A CONVEX REPRESENTATION OF TOTALLY BALANCED GAMES

J. M. BILBAO* AND J. E. MARTÍNEZ-LEGAZ**

*Dept. Matemática Aplicada II, Escuela Superior de Ingenieros
Camino de los Descubrimientos, 41092 Sevilla, Spain
http://www.esi.us.es/~mbilbao/ E-mail: mbilbao@us.es
**Dept. d'Economia i d'Història Econòmica,
Universitat Autònoma de Barcelona, 08193 Bellaterra, Spain
E-mail: JuanEnrique.Martinez.Legaz@uab.es

Abstract. We analyze the least increment function, a convex function of n variables associated to an n-person cooperative game. Several dual representations of cooperative games are proposed in [3, 4]. Chronologically, the first one is the indirect function [3], which is closely related to the conjugation theory of discrete convex analysis. The maximum average value function [4] constitutes another dual representation of a (nontrivial nonnegative) cooperative game by a convex function but, unlike the indirect function, which faithfully represents any game, the maximum average value function requires total balancedness of the game for the full validity of the duality relations. The least increment function shares with the maximum average value function the fact that it only preserves all the information on the game if the game is totally balanced.

Key words. Cooperative games, indirect function, least increment function

AMS subject classifications. 91A12

1. Introduction

A *cooperative game* is a pair (N, v), where $N = \{1, \ldots, n\}$ is a finite set and $v : 2^N \to \mathbb{R}$ is a function satisfying the condition $v(\emptyset) = 0$. Given a game (N, v) and a coalition $S \subseteq N$, the *subgame* (S, v) is obtained by restricting v to 2^S. To every coalition $S \subseteq N$ is associated its *characteristic vector* $\mathbf{1}_S \in \{0, 1\}^n$, where $\mathbf{1}_S(i) = 1$ if $i \in S$, and $\mathbf{1}_S(i) = 0$ if $i \notin S$.

In view of economic applications, it is convenient to distinguish between profit games and cost games. Since the definitions of most concepts relative to games depend on whether the game in question is regarded as a profit or a cost game, one should make that distinction explicit at a formal level. To this aim one can redefine, in a more rigorous way, a game as a pair (v, σ) consisting of a function $v : 2^N \to \mathbb{R}$ and a sign $\sigma \in \{1, -1\}$ indicating whether the game is a profit ($\sigma = 1$) or a cost ($\sigma = -1$) game. Even though we adopt this point of view, we shall always identify the game (v, σ) with v, and shall distinguish between profit and cost games by explicitly mentioning the type. When the type will not be mentioned, we shall be referring to games that may be of either type.

The second author acknowledges the support of the MICINN of Spain, Grant MTM2008-06695-C03-03, of the Barcelona Graduate School of Economics and of the Government of Catalonia. He is affiliated to MOVE (Markets, Organizations and Votes in Economics).

For a cost game $c : 2^N \to \mathbb{R}$ and a profit game $v : 2^N \to \mathbb{R}$ we define the polyhedra

$$\begin{aligned} P(c) &= \{x \in \mathbb{R}^n : x(S) \leq c(S) \text{ for all } S \subseteq N\}, \\ P(v) &= \{x \in \mathbb{R}^n : x(S) \geq v(S) \text{ for all } S \subseteq N\}, \\ C(c) &= \{x \in P(c) : x(N) = c(N)\}, \\ C(v) &= \{x \in P(v) : x(N) = v(N)\}, \end{aligned}$$

where $x(S) = \langle \mathbf{1}_S, x \rangle = \sum_{i \in S} x_i$. Note that $P(c) \neq \emptyset$ if and only if $c(\emptyset) \geq 0$, and $P(v) \neq \emptyset$ if and only if $v(\emptyset) \leq 0$. Thus, the polyhedra $P(c)$ and $P(v)$ associated to profit and cost games c and v, respectively, are nonempty. The polyhedra $C(c)$ and $C(v)$ are called the cores of the respective games. Games with a nonempty core are called *balanced games*. A game is *totally balanced* if each subgame is balanced. A useful reference for these concepts is [6].

For $C \subseteq \mathbb{R}^n$, we denote by *bd* C its boundary, and by

$$N(C, x) = \{x^* \in \mathbb{R}^n : \langle x^*, y - x \rangle \leq 0 \text{ for all } y \in C\}$$

its normal cone at $x \in \mathbb{R}^n$. Notice that if there are no supporting hyperplanes to C through x, then $N(C, x) = \{0\}$.

For a function $\varphi : C \subseteq \mathbb{R}^n \to \mathbb{R}$, we consider the convex conjugate function $\varphi^* : \mathbb{R}^n \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, given by (see [7])

$$\varphi^*(x) = \sup_{x^* \in C} \{\langle x^*, x \rangle - \varphi(x^*)\},$$

and its subdifferential at $x \in C$:

$$\partial f(x) = \{x^* \in \mathbb{R}^n : f(y) - f(x) \geq \langle x^*, y - x \rangle \text{ for all } y \in C\}.$$

2. The indirect function

In this section we study a representation of n-person cooperative games by functions of n variables. As in [3], where this representation was introduced, we shall restrict the presentation to profit games; of course, a parallel theory can be developed for cost games:

Definition 2.1. *The indirect function of a profit game* $v : 2^N \to \mathbb{R}$ *is the function* $\pi_v : \mathbb{R}^n \to \mathbb{R}$ *given by* $\pi_v(x) = \max\{v(S) - x(S) : S \subseteq N\}$ *for all* $x \in \mathbb{R}^n$.

