Editorial Manager(tm) for TOP
Manuscript Draft

Manuscript Number: TOPR-D-10-00116R2

Title: ON THE INFIMUM OF A QUASICONVEX VECTOR FUNCTION OVER AN INTERSECTION

Article Type: Special Issue: OVA4MLOPEZ

Keywords: Quasiconvex functions

Corresponding Author: JUAN ENRIQUE MARTÍNEZ-LEGAZ

Corresponding Author's Institution: Universidad Autónoma de Barcelona

First Author: JUAN ENRIQUE MARTÍNEZ-LEGAZ

Order of Authors: JUAN ENRIQUE MARTÍNEZ-LEGAZ;ANTONIO MARTINÓN

Manuscript Region of Origin: SPAIN

ON THE INFIMUM OF A QUASICONVEX VECTOR FUNCTION OVER AN INTERSECTION

JUAN-ENRIQUE MARTÍNEZ-LEGAZ AND ANTONIO MARTINÓN

Abstract. We give sufficient conditions for the infimum of a quasiconvex vector function f over an intersection $\bigcap_{i\in I} R_i$ to agree with the supremum of the infima of f over the R_i's.

1. Introduction

Let S be a set, $(R_i)_{i\in I}$ be a family of subsets of S, $C = \bigcup_{i\in I} R_i$ its union, $R = \bigcap_{i\in I} R_i$ its intersection and E be a complete ordered space. Given a map $f : C \longrightarrow E$, we are interested in finding conditions which ensure the following equality:

$$\inf_{x\in R} f(x) = \sup_{i\in I} \inf_{x\in R_i} f(x) . \tag{1.1}$$

Of course, the inequality $\geq$ is always satisfied.

In [13] we have considered several settings. Our main results about equality (1.1) were obtained in the case when $S = V$ is a linear space, C is a convex subset of V, $E = \overline{\mathbf{R}}$, f is a quasiconvex function and the family $(R_i)_{i\in I}$ satisfies certain conditions, basically $R_i \cap R_j = R$ $(i \neq j)$. These results were applied to the case when $S = X$ is a normed space, $C \subset X$ is a convex subset, $x \in X$ is a given point, and f is the distance function $d(x, \cdot) : C \longrightarrow \mathbf{R}$, which is convex, hence quasiconvex, and we obtained the equality

$$d(x, R) = \sup_{i\in I} d(x, R_i) . \tag{1.2}$$

Equality (1.2) has been studied by many authors in different situations [3, 4, 6, 12, 14, 15, 17, 18, 19].

In this paper we obtain analogous results to those of [13] for the case when f is an order bounded quasiconvex vector function taking values on an order complete Riesz space E. We observe some differences between the scalar and vector cases. Indeed, under certain conditions, the equality

$$\inf_{x\in R} f(x) = \inf_{x\in R_i} f(x) \tag{1.3}$$

holds for every $i \in I$ except at most one if f is scalar; however, under similar requirements, it is possible to have

$$\inf_{x\in R} f(x) \neq \inf_{x\in R_i} f(x)$$

2000 *Mathematics Subject Classification.* 06F20, 52A41, 54E99.

Key words and phrases. Quasiconvex functions, distance to the intersection.

Dedicated to Marco Antonio López Cerdà on the occasion of his 60th birthday.

The first author acknowledges the support of the MICINN of Spain, Grant MTM2008-06695-C03-03, of the Barcelona Graduate School of Economics and of the Government of Catalonia. He is affiliated to MOVE (Markets, Organizations and Votes in Economics).

for every $i \in I$ if f is a vector function, although (1.1) is still true. If $E \subset \mathbf{R}^D$ is a complete function space, for example if the positive cone of E is a Yudin cone (see definitions in Section 5), it is possible to obtain better results: the equality (1.3) holds for any $i \in I \setminus I_D$, where the cardinality of I_D is less than or equal to the cardinality of D.

These general results are applied to the vector distance (or cone distance) derived from a vector norm (or cone norm) and to obtain the validity of the equality (1.2) for vector distances. That is, in the vector case we prove that, under some requirements about the sets R_i, the distance from a point x to the intersection $\bigcap_{i\in I} R_i$ coincides with the supremum of the distances from x to each R_i. In [9] vector normed spaces, also called Kantorovich spaces, and their relation with certain optimization problems are considered.

2. Quasiconvex vector functions

A nonempty subset K of a (real) vector space E is said to be a *pointed convex cone* if it satisfies the following properties [2, Definition 1.2]: (1) $K + K \subset K$; (2) $\alpha K \subset K$, for all scalars $\alpha \geq 0$; (3) $K \cap (-K) = \{0\}$. In this case, K defines an order $x \leq y \iff y - x \in K$ on E which is compatible with its vector structure and we say that E is an ordered vector space. If every pair $x, y \in E$ has a maximum and a minimum we say that E is a Riesz space.

