\}$$

with the usual changes when $q = \infty$. When $p = q$, we write $\Lambda^p(X)$ instead of $\Lambda^{p,p}(X)$.

Notice that if $X = L^1$, then $\Lambda^{p,q}(L^1) = L^{p,q}$.

It is well known that for $0 < q_0 \leq q_1 \leq \infty$,

$$\Lambda^{p,q_0}(X) \subset \Lambda^{p,q_1}(X).$$

Moreover, if X is a Banach space, then

$$\Lambda^1(X) \subset X \subset \Lambda^{1,\infty}(X).$$

In fact, the spaces $\Lambda^1(X)$ and $\Lambda^{1,\infty}(X)$ are respectively the smallest and largest r.i. spaces with fundamental function equal to φ_X .

Let X be an r.i. space on \mathbb{R}^n . Maz'ya's classical method shows that

$$\|f\|_X \lesssim \|\nabla f\|_{L^1} \quad (f \in \text{Lip}_0(\mathbb{R}^n))$$

if and only if, for every Borel set A ,

$$\varphi_X(m_n(A)) \lesssim m_n^+(A),$$

where m_n^+ is Minkowski's perimeter (see [M11] or [EG]).

As shown in [MM2], the following self-improvement property follows for $f \in \text{Lip}_0(\mathbb{R}^n)$:

$$\|f\|_{\Lambda^{1,\infty}(X)} \lesssim \|\nabla f\|_{L^1} \Leftrightarrow \|f\|_X \lesssim \|\nabla f\|_{L^1} \Leftrightarrow \|f\|_{\Lambda^1(X)} \lesssim \|\nabla f\|_{L^1}.$$

This Sobolev self-improvement obtained in the case $q = 1$ is also extended to the case $q > 1$ as

$$\|f\|_{\Lambda^{q,\infty}(X)} \lesssim \|\nabla f\|_{L^q} \Leftrightarrow \|f\|_{\Lambda^{1,q}(X)} \lesssim \|\nabla f\|_{L^q}.$$

In particular, if X is q -convex, then the space

$$X_{(q)} = \{f : |f|^{1/q} \in X\}, \quad \|f\|_{X_{(q)}} = \||f|^{1/q}\|_X^q$$

is an r.i. space, and

$$\Lambda^{1,q}(X) = \Lambda^q(X_{(q)}) \subset X_{(q)} \subset \Lambda^{1,\infty}(X_{(q)}).$$

In summary, in terms of the $X_{(q)}$ scale of spaces, on Lipschitz functions we have the following equivalences (see [MMP]):

$$\|f\|_{\Lambda^{1,\infty}(X_{(q)})} \lesssim \|\nabla f\|_{L^q} \Leftrightarrow \|f\|_{\Lambda^q(X_{(q)})} \lesssim \|\nabla f\|_{L^q} \Leftrightarrow \|f\|_{X_{(q)}} \lesssim \|\nabla f\|_{L^q}.$$

In this section we shall extend this result to the setting of quasi-Banach r.i. spaces. As an application of Theorem 3.2, we characterize Sobolev-type estimates in terms of isocapacitary inequalities.

From now on, Ω will be a domain in \mathbb{R}^n , X a quasi-Banach function space on Ω , μ a Borel measure on Ω , and Y an r.i. space on (Ω, μ) . The notation $\mathcal{g} \Subset G$ means that \mathcal{g} is an open set whose closure is a compact subset of the open set G .

An isocapacitary inequality is an inequality of the form $\text{Cap}_X(K) \geq J(\mu(K))$, where J is a nonnegative function and K is any compact set in Ω .