The indirect function admits an economic interpretation. Let us regard the players of the profit game as workers, and $v(S)$ as the profit (measured in money units) that coalition S yields when its members work together, provided that they have available the resources needed for production. Suppose that an employer, owning these resources, wishes to choose those workers who would provide him with the maximum possible profit. If the subset S is selected then the total amount of money that S will yield is $v(S)$. If $x = (x_1, \ldots, x_n)$ is the vector of (possibly negative) salaries demanded by the workers then $\pi_v(x)$ represents the maximum net profit the employer can obtain under those given salaries.

Theorem 2.1. *Let* $v : 2^N \to \mathbb{R}$ *be a profit game. Then, for all* $S \subseteq N$, *one has*

$$v(S) = \min\{x(S) + \pi_v(x) : x \in \mathbb{R}^n\}. \tag{1}$$

The importance of the preceding theorem lies in that it shows that the indirect function π_v of a profit game v contains all the information on the game, as it allows to recover v from π_v.

Indirect functions of profit games are characterized in [3] by three properties, two of which are expressed in terms of the convex analytic subdifferential:

Theorem 2.2. *Let $\pi : \mathbb{R}^n \to \mathbb{R}$. There exists a profit game $v : 2^N \to \mathbb{R}$ such that $\pi = \pi_v$ if and only if π satisfies the following properties:*

(1) $\partial\pi(x) \bigcap \{-1, 0\}^n \neq \emptyset$, *for all* $x \in \mathbb{R}^n$.
(2) $\{-1, 0\}^n \subseteq \bigcup_{x \in \mathbb{R}^n} \partial\pi(x)$.
(3) $\min \{\pi(x) : x \in \mathbb{R}^n\} = 0$.

The following alternative characterization of indirect functions in terms of gradients, instead of subdifferentials, was given in [4].

Theorem 2.3. *Let $\pi : \mathbb{R}^n \to \mathbb{R}$ and let $\nabla\pi$ denote the gradient mapping of π. Then there exists a profit game $v : 2^N \to \mathbb{R}$ such that $\pi = \pi_v$ if and only if π satisfies the following properties:*

(1) *π is convex.*
(2) *The range of $\nabla\pi$ is $\{-1, 0\}^n$.*
(3) $\min \{\pi(x) : x \in \mathbb{R}^n\} = 0$.

Many concepts in the theory of cooperative profit games can be easily expressed in terms of indirect functions; we refer the reader for details to [3]. In particular, totally balanced (profit) games, an important class of games that will be dealt with in the subsequent sections, are characterized by their indirect functions in [5].

3. The least increment function

This section is devoted to a different way of representing profit games by convex functions, namely, by the so-called *least increment functions.* The interested reader can adapt this representation to the case of cost games.

Definition 3.1. *The least increment function of a profit game $v : 2^N \to \mathbb{R}$ is the function $\epsilon_v : \mathbb{R}^n \to \mathbb{R}$, given by*

$$\epsilon_v(x) = \min \{y(N) - x(N) : y \in P(v),\ y \geq x\}.$$

Since $y \geq x$ implies $y(N) - x(N) \geq 0$, we obtain $\epsilon_v(x) \geq 0$ for all $x \in \mathbb{R}^n$. Notice also that $P(v) \subseteq \epsilon_v^{-1}(0)$. To prove the reverse inclusion, suppose that $\epsilon_v(x) = 0$. Then $\sum_{i \in N} (y_i - x_i) = 0$ for some $y \in P(v)$ such that $y \geq x$, and hence $x = y$. This establishes

$$\epsilon_v^{-1}(0) = P(v).$$

The least increment function admits the following interpretation. Suppose that a payoff vector $x \in \mathbb{R}^n$ is offered to the players. Then $\epsilon_v(x)$ is the least amount $y(N) - x(N)$ by which the total payoff $x(N)$ should be incremented to make the resulting payoff vector acceptable by all coalitions ($y(S) \geq v(S)$ for all $S \subseteq N$) and preferred to the initial one by all players ($y \geq x$).

Example 3.1. *Let* $(\{1\}, v)$ *be a profit game. By identifying* v *with* $v(\{1\})$, *we get*

$$\begin{aligned}\epsilon_v(x) &= \min\{y - x : y \geq v \text{ and } y \geq x\} = \min\{y - x : y \geq \max\{v, x\}\} \\ &= \min\{y : y \geq \max\{v, x\}\} - x = \max\{v, x\} - x \\ &= \begin{cases} v - x & \text{if } x \leq v, \\ 0 & \text{if } x \geq v. \end{cases}\end{aligned}$$

Proposition 3.1. *The least increment function* ϵ_v *of a profit game* (N, v) *satisfies*

$$\epsilon_v(x) = \max\left\{\sum_{S \subseteq N} \lambda_S \left[v(S) - x(S)\right] : \sum_{S \subseteq N} \lambda_S \mathbf{1}_S \in [0,1]^n,\ \lambda_S \in \mathbb{R}_+^{2^N}\right\}. \tag{2}$$