Throughout this paper E denotes an order complete Riesz space; that is, E is a Riesz space such that any order bounded subset A of E has a supremum $\sup A$ and an infimum $\inf A$. Recall that the infinite distributive laws are valid ([2, Exercise 1.3.11] [11, Theorem 2.12.2]): given $a \in E$ and a family $(a_i)_{i\in I}$ of vectors of E, we have

$$\inf\{a, \sup_{i\in I} a_i\} = \sup_{i\in I} \inf\{a, a_i\} \quad , \quad \sup\{a, \inf_{i\in I} a_i\} = \inf_{i\in I} \sup\{a, a_i\} \ . \tag{2.4}$$

The next well-known notion is basic in this paper:

Let V be a linear space, $C \subset V$ a convex set and E be a Riesz space. The function $f : C \longrightarrow E$ is called quasiconvex if, for every $x, y \in C$ and $0 \leq t \leq 1$, we obtain

$$f(tx + (1-t)y) \leq \sup\{f(x), f(y)\} \ ;$$

equivalently, for every $e \in E$, the set $\{x \in C : f(x) \leq e\}$ is convex ([10, Prop. 5.9]).

Several authors (see, for example, [5]) introduce the following definition: given linear spaces V and W, a convex cone $K \neq \{0\}$ in W and a nonempty subset C of V, the map $f : C \longrightarrow W$ is called *quasiconvex* if, for any $b \in W$, the set $\{a \in C : f(a) \in b - K\}$ is a convex set. If K is a pointed convex cone, hence W is an ordered vector space, and if moreover W is a Riesz space and C is convex, then both definitions coincide.

Remark 2.1. *Assume that V is a linear space, $C \subset V$ a convex set, E a Riesz space and $f : C \longrightarrow E$ a map. Consider the following assertions:*

A. For every $e \in E$, the set $M_e = \{x \in C : f(x) < e\}$ is convex.
B. For every $e \in E$, the set $N_e = \{x \in C : f(x) \leq e\}$ is convex.

If $E = \mathbf{R}$, then it is well known that $A \Longleftrightarrow B$. In the general complete Riesz space case we have $A \Longrightarrow B$. In fact, for every $e \in E$, the equality

$$N_e = \bigcap_{g>e} M_g$$

holds. Indeed: for every $g > e$,

$$N_e = \{x \in C : f(x) \le e\} \subset \{x \in C : f(x) < g\} = M_g \ ,$$

hence $N_e \subset \bigcap_{g>e} M_g$; on the other hand, if $x \in \bigcap_{g>e} M_g$, then, for every $g > e$, we have $f(x) < g$, so $f(x)$ is a lower bound of $\{g : g > e\}$, hence $f(x) \le \inf\{g : g > e\} = e$. As the sets M_g are convex, we obtain that N_e is convex.

The implication $B \Longrightarrow A$ is not valid, as shown in the next example.

Example 2.2. *Let $V = \mathbf{R}$, $C = [0, 1] \subset V$, $E = \mathbf{R}^2$ with the usual order. Let $f : [0, 1] \longrightarrow \mathbf{R}^2$, $f = (f_1, f_2)$, defined by*

$$f_1(x) = \begin{cases} -4x + 1 & \text{if } 0 \le x < \frac{1}{4} \\ 4x - 1 & \text{if } \frac{1}{4} \le x < \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

$$f_2(x) = \begin{cases} 1 & \text{if } 0 \le x < \frac{1}{2} \\ -4x + 3 & \text{if } \frac{1}{2} \le x < \frac{3}{4} \\ 4x - 3 & \text{if } \frac{3}{4} \le x \le 1 \end{cases}$$

As f_1 and f_2 are quasiconvex, we have that f is quasiconvex. Moreover, the set

$$M_{(1,1)} = \{x \in [0, 1] : f(x) < (1, 1)\}$$

$$= \{x \in [0, 1] : f_1(x) < 1 \text{ or } f_2(x) < 1\} =]0, 1/2[\cup]1/2, 1[$$

is not convex.

3. Separation

Let V be a vector space. Denote by $[x, y]$ the segment with endpoints x and y, that is, $[x, y] := \{tx + (1 - t)y : 0 \le t \le 1\}$.

For nonempty subsets R_1, R_2, R of V, we say that R *separates* the sets R_1 and R_2 if $[r_1, r_2] \cap R \ne \emptyset$, for every $r_1 \in R_1$ and $r_2 \in R_2$ [16, page 489].

We generalize the above notion in the following way: for a family $(R_i)_{i\in I}$ of nonempty subsets of V containing at least two members, we say that a subset R of V *separates* the family $(R_i)_{i\in I}$ if R separates the sets R_i and R_j for every $i, j \in I$, $i \ne j$. Notice that in this situation $\bigcap_{i\in I} R_i \subset R$.

In [16, Proposition 2.3] it is proved that if A, B are nonempty convex algebraic closed subsets of a vector space V, the set $A \cap B$ separates the sets A and B if and only if the union $A \cup B$ is a convex set. Observing that the proof of the "only if" part of that statement does not use the algebraic closedness assumption, one easily gets the following result: given a family $(R_i)_{i\in I}$ of nonempty convex subsets of a vector space V such that $\bigcap_{i\in I} R_i$ separates $(R_i)_{i\in I}$, we have that $R_j \cup R_k$ is convex for every $j, k \in I$.

Similarly to [16, Proposition 2.3], the following result characterizes convexity in terms of separation. We leave the simple proof to the reader.

Proposition 3.1. *Let C be a nonempty subset of a vector space V. The following assertions are equivalent:*

(1) C is convex.