Proposition 4.1. *If*

$$\sup \frac{\varphi_Y(\mu(\mathcal{g}))}{\text{Cap}_X(\bar{\mathcal{g}}, G)} < \infty,$$

the supremum being taken over all sets \mathcal{g}, G such that $\mathcal{g} \Subset G \Subset \mathbb{R}^n$, then for every compact subset K in Ω ,

$$\varphi_Y(\mu(K)) \lesssim \text{Cap}_X(K).$$

Proof. Let K be a compact subset in Ω and $d := d(K, \Omega^c) > 0$. Denote $\lambda_n = 1/n$, and consider the smallest $n \in \mathbb{N}$, n^* , such that $1/n^* \leq d$. For each $n \geq n^*$, let

$$\mathcal{G}(\lambda_n) := \{x \in \Omega : d(K, x) < \lambda_n\}, \quad \mathcal{K}(\lambda_n) := \{x \in \Omega : d(K, x) \leq \lambda_n\}.$$

Then $\bigcap_{n \geq n^*} \overline{\mathcal{G}(\lambda_n)} = K$ and $\bigcap_{n \geq n^*} \mathcal{K}(\lambda_n) = K$. Since

$$\varphi_Y(\mu(\mathcal{G}(\lambda_k))) \lesssim \text{Cap}_X(\overline{\mathcal{G}(\lambda_k)}) \quad (k \geq n^*),$$

by the properties of Cap_X ,

$$\varphi_Y(\mu(K)) \leq \lim_{k \rightarrow \infty} \varphi_Y(\mu(\mathcal{G}(\lambda_k))) \lesssim \lim_{k \rightarrow \infty} \text{Cap}_X(\overline{\mathcal{G}(\lambda_k)}) = \text{Cap}_X(K),$$

and the result follows. □

Theorem 4.2. *Let $0 < p < \infty$, and assume that X satisfies a lower p -estimate. Then the following properties are equivalent:*

- (i) $\varphi_Y(\mu(K)) \lesssim \text{Cap}_X(K)$ for every compact set K on Ω .
- (ii) $\|f\|_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|_X$ ($f \in \text{Lip}_0(\Omega)$).
- (iii) $\|f\|_{\Lambda^{1,\infty}(Y)} \lesssim \|\nabla f\|_X$ ($f \in \text{Lip}_0(\Omega)$).

Moreover, for $q \geq p$, if Y is q -convex, or if Y satisfies an upper q -estimate and $\varphi_Y(t)/t^{1/p}$ is quasi-increasing, then, for every $f \in \text{Lip}_0(\Omega)$,

$$(4.1) \quad \|f\|_{\Lambda^{1,\infty}(Y)} \lesssim \|\nabla f\|_X \Leftrightarrow \|f\|_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|_X \Leftrightarrow \|f\|_Y \lesssim \|\nabla f\|_X.$$

Proof. (i) \Rightarrow (ii) If $a > 1$, by Theorem 3.2 we have

$$\begin{aligned} \|f\|_{\Lambda^{1,p}(Y)} &\leq \left(\int_0^\infty t^{p-1} \varphi_Y(\mu(\overline{M}_t))^p dt \right)^{1/p} \lesssim \left(\int_0^\infty t^{p-1} \text{Cap}_X(\overline{M}_t)^p dt \right)^{1/p} \\ &\leq a \left(\int_0^\infty s^{p-1} \text{Cap}_X(\overline{M}_{as}, M_s)^p ds \right)^{1/p} \lesssim \|\nabla f\|_X. \end{aligned}$$

(ii) \Rightarrow (iii) Observe that $\Lambda^{1,p}(Y) \subset \Lambda^{1,\infty}(Y)$.

(iii) \Rightarrow (i) This is trivial with use of Proposition 4.1.

To prove (4.1), if Y is q -convex, then $\varphi_Y(t)^q/t$ is quasi-decreasing and

$$\|f\|_Y = \| |f|^q \|_{Y(q)}^{1/q} \leq \| |f|^q \|_{\Lambda^q(Y(q))}^{1/q} = \| |f^q| \|_{\Lambda^{1,q}(Y)}^{1/q}.$$

Then (4.1) follows.