Proof. The duality theorem of linear programming implies

$$\begin{aligned}\epsilon_v(x) &= \min\left\{\langle \mathbf{1}_N, y - x\rangle : y - x \geq 0,\ \langle \mathbf{1}_S, y - x\rangle \geq v(S) - x(S),\ \forall S \subseteq N\right\} \\ &= \max\left\{\sum_{S \subseteq N} \lambda_S \left[v(S) - x(S)\right] : \sum_{S \subseteq N} \lambda_S \mathbf{1}_S \leq \mathbf{1}_N,\ \lambda_S \in \mathbb{R}_+^{2^N}\right\}.\end{aligned}$$

□

According to the preceding definition, the least increment function is the optimum value function of A parametric linear programming problem. Using the duality theorem of linear programming, it easily follows that, like the indirect function, the least increment function is a polyhedral convex function. The following proposition compares both functions.

Proposition 3.2. *If* $v : 2^N \to \mathbb{R}$ *is a profit game then, for all* $x \in \mathbb{R}^n$,

$$\epsilon_v(x) \geq \pi_v(x).$$

Proof. For each $S \subseteq N$ and for each $y \in P(v)$ such that $y \geq x$, we have

$$y(N) = y(S) + y(N \setminus S) \geq v(S) + x(N \setminus S),$$

and therefore

$$\min\{y(N) - x(N) : y \in P(v),\ y \geq x\} \geq \max\{v(S) - x(S) : S \subseteq N\} = \pi_v(x).$$

□

Example 3.2. *Let us consider* $N = \{1, 2\}$ *and* $v : 2^N \to \mathbb{R}$ *given by* $v(\emptyset) = 0$, $v(\{1\}) = v_1$, $v(\{2\}) = v_2$, *and* $v(\{1, 2\}) = \overline{v}$. *We consider at first the case in which* $v_1 + v_2 \leq \overline{v}$. *For the polyhedron*

$$P(v) = \left\{y \in \mathbb{R}^2 : y_1 \geq v_1,\ y_2 \geq v_2,\ y_1 + y_2 \geq \overline{v}\right\}$$

there are the following boundary lines:

$$y_1 = v_1, \quad y_2 = v_2, \quad y_1 + y_2 = \overline{v}.$$

Then there are two extreme points $(v_1, \overline{v} - v_1)$ *and* $(\overline{v} - v_2, v_2)$ *of* $P(v)$. *Furthermore, the core of* v *is the segment defined by these extreme points. Proposition 3.2 implies that*

$$\epsilon_v(x) \geq v_1 - x_1, \quad \epsilon_v(x) \geq v_2 - x_2, \quad \epsilon_v(x) \geq \overline{v} - x_1 - x_2,$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. *Taking suitable vectors in the boundary lines of* $P(v)$, *we obtain*

$$\epsilon_v(x) = \begin{cases} 0 & \text{if } x \in P(v), \\ v_1 - x_1 & \text{if } x_1 \leq v_1, \ \ x_2 \geq \overline{v} - v_1, \\ v_2 - x_2 & \text{if } x_1 \geq \overline{v} - v_2, \ \ x_2 \leq v_2, \\ \overline{v} - x_1 - x_2 & \text{if } x_1 \leq \overline{v} - v_2, \ \ x_2 \leq \overline{v} - v_1, \ \ x_1 + x_2 \leq \overline{v}. \end{cases}$$

In the case $v_1 + v_2 > \overline{v}$, *the only extreme point of* $P(v)$ *is* (v_1, v_2) *and the core of* v *is empty. Then the least increment function is given by*

$$\epsilon_v(x) = \begin{cases} 0 & \text{if } x \in P(v), \\ v_1 - x_1 & \text{if } x_1 \leq v_1, \ \ x_2 \geq v_2, \\ v_2 - x_2 & \text{if } x_1 \geq v_1, \ \ x_2 \leq v_2, \\ v_1 + v_2 - x_1 - x_2 & \text{if } x_1 \leq v_1, \ \ x_2 \leq v_2. \end{cases}$$

Shapley [8] introduced convex (profit) games as follows:

Definition 3.2. *A profit game* $v : 2^N \to \mathbb{R}$ *is called convex if*

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$$

for all $S, T \subseteq N$. *A cost game* $c : 2^N \to \mathbb{R}$ *is concave if the reverse inequality holds.*

Definition 3.3. *The vector rank function of a concave cost game* $c : 2^N \to \mathbb{R}$ *is* $r_c : \mathbb{R}^n \to \mathbb{R}$, *given by* $r_c(u) = \min\{c(S) + u(N \setminus S) : S \subseteq N\}$.