(2) There exists a family $(R_i)_{i\in I}$ of nonempty convex subsets of V containing at least two members such that $\bigcup_{i\in I} R_i = C$ and $\bigcap_{i\in I} R_i$ separates $(R_i)_{i\in I}$.

(3) There exists a family $(R_i)_{i\in I}$ of nonempty convex subsets of V containing at least two members such that $\bigcup_{i\in I} R_i = C$ and $R_j \cup R_k$ is convex for every $j, k \in I$.

4. The infimum over an intersection

In this section we give the main results of this paper: under certain conditions we obtain that equality (1.1) holds.

Proposition 4.1. *Let V be a vector space, $C \subset V$ be a convex set, $(R_i)_{i\in I}$ be a family of nonempty subsets of C containing at least two members, $R := \bigcap_{i\in I} R_i$, E be an order complete Riesz space, and $f : C \longrightarrow E$ be an order bounded quasiconvex function. If R separates $(R_i)_{i\in I}$ then, for $i, j \in I$ with $i \neq j$,*

$$\inf_{x\in R} f(x) = \sup\{\inf_{x\in R_i} f(x), \inf_{x\in R_j} f(x)\} . \tag{4.5}$$

Hence equality (1.1) holds.

Proof. Let $i, j \in I$ with $i \neq j$, and let $r_i \in R_i$ and $r_j \in R_j$. There exists $z \in [r_i, r_j] \cap R$. As f is quasiconvex we obtain

$$f(z) \leq \sup\{f(r_i), f(r_j)\} ;$$

that is, for all $r_i \in R_i$ and $r_j \in R_j$,

$$\inf_{x\in R} f(x) \leq \sup\{f(r_i), f(r_j)\} .$$

Fix $r_i \in R_i$, apply the infinite distributive laws (2.4) and obtain

$$\inf_{x\in R} f(x) \leq \inf_{x\in R_j} \sup\{f(r_i), f(x)\} = \sup\{f(r_i), \inf_{x\in R_j} f(x)\} .$$

Also

$$\inf_{x\in R} f(x) \leq \inf_{x\in R_i} \sup\{f(x), \inf_{x\in R_j} f(x)\} = \sup\{\inf_{x\in R_i} f(x), \inf_{x\in R_j} f(x)\} .$$

As $R \subset R_i$, we have $\inf_{x\in R_i} f(x) \leq \inf_{x\in R} f(x)$ for all $i \in I$, from which we obtain the converse inequality; so (4.5) is proved. Equality (1.1) follows from (4.5) and the assumption that the index set I has at least two elements. □

The following example shows that, even in the case when the index set I is finite, under the assumptions of the preceding proposition one may have $\inf_{x\in R} f(x) \neq \inf_{x\in R_i} f(x)$ for every $i \in I$.

Example 4.2. *Let L denote the vector subspace of $\mathbf{R}^{[0,1]}$ consisting of all continuous functions of bounded total variation, and consider the cone*

$$K = \{f \in L : f \geq 0, f \textit{ is increasing}\};$$

then L is order complete [2, Problem 1.3.20]*. We write $f \leq g$ if $f(x) \leq g(x)$ for any $x \in [0,1]$. Notice that the order $\preceq$ associated to K is defined by*

$$f \preceq g \Longleftrightarrow f \leq g \textit{ and } g - f \textit{ is increasing.}$$

We take the functions $a, b, c \in L$ defined in the following way:

$$a(x) = \begin{cases} 2x & \textit{if} \quad 0 \leq x < \frac{1}{2} \\ -2x + 2 & \textit{if} \quad \frac{1}{2} \leq x \leq 1, \end{cases}$$

$$b(x) = \begin{cases} 2x - 1 & \textit{if} \quad 0 \leq x < \frac{1}{2} \\ 0 & \textit{if} \quad \frac{1}{2} \leq x \leq 1, \end{cases}$$

$$c(x) = \begin{cases} -2x & \textit{if} \quad 0 \leq x < \frac{1}{2} \\ 2x - 2 & \textit{if} \quad \frac{1}{2} \leq x \leq 1. \end{cases}$$

Let $f : [0,1] \longrightarrow L$ be defined by

$$f(t) = \begin{cases} a & \textit{if} \quad 0 \leq t < \frac{1}{2} \\ b & \textit{if} \quad t = \frac{1}{2} \\ c & \textit{if} \quad \frac{1}{2} < t \leq 1. \end{cases}$$

The function f is order bounded and quasiconvex, because from $\sup\{a, c\}(x) = 2x$ for all $x \in [0,1]$ it follows that $b \preceq \sup\{a, c\}$. Consider the sets $R_1 = [0, 1/2]$ and $R_2 = [1/2, 1]$, so $R = R_1 \cap R_2 = \{1/2\}$ and R separates R_1 and R_2. We obtain

$$\inf_{t \in R_1} f(t)(x) = \inf\{a, b\}(x) = \begin{cases} 2x - 1 & \textit{if} \quad 0 \leq x < \frac{1}{2} \\ -2x + 1 & \textit{if} \quad \frac{1}{2} \leq x \leq 1 \end{cases}$$

and

$$\inf_{t \in R_2} f(t)(x) = \inf\{b, c\}(x) = \begin{cases} -2x - 1 & \textit{if} \quad 0 \leq x < \frac{1}{2} \\ -2 & \textit{if} \quad \frac{1}{2} \leq x \leq 1. \end{cases}$$