If Y satisfies a upper q -estimate and $\varphi_Y(t)/t^{1/p}$ is quasi-increasing, then it also satisfies a upper p -estimate, and then, for every simple function $s = \sum_i a_i \chi_{A_i}$ with $A_i \cap A_j = \emptyset$ if $i \neq j$, we obtain

$$\begin{aligned} \|s\|_Y &= \left\| \sum_i a_i \chi_{A_i} \right\|_Y = \left\| \left(\sum_i a_i^p \chi_{A_i} \right)^{1/p} \right\|_Y \\ &\leq M^{(p)}(X) \left(\sum_i \|a_i \chi_{A_i}\|_Y^p \right)^{1/p} = M^{(p)}(X) \left(\sum_i |a_i|^p \varphi_Y(\mu(A_i))^p \right)^{1/p}. \end{aligned}$$

Since φ_Y is also the fundamental function of $\Lambda^{1,p}(Y)$ and $\varphi_Y(t)/t^{1/p}$ is quasi-increasing, we know that $\Lambda^{1,p}(Y)$ satisfies a lower p -estimate (see [KM, Theorem 8]). Hence

$$\begin{aligned} \left(\sum_i a_i^p \varphi_Y(\mu(A_i))^p \right)^{1/p} &= \left(\sum_i \|a_i \chi_{A_i}\|_{\Lambda^{1,p}(Y)}^p \right)^{1/p} \\ &\leq \left\| \left(\sum_i a_i^p \chi_{A_i} \right)^{1/p} \right\|_{\Lambda^{1,p}(Y)} = \|s\|_{\Lambda^{1,p}(Y)}. \end{aligned}$$

Then, by the Fatou property, for every positive function f we have

$$\|f\|_Y \lesssim \|f\|_{\Lambda^{1,p}(Y)}.$$

Therefore, if $\|f\|_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|_X$, then $\|f\|_Y \lesssim \|\nabla f\|_X$. Conversely, if $\|f\|_Y \lesssim \|\nabla f\|_X$, then, since $Y \hookrightarrow \Lambda^{1,\infty}(Y)$, it follows that $\|f\|_{\Lambda^{1,\infty}(Y)} \lesssim \|\nabla f\|_X$, and we conclude that $\|f\|_{\Lambda^{1,p}(Y)} \lesssim \|\nabla f\|_X$. \square

Remark 4.3. Theorem 4.2 is to be compared with the results in Section 1 of [MM2].

Let us consider some examples: It is well known that the Gagliardo-Nirenberg inequality

$$\|f\|_{L^{n/(n-1)}} \lesssim \|\nabla f\|_{L^1} \quad (f \in \text{Lip}_0(\Omega))$$

allows us to see that if $p \in (1, n)$, $s = np/(n - p)$ and $\alpha = (n - 1)s/n$, since $\|f\|_{L^s}^\alpha = \| |f|^\alpha \|_{L^{n/(n-1)}}^{n/(n-1)}$, then

$$\|f\|_{L^s}^{s(n-1)/n} \lesssim \|\alpha |f|^{\alpha-1} |\nabla f|\|_{L^1} \lesssim \|f\|_{L^s}^{s/p'} \|\nabla f\|_{L^p},$$

where p' is the conjugate exponent of p . Hence $\|f\|_{L^s} \lesssim \|\nabla f\|_{L^p}$. Therefore, since $L^s \hookrightarrow L^{s,\infty}$, it follows that

$$\|f\|_{L^{s,\infty}} \lesssim \|\nabla f\|_{L^p}.$$

But $\|f\|_{\Lambda^{1,\infty}(L^s)} = \|f\|_{L^{s,\infty}} \lesssim \|\nabla f\|_{L^p}$, and then, since L^p satisfies a lower p -estimate, from Theorem 4.2, we conclude that

$$\|f\|_{L^{s,p}} = \|f\|_{\Lambda^{1,p}(L^s)} \lesssim \|\nabla f\|_{L^p} \quad (f \in \text{Lip}_0(\Omega)),$$

and we have obtained a self-improvement.