The function r_c is related to $P(c)$ by the min-max equation

$$r_c(u) = \max\{x(N) : x \in P(c), \ x \leq u\}, \tag{3}$$

which is an immediate consequence of the following generalization of the intersection theorem for polymatroids due to Edmonds [1]:

Theorem 3.3. *For distributive lattices* $\mathcal{F}_1, \mathcal{F}_2 \subseteq 2^N$, *let* $c_1 : \mathcal{F}_1 \to \mathbb{R}, \ c_2 : \mathcal{F}_2 \to \mathbb{R}$ *be concave cost games. If there exists a set* $S \in \mathcal{F}_1$ *such that* $N \setminus S \in \mathcal{F}_2$, *then we have*

$$\begin{aligned} &\min\{c_1(S) + c_2(N \setminus S) : S \in \mathcal{F}_1, \ N \setminus S \in \mathcal{F}_2\} \\ &\quad = \max\{x(N) : x \in P(c_1) \cap P(c_2)\}. \end{aligned}$$

Moreover, if c_1 *and* c_2 *are integer valued, then the maximum is attained by an integral vector.*

Proof. See Fujishige [2, Theorem 4.9]. □

We shall next give a sufficient condition for the inequality in the preceding proposition to hold with the equal sign.

Theorem 3.4. *If* $v : 2^N \to \mathbb{R}$ *is a convex profit game then* $\epsilon_v = \pi_v$.

Proof. The result follows from (3) applied to the concave cost game $-v$:

$$\begin{aligned}
\epsilon_v(x) &= \min\{y(N) - x(N) : y \in P(v),\ y \geq x\} \\
&= \min\{y(N) : y \in P(v),\ y \geq x\} - x(N) \\
&= \min\{y(N) : -y \in P(-v),\ y \geq x\} - x(N) \\
&= -\max\{z(N) : z \in P(-v),\ z \leq -x\} - x(N) \\
&= -\min\{-v(S) - x(N \setminus S) : S \subseteq N\} - x(N) \\
&= \max\{v(S) + x(N \setminus S) : S \subseteq N\} - x(N) \\
&= \max\{v(S) - x(S) : S \subseteq N\} \\
&= \pi_v(x).
\end{aligned}$$

□

A natural question to ask is whether the convexity assumption can be removed in the preceding theorem. In other words, do the indirect function and the least increment function of any profit game coincide? If this were the case, according to (1) the expression $\min\{x(S) + \epsilon_v(x) : x \in \mathbb{R}^n\}$ would coincide with $v(S)$ for any profit game and any coalition S. However, from the next theorem it follows that the equality $v(S) = \min\{x(S) + \epsilon_v(x) : x \in \mathbb{R}^n\}$ is characteristic of totally balanced profit games. To give the definition of this class of games, we first need to recall the notion of x-balanced collection:

Definition 3.4. *For $x \in \mathbb{R}^n_+$, $\{\lambda_T\}_{T\subseteq N}$ is an x-balanced collection if $\lambda_T \geq 0$ for all $T \subseteq N$ and $\sum_{T\subseteq N} \lambda_T \mathbf{1}_T = x$.*

Definition 3.5. *A profit game $v : 2^N \to \mathbb{R}$ is totally balanced if for all $S \in 2^N$ and all $\mathbf{1}_S$-balanced collection $\{\lambda_T\}_{T\subseteq N}$ it satisfies $\sum_{T\subseteq N} \lambda_T v(T) \leq v(S)$.*

The class of totally balanced profit games is closed under pointwise infimum, that is, if $\{v_i\}_{i\in I}$ is an arbitrary nonempty family of totally balanced profit games then the profit game $v : 2^N \to \mathbb{R}$ defined by $v(S) = \inf_{i\in I} v_i(S)$ for all $S \in 2^N$ is totally balanced, too. Besides, any profit game $v : 2^N \to \mathbb{R}$ admits a totally balanced majorant, i.e., a totally balanced profit game $w : 2^N \to \mathbb{R}$ satisfying $w(S) \geq v(S)$ for all $S \in 2^N$ Indeed, one can take, e.g., the (additive) game defined by $w(S) = k\,|S|$ for all $S \in 2^N$, with

$$k \geq \max\left\{\frac{v(S)}{|S|} : S \in 2^N \setminus \{\emptyset\}\right\}.$$

In view of these properties, the following concept is well defined:

Definition 3.6. *The totally balanced cover of a profit game $v : 2^N \to \mathbb{R}$ is the profit game $\widetilde{v} : 2^N \to \mathbb{R}$ defined by*

$$\widetilde{v}(S) = \inf\left\{w(S) :\ w : 2^N \to \mathbb{R} \text{ is a totally balanced majorant of } v\right\}.$$

From this definition the following result immediately follows:

Proposition 3.5. *The totally balanced cover $\widetilde{v} : 2^N \to \mathbb{R}$ of the profit game $v : 2^N \to \mathbb{R}$ is the smallest (in the pointwise sense) totally balanced majorant of v. Therefore, v is totally balanced if and only if $\widetilde{v} = v$.*

The next proposition is well known [9, formula (4-4)] and easy to prove:

Proposition 3.6. *The totally balanced cover* $\widetilde{v} : 2^N \to \mathbb{R}$ *of the profit game* $v : 2^N \to \mathbb{R}$ *is given by*

$$\widetilde{v}(S) = \max\left\{ \sum_{T \subseteq N} \lambda_T v(T) : \sum_{T \subseteq S} \lambda_T \mathbf{1}_T = \mathbf{1}_S,\ \lambda_T \geq 0,\ \forall T \subseteq N \right\}.$$