One has

$$\sup\{\inf_{t \in R_1} f(t), \inf_{t \in R_2} f(t)\} = \sup\{\inf\{a, b\}, \inf\{b, c\}\} = \inf\{b, \sup\{a, c\}\} = b = \inf_{t \in R} f(t),$$

but

$$\inf_{t \in R} f(t) \neq \inf_{t \in R_1} f(t) \quad \textit{and} \quad \inf_{t \in R} f(t) \neq \inf_{t \in R_2} f(t).$$

The next proposition deals with sequences of linearly closed sets. We recall that a subset S of a set C in a vector space V is said to be linearly closed in C if the intersection of S with every straight line L is closed in $C \cap L$, this latter set being endowed with its natural topology as a subset of the straight line L.

Proposition 4.3. *Let V be a vector space, $C \subset V$ be a convex set, $R \subset C$, $(R_i)_{i\in\mathbf{N}}$ be a sequence of nonempty subsets of C consisting of linearly closed sets in C such that $\bigcup_{i\in\mathbf{N}} R_i = C$ and $R_i \cap R_j = R$, for $i, j \in \mathbf{N}$ with $i \neq j$, E be an order complete Riesz space, and $f : C \longrightarrow E$ be an order bounded quasiconvex function. Then $R \neq \emptyset$ and, for $i, j \in \mathbf{N}$ with $i \neq j$,*

$$\inf_{x\in R} f(x) = \sup\{\inf_{x\in R_i} f(x), \inf_{x\in R_j} f(x)\} . \tag{4.6}$$

Hence equality (1.1) holds.

Proof. In view of Prop. 4.1, we only have to prove that if $r_h \in R_h$ and $r_k \in R_k$ with $h, k \in \mathbf{N}$ and $h \neq k$, then the segment $[r_h, r_k]$ satisfies $[r_h, r_k] \cap R \neq \emptyset$. For $i \in \mathbf{N}$ we denote $S_i := R_i \cap [r_h, r_k]$. As $[r_h, r_k] = \bigcup_{i\in\mathbf{N}} S_i$ is a continuum (compact, connected and Hausdorff topological space) and no continuum can be written as the union of countably many disjoint closed sets [21, Problem 28E.2], we have $S_i \cap S_j \neq \emptyset$ for some $i, j \in \mathbf{N}$ with $i \neq j$, hence we obtain $[r_h, r_k] \cap R = S_i \cap S_j \neq \emptyset$. □

Proposition 4.4. *Let V be a vector space, $C \subset V$ be a convex set, E be an order complete Riesz space, $f : C \longrightarrow E$ be an order bounded quasiconvex function, $R \subset C$, $R_1, R_2, ..., R_n \subset C$ $(n \geq 2)$ be nonempty and linearly closed in C satisfying $\bigcup_{i=1}^{n} R_i = C$ and such that, for a certain $h = 1, ..., n$ and for every $j = 1, ..., n$ with $j \neq h$, $R_h \cap R_j = R$ and*

$$\inf_{x\in R_h} f(x) \leq \inf_{x\in R_j} f(x) . \tag{4.7}$$

Then $R \neq \emptyset$ and

$$\forall j = 1, ..., n; j \neq h, \quad \inf_{x\in R} f(x) = \inf_{x\in R_j} f(x),$$

hence equality (1.1) holds.

Proof. Let $r \in R_h$, $s \in R_j$ $(1 \leq j \leq n,\ j \neq h)$ and, assuming that $r \neq s$, let L be the straight line that contains the segment $[r, s] \subset C$. Define

$$\overline{\lambda} := \sup\{\lambda \in [0, 1] : (1 - \lambda)r + \lambda s \in R_h\}$$

and $t := (1 - \overline{\lambda})r + \overline{\lambda}s$. We have $t \in R_h$. If $\overline{\lambda} = 1$, then $t = s \in [r, s] \cap R_h \cap R_j$. Let $S := (\bigcup_{k\neq h} R_k) \cap [r, s]$. If $\overline{\lambda} < 1$, for every $p = 1, 2, 3...$ there exists $y_p \in S$ such that $y_p := (1 - \lambda_p)r + \lambda_p s$, being $\overline{\lambda} < \lambda_p \leq \overline{\lambda} + 1/p$. As S is the finite union of the sets $R_k \cap [r, s]$, there is a subsequence $(y_{p_m})_{m\geq 1}$ of $(y_p)_{p\geq 1}$ contained in some $R_k \cap [r, s]$. Since $R_k \cap [r, s]$ is closed in $C \cap L$, it is clear that t is the limit of a sequence of points of $R_k \cap L$ and, consequently, $t \in R_k$; hence $t \in R_h \cap R_k = R = R_h \cap R_j$. Hence R separates the family $\{R_h, R_j\}$. Then apply Proposition 4.1 to this family $\{R_h, R_j\}$ to conclude that, for $j \neq h$,

$$\inf_{x\in R} f(x) = \sup\left\{\inf_{x\in R_h} f(x), \inf_{x\in R_j} f(x)\right\} = \inf_{x\in R_j} f(x) ,$$

due to (4.7). From this we obtain the result. □

5. Function spaces

If E is a coordinate space we can work with the components of the function f. More precisely, in [1, 1.1 Definition] the following definition is given: a set E of real functions on a nonempty set D is a *function space* if it is a vector subspace of $\mathbf{R}^D$ under pointwise addition and scalar multiplication and is closed under finite pointwise suprema and infima. Every function space in the sense of the above definition is a Riesz space under the pointwise ordering [1, 8.1 (8) Example]. We say that the function space E is a *complete function space* if E is an order complete Riesz space under the pointwise ordering.