If $p = n$, then we start from the Trudinger inequality,

$$\left(\frac{\int_0^t f^*(s)^{n/(n-1)} ds}{t \left(1 + \log \frac{1}{t}\right)} \right)^{(n-1)/n} \lesssim \|\nabla f\|_{L^n},$$

which gives the estimate

$$\varphi(\mu(K)) = \left(1 + \log \frac{1}{\mu(K)} \right)^{(1-n)/n} \leq \text{Cap}_{L^n}(K),$$

and then

$$\|f\|_{\Lambda^{1,n}(\varphi)} \lesssim \|\nabla f\|_{L^n}.$$

But,

$$\Lambda^{1,n}(\varphi) = \left(\int_0^\infty t^{n-1} (\varphi(\mu_f(t)))^n dt \right)^{1/n} = \left(\int_0^1 \left(\frac{f^*(s)}{(1 + \log(1/s))} \right)^n \frac{ds}{s} \right)^{1/n}.$$

If $r \leq s < p$, then $L^{s,r}$ satisfies an upper p -estimate and $\varphi_{L^{s,r}}(t)/t^{1/p}$ is quasi-increasing, so that, since $\|f\|_{L^{s,\infty}} = \|f\|_{\Lambda^{1,\infty}(L^s)} \lesssim \|\nabla f\|_{L^p}$,

$$\|f\|_{L^{s,\infty}} \simeq \|f\|_{\Lambda^{1,\infty}(L^{s,r})} \lesssim \|\nabla f\|_{L^p} \lesssim \|\nabla f\|_{L^{p,q}} \quad (q \leq p),$$

and then $\|f\|_{\Lambda^{1,p}(L^{s,r})} \lesssim \|\nabla f\|_{L^p}$. Therefore, if $q \leq p$, then we obtain the self-improvement

$$\|f\|_{L^{s,p}} \simeq \|f\|_{\Lambda^{1,p}(L^{s,r})} \lesssim \|\nabla f\|_{L^{p,q}} \quad (f \in \text{Lip}_0(\Omega)).$$

5. SOBOLEV-POINCARÉ ESTIMATES FOR TWO MEASURE SPACES

In [CoMa], characterizations for Sobolev-Lorentz-type inequalities involving two measures are proved, extending results obtained in [M05] and [M06]. Here, we extend those results and derive, with similar methods, necessary and sufficient conditions for such Sobolev-type inequalities involving two rearrangement invariant spaces subjected to appropriate convexity conditions.

From now on, μ and ν are two Borel measures on Ω , and $0 < p < \infty$. Let X be a quasi-Banach function space on Ω , Y be an r.i. space on (Ω, μ) , and Z be an r.i. space on (Ω, ν) .

Theorem 5.1. *Suppose that X satisfies a lower p -estimate. Then, the following properties are equivalent:*

(i) *There is a constant $A > 0$ such that*

$$\|f\|_{\Lambda^{1,p}(Y)} \leq A(\|\nabla f\|_X + \|f\|_{\Lambda^{1,p}(Z)}) \quad (f \in \text{Lip}_0(\Omega)).$$

(ii) *There exists a constant $B > 0$ such that*

$$\varphi_Y(\mu(g)) \leq B(\text{Cap}_X(\bar{g}, G) + \varphi_Z(\nu(G))) \quad (g \in G \in \Omega).$$