We are now in a position to state the theorem announced above:

Theorem 3.7. *For any profit game* $v : 2^N \to \mathbb{R}$*, one has*

$$\widetilde{v}(S) = \min\left\{ x(S) + \epsilon_v(x) : x \in \mathbb{R}^n \right\} \quad \textit{for all} \quad S \in 2^N.$$

Proof. Let $x \in \mathbb{R}^n$. We recall that

$$\epsilon_v(x) = \min\left\{ \langle \mathbf{1}_N, z \rangle : z \geq 0,\ z(T) \geq v(T) - x(T),\ \forall T \subseteq N \right\}.$$

Then, for every coalition $S \in 2^N$,

$$\begin{aligned}
&\min\left\{ x(S) + \epsilon_v(x) : x \in \mathbb{R}^n \right\} \\
&= \min\left\{ \langle \mathbf{1}_S, x \rangle + \langle \mathbf{1}_N, z \rangle : x, z \in \mathbb{R}^n,\ z \geq 0,\ x(T) + z(T) \geq v(T),\ \forall T \subseteq N \right\} \\
&= \max\left\{ \sum_{T \subseteq N} \lambda_T v(T) : \sum_{T \subseteq N} \lambda_T \mathbf{1}_T = \mathbf{1}_S,\ \sum_{T \subseteq N} \lambda_T \mathbf{1}_T \leq \mathbf{1}_N,\ \lambda_T \geq 0,\ \forall T \subseteq N \right\} \\
&= \max\left\{ \sum_{T \subseteq N} \lambda_T v(T) : \sum_{T \subseteq N} \lambda_T \mathbf{1}_T = \mathbf{1}_S,\ \lambda_T \geq 0,\ \forall T \subseteq N \right\} \\
&= \max\left\{ \sum_{T \subseteq N} \lambda_T v(T) : \sum_{T \subseteq S} \lambda_T \mathbf{1}_T = \mathbf{1}_S,\ \lambda_T \geq 0,\ \forall T \subseteq N \right\} = \widetilde{v}(S);
\end{aligned}$$

for the latter equality see Proposition 3.6. □

In view of Theorem 3.7, one can say that the least increment function provides a dual representation of totally balanced profit games. Indeed, unlike the indirect function, the least increment function of an arbitrary profit game does not contain all the information on the game, but only on its totally balanced cover, since one can prove that profit games having the same totally balanced cover have also the same least increment function.

Based on Theorem 3.7, a plausible conjecture is that, even though indirect functions and least increment functions do not generally coincide, they do in the case of totally balanced profit games. However, the following example of a 4-players totally balanced (but not convex, of course) profit game shows that this conjecture is wrong.

Example 3.3. *Let* (N, v) *be the game given by* $N = \{1, 2, 3, 4\}$ *and*

$$\begin{aligned}
v(\emptyset) &= 0,\ v(\{i\}) = -2 \ \textit{for all}\ i \in N,\ v(\{1,2\}) = v(\{2,3\}) = 1, \\
v(\{1,3\}) &= v(\{1,4\}) = v(\{2,4\}) = v(\{3,4\}) = -2,\ v(\{1,2,3\}) = 0, \\
v(\{1,2,4\}) &= -1,\ v(\{1,3,4\}) = 1,\ v(\{2,3,4\}) = -1,\ v(N) = 1.
\end{aligned}$$

We prove that this game is totally balanced, i.e., all the subgames (S, v_S) *are balanced. Notice that* $v(\{i,j\}) \geq v(\{i\}) + v(\{j\})$ *for all* $i \neq j$*, and hence the subgames*

(S, v_S) are convex and balanced for coalitions S such that $|S| \leq 2$. Moreover,

$$\begin{array}{lllll} \textit{if } S & = & \{1,2,3\} & \textit{then} & (x_1,x_2,x_3) = (-1,2,-1) \in C(v_S); \\ \textit{if } S & = & \{1,2,4\} & \textit{then} & (x_1,x_2,x_4) = (0,1,-2) \in C(v_S); \\ \textit{if } S & = & \{1,3,4\} & \textit{then} & (x_1,x_3,x_4) = (1/3,1/3,1/3) \in C(v_S); \\ \textit{if } S & = & \{2,3,4\} & \textit{then} & (x_2,x_3,x_4) = (1,0,-2) \in C(v_S); \\ \textit{if } S & = & N & \textit{then} & (x_1,x_2,x_3,x_4) = (3/2,0,3/2,-2) \in C(v). \end{array}$$

The indirect function $\pi_v(0) = \max\{v(S) : S \subseteq N\} = 1$. Proposition 3.1 implies

$$\epsilon_v(0) = \max\left\{ \sum_{S\subseteq N} \lambda_S v(S) : \sum_{S\subseteq N} \lambda_S \mathbf{1}_S \leq \mathbf{1}_N,\ \lambda_S \geq 0 \textit{ for all } S \subseteq N \right\}.$$

We consider the collection $\mathcal{F} = \{\{1,2\},\{2,3\},\{1,3,4\}\}$ and define

$$\lambda_S = \begin{cases} 1/2 & \textit{if } S \in \mathcal{F}, \\ 0 & \textit{if } S \notin \mathcal{F}. \end{cases}$$