Assume that E is a complete function space with $E \subset \mathbf{R}^D$. Let $(g_j)_{j\in J}$ be a family of functions in E. For every $d \in D$ we have

$$\left(\inf_{j\in J} g_j\right)(d) = \inf_{j\in J} g_j(d) \quad , \quad \left(\sup_{j\in J} g_j\right)(d) = \sup_{j\in J} g_j(d).$$

Consider now a convex subset C of a vector space V. Given a function $f : C \longrightarrow E$, we consider its components f_d $(d \in D)$:

$$f_d : C \longrightarrow \mathbf{R} \quad , \quad f_d(x) = f(x)(d) \ (x \in C, d \in D) \quad .$$

It is easy to prove that

$$f \text{ is quasiconvex } \Longleftrightarrow \forall d \in D, f_d \text{ is quasiconvex}$$

and

$$f \text{ is order bounded } \Longleftrightarrow \forall d \in D, f_d \text{ is bounded .}$$

We will next improve Proposition 4.1 in the case E is a complete function space. In order to stress the differences between the scalar and vector cases, we first recall the following scalar result:

Proposition 5.1. [13, Proposition 3.1] *Let V be a vector space, $C \subset V$ be a convex set, $(R_i)_{i\in I}$ be a family of nonempty subsets of C containing at least two members, $R := \bigcap_{i\in I} R_i$ and $f : C \longrightarrow \overline{\mathbb{R}} = [-\infty, \infty]$ be a quasiconvex function. If R separates $(R_i)_{i\in I}$, then*

$$\inf_{x\in R} f(x) = \inf_{x\in R_i} f(x) \tag{5.8}$$

for every $i \in I$, except perhaps only one i. Hence equality (1.1) holds.

Proposition 5.2. *Let V be a vector space, $C \subset V$ be a convex set, $(R_i)_{i\in I}$ be a family of nonempty subsets of C containing at least two members, $R = \bigcap_{i\in I} R_i$, E be a complete function space with $E \subset \mathbf{R}^D$, for some set $D \neq \emptyset$, and $f : C \longrightarrow E$ be an order bounded quasiconvex function. If R separates $(R_i)_{i\in I}$, then (1.1) holds. Moreover, for some set $I_D \subset I$ with cardinality less than or equal to the cardinality of D one has*

$$\inf_{x\in R} f(x) = \inf_{x\in R_k} f(x) \tag{5.9}$$

for every $k \in I \setminus I_D$.

Proof. Use Proposition 4.1 to conclude that (1.1) holds. Fix $d \in D$ and apply Proposition 5.1 to get $h_d \in I$ such that

$$\inf_{x\in R} f_d(x) = \inf_{x\in R_i} f_d(x) \ ,$$

for $i \in I \setminus \{h_d\}$. Put $I_D := \{h_d : d \in D\}$. Then

$$\inf_{x\in R} f_d(x) = \inf_{x\in R_i} f_d(x) \ ,$$

for every $d \in D$ and $i \in I \setminus I_D$. This shows (5.9). □

In the vector case it is not possible to guarantee equality (5.8) for every $i \in I$ except only one i, as we show in the next examples.

Example 5.3. *Let $V = \mathbf{R}$, $C = [-1, 1]$, $E = \mathbf{R}^2$ endowed with the usual (pointwise) order, $f : C \longrightarrow E$ defined by $f(x) = (x + 1, x^2 + 1)$, which is quasiconvex and order bounded. Let*

$$R_{-1} = \left[-1, -\frac{1}{3}\right] \bigcup \left\{\frac{1}{3}\right\} \quad , \quad R_0 = \left[-\frac{1}{3}, \frac{1}{3}\right] \quad , \quad R_1 = \left\{-\frac{1}{3}\right\} \bigcup \left[\frac{1}{3}, 1\right] \quad ,$$

hence

$$R = R_{-1} \cap R_0 \cap R_1 = \left\{-\frac{1}{3}, \frac{1}{3}\right\} \quad .$$

It is clear that R separates the family (R_{-1}, R_0, R_1). Then

$$\inf_{x\in R} f_1(x) = \frac{2}{3} = \inf_{x\in R_0} f_1(x) = \inf_{x\in R_1} f_1(x) > 0 = \inf_{x\in R_{-1}} f(x) \quad ,$$

$$\inf_{x\in R} f_2(x) = \frac{10}{9} = \inf_{x\in R_{-1}} f_2(x) = \inf_{x\in R_1} f_2(x) > 1 = \inf_{x\in R_0} f(x) \quad .$$

Consequently,

$$\inf_{x\in R} f(x) = \left(\frac{2}{3}, \frac{10}{9}\right) = \inf_{x\in R_1} f(x) \quad ,$$

but

$$\inf_{x\in R} f(x) \neq \inf_{x\in R_{-1}} f(x) \quad \text{and} \quad \inf_{x\in R} f(x) \neq \inf_{x\in R_0} f(x) \quad .$$

Therefore, $\inf_{x\in R} f(x)$ coincides with only one of the values $\inf_{x\in R_k} f(x)$.