Proof. (i) \Rightarrow (ii) Choose $g \in G \in \Omega$, and consider $f \in W(\bar{g}, G)$ arbitrary. Since $g \subset \{f \geq 1\}$, it follows that

$$\begin{aligned} \varphi_Y(\mu(g))^p &\leq \int_0^1 \varphi_Y(\mu(\{f \geq t\}))^p dt^p \leq \int_0^1 \varphi_Y(\mu(\{f > t\}))^p dt^p \\ &\leq p\|f\|_{\Lambda^{1,p}(Y)}^p \lesssim \|\nabla f\|_X^p + \|f\|_{\Lambda^{1,p}(Z)}^p, \end{aligned}$$

with

$$\|f\|_{\Lambda^{1,p}(Z)}^p \leq \int_0^1 t^{p-1} \varphi_Z(\nu(G))^p dt = \frac{1}{p} \varphi_Z(\nu(G))^p,$$

and (ii) follows by taking infimum over all functions $f \in W(\bar{g}, G)$.

(ii) \Rightarrow (i) From $M_t = \{|f| > t\} \subset \text{supp}(f)$ and $M_{at} \subset M_t$ if $a > 1$, $\varphi_Y(\mu(M_{at}))^p \lesssim \text{Cap}_X(\bar{M}_{at}, M_t)^p + \varphi_Z(\nu(M_t))^p$, and Theorem 3.2 yields

$$\begin{aligned} \|f\|_{\Lambda^{1,p}(Y)} &= \left(\int_0^\infty a^{p-1} s^{p-1} \varphi_Y(\mu(M_{as}))^p a \, ds \right)^{1/p} \\ &\lesssim a \left\{ \left(\int_0^\infty s^{p-1} \text{Cap}_X(\bar{M}_{as}, M_s)^p \, ds \right)^{1/p} \right. \\ &\quad \left. + \left(\int_0^\infty s^{p-1} \varphi_Z(\nu(M_s))^p \, ds \right)^{1/p} \right\} \\ &\lesssim \|\nabla f\|_X + \|f\|_{\Lambda^{1,p}(Z)}. \end{aligned} \quad \square$$

6. EXTENSION TO CAPACITARY FUNCTION SPACES

Let us now extend our results to the capacitary function spaces setting considered in [Ce], [CMS] and [CMS1].

By a capacity C on a measurable space (Ω, Σ) we mean a set function defined on Σ satisfying at least the following properties:

- (a) $C(\emptyset) = 0$,
- (b) $0 \leq C(A) \leq \infty$,
- (c) $C(A) \leq C(B)$ if $A \subset B$, and
- (d) (Quasi-subadditivity) $C(A \cup B) \leq c(C(A) + C(B))$, where $c \geq 1$ is a constant.

Then the Lorentz spaces $L^{p,q}(C)$ are defined by the condition

$$\|f\|_{L^{p,q}(C)} := \left(q \int_0^\infty t^{q-1} C\{|f| > t\}^{q/p} \, dt \right)^{1/q} < \infty.$$

With this notation, Theorem 3.2 states that if X satisfies a lower p -estimate, then

$$\|f\|_{L^{1,p}(\text{Cap}_X)} \lesssim \|\nabla f\|_X \quad (f \in \text{Lip}_0(\Omega)).$$

Let us denote by $C^{(p)} := C^{1/p}$ the p -convexification of C (see [Ce]).

Theorem 6.1. *Suppose $0 < p, s, q < \infty$, and let C and \tilde{C} be two capacities on (Ω, Σ) . If X satisfies a lower q -estimate, then the following properties are equivalent:*

- (i) $\|f\|_{L^{p,q}(C)} \lesssim \|\nabla f\|_X + \|f\|_{L^{s,q}(\tilde{C})}$ for every $f \in \text{Lip}_0(\Omega)$.
- (ii) $C^{(p)}(g) \lesssim \text{Cap}_X(\bar{g}, G) + \tilde{C}^{(s)}(G)$ for all sets g and G such that $g \in G \in \Omega$.