Since $\sum_{S\subseteq N} \lambda_S \mathbf{1}_S = (1,1,1,1/2) \leq \mathbf{1}_N$, we obtain

$$\epsilon_v(0) \geq \sum_{S\in\mathcal{F}} \lambda_S v(S) = \frac{3}{2}.$$

In order to characterize least increment functions we will need the following lemma:

Lemma 3.8. *Let $C \subset \mathbb{R}^n$ be a convex polyhedron with nonempty interior and let $W \subset \mathbb{R}^n \setminus \{0\}$ be such that $N(C,x) \cap W \neq \emptyset$ for all $x \in bd\ C$. Then there exist $w_i \in W$ and $r_i \in \mathbb{R}$, $i = 1,...,p$, such that*

$$C = \{x \in \mathbb{R}^n : \langle x, w_i\rangle \leq r_i,\ i = 1,...,p\}.$$

Proof. Let $\langle x, x_i^*\rangle \leq b_i$, $i = 1,...,p$, be a minimal system representing C, that is,

$$C = \{x \in \mathbb{R}^n : \langle x, x_i^*\rangle \leq b_i,\ i = 1,...,p\}$$

and

$$C \neq \{x \in \mathbb{R}^n : \langle x, x_i^*\rangle \leq b_i,\ i \in I\}$$

if I is a proper subset of $\{1,...,p\}$. Let $H_i = \{x \in \mathbb{R}^n : \langle x, x_i^*\rangle \leq b_i\}$, $i = 1,...,p$. Let $i \in \{1,...,p\}$. We have $\bigcap_{j\neq i} int\ H_j \nsubseteq int\ H_i$, since otherwise we would have $\bigcap_{j\neq i} H_j = cl\ int \bigcap_{j\neq i} H_j = cl \bigcap_{j\neq i} int\ H_j \subseteq cl\ int\ H_i = H_i$ and hence $C = \bigcap_{j=1}^{p} H_j = \bigcap_{j\neq i} H_j$, which is a contradiction with the minimality of the representation of C. Thus there exists $x_i \in \mathbb{R}^n$ such that $\langle x_i, x_j^*\rangle < b_i$ for all $j \neq i$ and $\langle x_i, x_i^*\rangle \leq b_i$. Since C has a nonempty interior, without loss of generality we can assume that $\langle x_i, x_i^*\rangle = b_i$. Then $x_i \in bd\ C$. Let $c \in N(C, x_i) \setminus \{0\}$ and $d \in \mathbb{R}^n$ be such that $\langle d, x_i^*\rangle = 0$. For sufficiently small $\lambda \in \mathbb{R}$, $\lambda\langle d, x_j^*\rangle < b_i - \langle x_i, x_j^*\rangle$ and hence $x_i + \lambda d \in C$. Therefore $\lambda\langle d, c\rangle \leq 0$, which implies that $\langle d, c\rangle = 0$. It thus follows that $c = \alpha x_i^*$ for some $\alpha \in \mathbb{R} \setminus \{0\}$. Given that $x_i - \beta x_i^* \in C$ for small enough $\beta > 0$, we have $0 \geq \langle -\beta x_i^*, c\rangle = \langle -\beta x_i^*, \alpha x_i^*\rangle = -\beta\alpha \|x_i^*\|^2$, so that $\alpha > 0$. We have thus proved that $N(C, x_i)$ is the cone generated by x_i^*, so that $\gamma_i x_i^* \in W$ for some $\gamma_i > 0$. The statement follows by setting $w_i = \gamma_i x_i^*$ and $r_i = \gamma_i b_i$. $\square$

Theorem 3.9. *Let $\epsilon : \mathbb{R}^n \to \mathbb{R}$. There exists a profit game $v : 2^N \to \mathbb{R}$ such that $\epsilon = \epsilon_v$ if and only if ϵ satisfies the following properties:*

1. $\partial\epsilon(x) \cap [-1,0]^n \neq \emptyset$*, for all* $x \in \mathbb{R}^n$.
2. $[-1,0]^n \subseteq \bigcup_{x\in\epsilon^{-1}(0)} \partial\epsilon(x)$.
3. $\min\{\epsilon(x) : x \in \mathbb{R}^n\} = 0$.
4. *int* $\epsilon^{-1}(0) \neq \emptyset$.
5. $N(\epsilon^{-1}(0), x) \cap (\{-1,0\}^n \setminus \{0\}) \neq \emptyset$*, for all* $x \in$ *bd* $\epsilon^{-1}(0)$.