Example 5.4. *Let $V = \mathbf{R}$, $C = [0, 1[\subset V$, $E = \ell_\infty$ with the usual order. Let*

$$R = \left\{\frac{2k}{2k+1} : k = 1, 2, 3...\right\} \text{ and } R_k = R \cup \left[\frac{2k-2}{2k-1}, \frac{2k}{2k+1}\right] \ (k = 1, 2, 3...) \ .$$

Then $[r_i, r_j] \cap R_i \cap R_j \neq \emptyset$, for every $r_i \in R_i$ and $r_j \in R_j$ $(i, j = 1, 2, 3...$ with $i \neq j)$, and $C = \bigcup_{k=0}^{\infty} R_k$; moreover $R = \bigcap_{k=0}^{\infty} R_k$ and $R_i \cap R_j = R$ for $i, j = 1, 2, 3...$ with $i \neq j)$. Let

$f : [0,1[\longrightarrow \ell_\infty$ be defined by $f = (f_n)_{n=1}^{\infty}$, where $f_n : [0,1[\longrightarrow \mathbf{R}$ is given by

$$f_n(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{2n-2}{2n-1} \\ -2n(2n-1)x + (2n-1)^2 & \text{if } \frac{2n-2}{2n-1} < x \le \frac{2n-1}{2n} \\ 2n(2n+1)x - (2n-1)(2n+1) & \text{if } \frac{2n-1}{2n} < x \le \frac{2n}{2n+1} \\ 1 & \text{if } \frac{2n}{2n+1} < x < 1 \end{cases}$$

The function f is quasiconvex since all the f_n are quasiconvex. Moreover, f is order bounded: for every $x \in [0,1[$ we have $(0,0,0...) \le f(x) \le (1,1,1...)$. Note that

$$\inf_{x\in R} f(x) = (1,1,1...) \text{ and } \inf_{x\in R_k} f(x) = (1,1,...,1,0,1...) \ ,$$

being $0 = f_k(x)$. Then, for every $n = 1,2,3...$ and every $k \neq n$,

$$\inf_{x\in R} f_n(x) = 1 \text{ and } \inf_{x\in R_k} f_n(x) = 1 > 0 = \inf_{x\in R_n} f_n(x) \ .$$

Hence, for every $k = 1,2,3...$,

$$\inf_{x\in R} f(x) = \sup_{k\ge 1} \inf_{x\in R_k} f(x) > \inf_{x\in R_k} f(x) \ .$$

Consequently, none of the values $\inf_{x\in R_k} f(x)$ coincides with $\inf_{x\in R} f(x)$.

A cone K of a vector space E is called a *Yudin cone* if K is generated by a Hamel Basis $(e_d)_{d\in D}$ of E; that is, K is the smallest cone that includes all the e_d $(d \in D)$ [2, Definition 3.15]. If K is a Yudin cone of the vector space E, then E ordered by K is an order complete Riesz space [2, Theorem 3.17]. For $y \in E$ we write

$$y = \sum_{d\in D} y_d e_d \ (y_d \in \mathbf{R}) \ . \tag{5.10}$$

By [2, Theorem 3.17] we have $y \ge 0$ if and only if $y_d \ge 0$ for all $d \in D$. Moreover, if $A \subset E$ is a bounded from above set, then

$$\sup_{y\in A} y = \sum_{d\in D} \left(\sup_{y\in A} y_d \right) e_d \ ; \tag{5.11}$$

analogously, if $A \subset E$ is a bounded from below set, then

$$\inf_{y\in A} y = \sum_{d\in D} \left(\inf_{y\in A} y_d \right) e_d \ . \tag{5.12}$$

Given a vector space E, we fix a Hamel basis $(e_d)_{d\in D}$. We can identify E with the subspace of $\mathbf{R}^D$ consisting of all “eventually zero" real functions on D:

$$E = \{(e_d)_{d\in D} \in \mathbf{R}^d : y_d = 0 \text{ for all but a finite number of } d\} \ ;$$

that is, $E = c_{00}(D)$ [2, p.129]. Hence we can consider a vector space E endowed with a Yudin cone K as a complete function space and then we can apply the above result.

The following proposition can be proved in analogous way to Proposition 5.2

Proposition 5.5. *Let V be a vector space, $C \subset V$ be a convex set, $R \subset C$, $(R_i)_{i\in\mathbf{N}}$ be a sequence of nonempty subsets of C consisting of linearly closed sets in C such that $\bigcup_{i\in\mathbf{N}} R_i = C$ and $R_i \cap R_j = R$, for $i, j \in \mathbf{N}$ with $i \neq j$, E be a complete function space with $E \subset \mathbf{R}^D$, for some set $D \neq \emptyset$, and $f : C \longrightarrow E$ be an order bounded quasiconvex function. Then $R \neq \emptyset$ and the equality (1.1) holds. Moreover, for some set $\mathbf{N}_D \subset \mathbf{N}$ with cardinality less than or equal to the cardinality of D one has (5.9) for every $k \in \mathbf{N} \setminus \mathbf{N}_D$.*

6. A topological result

In [12, Theorem 5] it is proved that if X is a metric space with distance d, $C \subset X$ is a closed subset and $x \in X$, then the following assertions are equivalent:

(1) C is x-boundedly connected; that is, for every $\varepsilon > 0$, the set $\{y \in C : d(y, x) < \varepsilon\}$ is connected.