Proof. (i) \Rightarrow (ii) Choose $g \in G \in \Omega$ and any $f \in W(\bar{g}, G)$. Then $\|f\|_{L^{p,q}(C)} \lesssim \|\nabla f\|_X + \|f\|_{L^{s,p}(\tilde{C})}$, so that

$$C^{(p)}(g) \leq \left(\int_0^1 C^{(p)}(\{f > t\})^q \, dt^q \right)^{1/q} \lesssim \|\nabla f\|_X + \|f\|_{L^{s,p}(\tilde{C})},$$

and $\|f\|_{L^{s,p}(\tilde{C})} = (\int_0^1 \tilde{C}\{|f| > s\}^{p/s} ds)^{1/p} \leq \tilde{C}^{(s)}(G)$. Taking the infimum over all $f \in W(\tilde{g}, G)$, we conclude that

$$C^{(p)}(g) \lesssim \text{Cap}_X(\tilde{g}, G) + \tilde{C}^{(s)}(G).$$

(ii) \Rightarrow (i) Consider $f \in \text{Lip}_0(\Omega)$, and take for $a > 1$ and $t > 0$ the open sets, $g := M_{at}$ and $G := M_t$. By hypothesis we have $C^{(p)}(M_{at}) \lesssim \text{Cap}_X(\bar{M}_{at}, M_t) + \tilde{C}^{(s)}(M_t)$, and then, by Theorem 3.2,

$$\begin{aligned} \|f\|_{L^{p,q}(C)} &\lesssim \left(\int_0^\infty s^{q-1} \text{Cap}_X(\bar{M}_{as}, M_s)^q ds \right)^{1/q} \\ &\quad + \left(\int_0^\infty s^{q-1} \tilde{C}^{(s)}(M_s)^q ds \right)^{1/q} \\ &\leq \|\nabla f\|_X + \|f\|_{L^{s,q}(\tilde{C})}. \end{aligned} \quad \square$$

In a similar way, we obtain the following result.

Theorem 6.2. *Let $0 < p, q < \infty$. Suppose that X satisfies a lower q -estimate, and let C be a capacity on (Ω, Σ) . The following properties are equivalent:*

- (i) $\|f\|_{L^{p,q}(C)} \lesssim \|\nabla f\|_X$ for every $f \in \text{Lip}_0(\Omega)$.
- (ii) $C^{(p)}(g) \lesssim \text{Cap}_X(\tilde{g}, G)$ if $g \in G \in \Omega$.

We say that the capacity C is Fatou if $C(A_n) \rightarrow C(A)$ whenever $A_n \uparrow A$ and that it is concave if

$$C(A \cup B) + C(A \cap B) \leq C(A) + C(B) \quad (A, B \in \Sigma).$$

The capacity C is said to be μ -invariant, where μ is a measure on (Ω, Σ) , if $C(A) = C(B)$ whenever $\mu(A) = \mu(B)$, and it is said to be quasi-concave with respect to μ if there exists a constant $\gamma \geq 1$ such that, whenever $\mu(A) \leq \mu(B)$, the following two conditions are satisfied:

- (a) $C(A) \leq \gamma C(B)$, and
- (b) $C(B)/\mu(B) \leq \gamma C(A)/\mu(A)$.

Assume that X is an r.i. quasi-Banach space. Now, if we define $C(A) := \varphi_X(m_n(A))$, then $L^p(C) = \Lambda^p(X)$ is a Banach space.

Indeed, since φ_X is continuous except possibly at the origin, C is a Fatou capacity on (Ω, m_n) and $L^p(C)$ is complete. Moreover, C is m_n -invariant and quasi-concave with respect to m_n , and, by [CMS, Theorem 8], there exists a Fatou concave capacity C_1 which is equivalent to C . For such a capacity, $L^p(C_1)$ is a normed space.

Acknowledgments. We thank Mario Milman for having informed us about the article [CoMa]. This research was partially supported by grant MTM2010-14946.

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KEY WORDS AND PHRASES: convexity, lower estimates, Sobolev spaces, rearrangement invariant spaces, Sobolev-type inequalities.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 46E30 (28A12).

Received: May 16, 2011.