Among the games v satisfying $\epsilon = \epsilon_v$ there is exactly one that is totally balanced, namely, the one defined by

$$v(S) = \min\{\epsilon(x) + x(S) : x \in \mathbb{R}^n\} = \min\left\{x(S) : x \in \epsilon^{-1}(0)\right\}.$$

Proof. Let us first assume that $\epsilon = \epsilon_v$ for some game v. Let $x \in \mathbb{R}^n$. Then there exist nonnegative coefficients $\{\lambda_S\}_{S\subseteq N}$ with $\sum_{S\subseteq N} \lambda_S \mathbf{1}_S \in [0,1]^n$ such that the maximum in (2) is attained. For every $y \in \mathbb{R}^n$ we have

$$\begin{aligned}
\epsilon(y) &\geq \sum_{S\subseteq N} \lambda_S [v(S) - y(S)] \\
&= \sum_{S\subseteq N} \lambda_S [v(S) - x(S)] + \sum_{S\subseteq N} \lambda_S [x(S) - y(S)] \\
&= \epsilon(x) + \left\langle -\sum_{S\subseteq N} \lambda_S \mathbf{1}_S, y - x \right\rangle.
\end{aligned}$$

Thus $-\sum_{S\subseteq N} \lambda_S \mathbf{1}_S \in \partial\epsilon(x)$, and this proves property 1.

To prove property 2, we define $\varphi : [-1,0]^n \to \mathbb{R}$ by

$$\varphi(x^*) = \max\{\langle x^*, y\rangle : y \in P(v)\}.$$

For every $x \in \mathbb{R}^n$ we have

$$\begin{aligned}
\varphi^*(x) &= \max_{x^*\in[-1,0]^n} \{\langle x^*, x\rangle - \varphi(x^*)\} \\
&= \max_{x^*\in[-1,0]^n} \{\langle x^*, x\rangle - \max\{\langle x^*, y\rangle : y \in P(v)\}\} \\
&= \max_{x^*\in[-1,0]^n} \{\langle x^*, x\rangle + \min\{\langle -x^*, y\rangle : y \in P(v)\}\}.
\end{aligned}$$

From linear programming duality it follows that

$$\begin{aligned}
&\min\{\langle -x^*, y\rangle : y \in P(v)\} \\
&\quad = \max\left\{\sum_{S\subseteq N} \lambda_S v(S) : \sum_{S\subseteq N} \lambda_S \mathbf{1}_S = -x^*,\ \lambda_S \geq 0,\ \forall S \subseteq N\right\}.
\end{aligned}$$

Thus, by using Proposition 3.1, we have

$$\begin{aligned}
\varphi^*(x) &= \max_{x^*\in[-1,0]^n}\left\{\max\left\{\sum_{S\subseteq N}\lambda_S v(S)+\langle x^*,x\rangle : \sum_{S\subseteq N}\lambda_S \mathbf{1}_S=-x^*,\ \lambda_S\in\mathbb{R}_+^{2^N}\right\}\right\} \\
&= \max_{x^*\in[-1,0]^n}\left\{\max\left\{\sum_{S\subseteq N}\lambda_S [v(S)-x(S)] : \sum_{S\subseteq N}\lambda_S \mathbf{1}_S=-x^*,\ \lambda_S\in\mathbb{R}_+^{2^N}\right\}\right\} \\
&= \max\left\{\sum_{S\subseteq N}\lambda_S [v(S)-x(S)] : \sum_{S\subseteq N}\lambda_S \mathbf{1}_S\in[0,1]^n,\ \lambda_S\in\mathbb{R}_+^{2^N}\right\} \\
&= \epsilon(x).
\end{aligned}$$

Let $x^*\in[-1,0]^n$. Since φ is convex, proper and lower semicontinuous, we have $\epsilon^*(x^*)=\varphi^{**}(x^*)=\varphi(x^*)=\langle x^*,y\rangle$ for some $y\in P(v)=\epsilon^{-1}(0)$, and hence

$$\langle x^*,y\rangle\geq\langle x,x^*\rangle-\epsilon(x)\quad\text{for all } x\in\mathbb{R}^n.$$

This inequality and $\epsilon(y)=0$ imply

$$\epsilon(x)-\epsilon(y)\geq\langle x^*,x-y\rangle\quad\text{for all } x\in\mathbb{R}^n.$$

Consequently $x^*\in\partial\epsilon(y)$, which proves property 2.

Since ϵ is nonnegative and takes the value 0 on the nonempty set $P(v)$, we obtain property 3.

Properties 4 and 5 are immediate consequences of the equality $\epsilon^{-1}(0)=P(v)$.

To prove the converse, we asssume that ϵ satisfies properties 1–5 and define the game $v:2^N\to\mathbb{R}$ by

$$v(S)=-\epsilon^*(-\mathbf{1}_S)=\inf_{x\in\mathbb{R}^n}\{x(S)+\epsilon(x)\}.$$

For every $S\subseteq N$, we have $-\mathbf{1}_S\in[-1,0]^n$. By property 2 there exists $y\in\epsilon^{-1}(0)$ such that $-\mathbf{1}_S\in\partial\epsilon(y)$. Thus

$$\epsilon(x)-\epsilon(y)\geq\langle-\mathbf{1}_S,x-y\rangle\quad\text{for all } x\in\mathbb{R}^n,$$

which is equivalent to

$$x(S)+\epsilon(x)\geq y(S)\quad\text{for all } x\in\mathbb{R}^n.$$

Therefore, for every $S\subseteq N$, we obtain

$$v(S)=\min_{x\in\mathbb{R}^n}\{x(S)+\epsilon(x)\}=\min\left\{x(S):x\in\epsilon^{-1}(0)\right\}. \tag{4}$$

Since v is the minimum game of a collection of additive games, we deduce that v is totally balanced. Notice also that for every $x^*\in[-1,0]^n$, property 2 implies the existence of $x\in\epsilon^{-1}(0)$ such that

$$\epsilon(y)\geq\langle x^*,y-x\rangle\quad\text{for all } y\in\mathbb{R}^n.$$

This proves that

$$\epsilon^*(x^*)=\sup_{y\in\mathbb{R}^n}\{\langle x^*,y\rangle-\epsilon(y)\}=\max_{x\in\epsilon^{-1}(0)}\langle x^*,x\rangle, \tag{5}$$

for all $x^*\in[-1,0]^n$.