(2) If R and S are closed subsets of X with $R \cup S = C$, then the equality $d(x, R \cap S) = \max\{d(x, R), d(x, S)\}$ holds.

The next result is an analogue to the above one. If X is a topological space and E is an ordered linear space, the map $f : X \longrightarrow E$ will be called upper level closed if the upper level set $\{x \in X : e \leq f(x)\}$ is closed for any $e \in E$.

Proposition 6.1. *Let X be a topological space, E be an order complete Riesz space, and $f : X \longrightarrow E$ be an upper level closed and order bounded function. The following assertions are equivalent:*

(i) The set $\{x \in X : e \nleq f(x)\}$ is connected for every $e \in E$.

(ii) If R and S are nonempty closed subsets of X and $R \cup S = X$ then at least one of the equalities

$$\inf_{x\in R\cap S} f(x) = \inf_{x\in R} f(x), \quad \inf_{x\in R\cap S} f(x) = \inf_{x\in S} f(x). \tag{6.13}$$

holds true.

Proof. (i) $\Longrightarrow$ (ii). Let $e := \inf_{x\in R\cap S} f(x)$. We clearly have $\inf_{x\in R} f(x) \leq e$ and $\inf_{x\in S} f(x) \leq e$. Assume that both inequalities hold strictly and set $U := \{x \in X : e \nleq f(x)\}$. By (i), U is connected; hence, as $R \cap U$ and $S \cap U$ are closed in U and nonempty and $(R \cap U) \cup (S \cap U) = (R \cup S) \cap U = X \cap U = U$, we have $R \cap S \cap U \neq \emptyset$. But this is impossible, since every $y \in R \cap S$ satisfies $e = \inf_{x\in R\cap S} f(x) \leq f(y)$ and hence does not belong to U.

(ii) $\Longrightarrow$ (i). Suppose that $U := \{x \in X : e \nleq f(x)\}$ is not connected for some $e \in E$. Then there exist two closed sets $\widetilde{R}, \widetilde{S} \subset X$ such that $\widetilde{R} \cap U \neq \emptyset$, $\widetilde{S} \cap U \neq \emptyset$, $\left(\widetilde{R} \cup \widetilde{S}\right) \cap U = U$ and $\widetilde{R} \cap \widetilde{S} \cap U = \emptyset$. Define $R := \widetilde{R} \cup (X \setminus U)$ and $S := \widetilde{S} \cup (X \setminus U)$. These sets are closed; moreover, $R \cap U = \widetilde{R} \cap U \neq \emptyset$, $S \cap U = \widetilde{S} \cap U \neq \emptyset$, $R \cup S = \widetilde{R} \cup \widetilde{S} \cup (X \setminus U) \supset \left(\left(\widetilde{R} \cup \widetilde{S}\right) \cap U\right) \cup (X \setminus U) = U \cup (X \setminus U) = X$ and $R \cap S \cap U = \widetilde{R} \cap \widetilde{S} \cap U = \emptyset$. Take $y \in R \cap U$ and $z \in S \cap U$. We have $\inf_{x\in R} f(x) \leq f(y)$ and $\inf_{x\in S} f(x) \leq f(z)$; hence, as $y, z \in U$, we have

$$e \nleq \inf_{x\in R} f(x) \text{ and } e \nleq \inf_{x\in S} f(x). \tag{6.14}$$

On the other hand, since $R \cap S \cap U = \emptyset$, for every $x \in R \cap S$ one has $x \notin U$, that is, $e \leq f(x)$; therefore $e \leq \inf_{x\in R\cap S} f(x)$, which, together with (6.14), contradicts (ii). □

7. Application to the distance function on a Kantorovich space

In this last section we apply the results of the above sections to the distance function defined on a vector (or cone) normed space or Kantorovich space [8]. Recently research in Kantorovich spaces has had a great development: see, for example, [7], [20] and [22]. In [9] the interaction between certain optimization problems and such spaces is shown.

Let us begin with the definition: let X be a vector space and E be an order complete Riesz space. A map $\|\cdot\| : X \longrightarrow E$ is said a *vector norm* (or *cone norm*) on X if, for $x, y \in X$ and $\alpha \in \mathbf{R}$,

(1) $\|x\| \geq 0$,
(2) $\|x\| = 0 \Longleftrightarrow x = 0$,
(3) $\|\alpha x\| = |\alpha|\|x\|$,
(4) $\|x+y\| \leq \|x\| + \|y\|$.

In this situation we say that X endowed with $\|\cdot\|$ is a *vector normed space* (or *cone normed space*, or *Kantorovich space*). From a vector norm $\|\cdot\|$ we can derive a *vector distance* (or *cone distance*) $d(x,y) = \|x-y\|$ which satisfies the usual properties of a distance.

Given a vector normed space X with $\|\cdot\|$ taking values in a order complete Riesz space, if $x \in X$ and $\emptyset \neq C \subset X$, then one defines the vector

$$d(x,C) = \inf_{y\in C} d(x,y)\ .$$

Consider now the map

$$d(x,\cdot) : C \longrightarrow E \quad , \quad y \in C \longmapsto d(x,y) \in E\ .$$

If C is convex, then $d(x,\cdot)$ is convex, hence quasiconvex, for any x.

The propositions of the previous sections can be applied in this context. For example, from Proposition 4.1 we obtain the next result:

Proposition 7.1. *Let X be a vector normed space with norm $\|\cdot\|$ taking values on an order complete Riesz space. Let $C \subset X$ be a convex set, $(R_i)_{i\in I}$ be a family of nonempty subsets of C containing at least two members and $R := \bigcap_{i\in I} R_i$. If R separates $(R_i)_{i\in I}$, then, for $i, j \in I$ with $i \neq j$,*

$$d(x,R) = \sup\{d(x,R_i), d(x,R_j)\ ,$$

for every $x \in X$. Hence equality (1.2) holds.

Similar results to propositions 4.3, 4.4, 5.2, 5.5 and 6.1 are obtained.

Acknowledgments The authors are thankful to the referees for their helpful comments.

References

[1] C. D. Aliprantis, K. C. Border: *Infinite Dimensional Analysis,* third edition. Springer, 2006.
[2] C. D. Aliprantis, R. Tourky: *Cones and Duality.* American Mathematical Society, 2007.
[3] J. M. F. Castillo, P. L. Papini: Approximation of the limit distance function in Banach spaces. *J. Math. Anal. Appl.* **328** (2007) 577-589.

[4] G. Chacón, V. Montesinos, A. Octavio: A note on the intersection of Banach subspaces. *Publ. Res. Inst. Math. Sci.* **40** (2004) 1-6.

[5] G. Y. Chen, X. Q. Yang, H. Yu: A nonlinear scalarization function and generalized quasi-vector equilibrium problems. *J. Global Optim.* **32** (2005) 451-466.

[6] A. Hoffmann: The distance to the intersection of two convex sets expressed by the distance to each of them. *Math. Nachr.* **157** (1992) 81–98.

[7] A. G. Kusraev: Banach-Kantorovich spaces. *Siberian Math. J.* **26** (1985) 254-259.

[8] A. G. Kusraev: Kantorovich spaces and the metrization problem. *Siberian Math. J.* **34** (1993) 688-694.

[9] A. G. Kusraev, S. S. Kutateladze: Kantorovich spaces and optimization. *J. Math. Sciences* **133** (2006) 1449-1455.

[10] D.T. Luc: Generalized convexity in vector optimization. In: N. Hadjisavvas, S. Komlósi and S. Schaible (EDS.), *Handbook of generalized convexity and generalized monotonicity,* pp. 195–236, Springer, 2005.

[11] W. A. J. Luxemburg, A. C. Zaanen: *Riesz Spaces. Volume I.* North-Holland, 1971.

[12] J.-E. Martínez-Legaz, A. Martinón: Boundedly connected sets and the distance to the intersection of two sets. *J. Math. Anal. Appl.* **332** (2007) 400-406.

[13] J.-E. Martínez-Legaz, A. Martinón: On the infimum of a quasiconvex function on an intersection. Application to the distance function. *J. Convex Anal.* **18** (2011), to appear.

[14] J.-E. Martínez-Legaz, A. M. Rubinov, I. Singer: Downward sets and their separation and approximation properties. *J. Global Optim.* **23** (2002) 111-137.

[15] A. Martinón: Distance to the intersection of two sets. *Bull. Austral. Math. Soc.* **70** (2004) 329-341.

[16] D. Pallaschke, R. Urbanski: On the separation and order law of cancellation for bounded sets. *Optimization* **51(3)** (2002) 487-496.

[17] A. M. Rubinov: Distance to the solution set of an inequality with an increasing function. In: A. Maugeri and P. Danilele (eds.), *Equilibrium Problems and Variational Models,* pp. 417-431, Kluwer Acad. Pub., 2003.

[18] A. M. Rubinov, I. Singer: Best approximation by normal and conormal sets. *J. Approx. Theor.* **107** (2000) 212-243.

[19] A. M. Rubinov, I. Singer: Distance to the intersection of normal sets and applications. *Numer. Funct. Anal. and Optimiz.* **21** (2000) 521-535.

[20] A. Sonmez, H. Cakalli: Cone normed spaces and weighted means. *Math. Comp. Modelling* **52** (2010) 1660-1666.

[21] S. Willard: *General Topology.* Addison-Wesley, 1970.

[22] P. P. Zabrejko: K-metric and K-normed linear spaces: survey. *Collecttanea Math.* **48** (1997) 825-859.

Juan-Enrique Martínez-Legaz
Departament d'Economia i d'Història Econòmica
Universitat Autònoma de Barcelona
08193 Bellaterra (Barcelona), Spain
e-mail: JuanEnrique.Martinez.Legaz@uab.es

Antonio Martinón
Departamento de Análisis Matemático
Universidad de La Laguna
38271 La Laguna (Tenerife), Spain
e-mail: anmarce@ull.es