Let $x\in\mathbb{R}^n$. We show that

$$\epsilon(x)=\max_{x^*\in[-1,0]^n}\{\langle x^*,x\rangle-\epsilon^*(x^*)\}. \tag{6}$$

By definition $\epsilon^*(x^*) \geq \langle x^*, x\rangle - \epsilon(x)$ for all $x^* \in \mathbb{R}^n$, and hence

$$\epsilon(x) \geq \langle x^*, x\rangle - \epsilon^*(x^*) \quad \text{for all } x^* \in \mathbb{R}^n. \tag{7}$$

It follows from property 1 that there exists $x_0^* \in [-1,0]^n$ such that $x_0^* \in \partial\epsilon(x)$. Then $\epsilon(x) \leq \langle x_0^*, x - y\rangle + \epsilon(y)$ for all $y \in \mathbb{R}^n$, which implies

$$\epsilon(x) \leq \langle x_0^*, x\rangle - \langle x_0^*, y\rangle \quad \text{for all } y \in \epsilon^{-1}(0),$$

and therefore, by (5), $\epsilon(x) \leq \langle x_0^*, x\rangle - \epsilon^*(x_0^*)$, which, in view of (7), proves (6).

We will next prove that $\epsilon^{-1}(0) = P(v)$. Since ϵ is a convex (by property 1) and continuous function (as it is a finite-valued), by property 3 the set $\epsilon^{-1}(0)$ is closed and convex. Moreover, by (4), $\epsilon^{-1}(0) \subseteq P(v)$. To prove the reverse inclusion, suppose $x \notin \epsilon^{-1}(0)$. Consequently, $\epsilon(x) > 0$ and therefore, by properties 4 and 5 and Lemma 3.8, there exists $S \subseteq N$ such that $\langle -\mathbf{1}_S, y\rangle < \langle -\mathbf{1}_S, x\rangle$ for all $y \in \epsilon^{-1}(0)$. Since

$$\epsilon^*(-\mathbf{1}_S) = \max_{y\in\epsilon^{-1}(0)} \langle -\mathbf{1}_S, y\rangle < -x(S),$$

we obtain $v(S) = -\epsilon^*(-\mathbf{1}_S) > x(S)$ and hence $x \notin P(v)$. Thus, $\epsilon^{-1}(0) = P(v)$ is proved. Combining this equality with linear programming duality, we get

$$\begin{aligned}
\epsilon^*(x^*) &= \max\left\{\langle x^*, y\rangle : y \in \epsilon^{-1}(0)\right\} \\
&= \max\left\{\langle x^*, y\rangle : y \in P(v)\right\} \\
&= \max\left\{\langle x^*, y\rangle : -y(S) \leq -v(S),\ \forall S \subseteq N\right\} \\
&= \min\left\{-\sum_{S\subseteq N} \lambda_S v(S) : -\sum_{S\subseteq N} \lambda_S \mathbf{1}_S = x^*,\ \lambda_S \in \mathbb{R}_+^{2^N}\right\}
\end{aligned}$$

for all $x^* \in [-1,0]^n$. Thus, inserting the above into (3) yields

$$\begin{aligned}
\epsilon(x) &= \max_{x^*\in[-1,0]^n}\left\{\langle x^*, x\rangle - \min\left\{-\sum_{S\subseteq N} \lambda_S v(S) : -\sum_{S\subseteq N} \lambda_S \mathbf{1}_S = x^*,\ \lambda_S \in \mathbb{R}_+^{2^N}\right\}\right\} \\
&= \max\left\{\langle x^*, x\rangle + \sum_{S\subseteq N} \lambda_S v(S) : -\sum_{S\subseteq N} \lambda_S \mathbf{1}_S = x^* \in [-1,0]^n,\ \lambda_S \in \mathbb{R}_+^{2^N}\right\} \\
&= \max\left\{\left\langle -\sum_{S\subseteq N} \lambda_S \mathbf{1}_S, x\right\rangle + \sum_{S\subseteq N} \lambda_S v(S) : \sum_{S\subseteq N} \lambda_S \mathbf{1}_S \in [0,1]^n,\ \lambda_S \in \mathbb{R}_+^{2^N}\right\} \\
&= \max\left\{\sum_{S\subseteq N} \lambda_S\left[v(S) - x(S)\right] : \sum_{S\subseteq N} \lambda_S \mathbf{1}_S \in [0,1]^n,\ \lambda_S \in \mathbb{R}_+^{2^N}\right\}
\end{aligned}$$

for all $x \in \mathbb{R}^n$. Therefore, Proposition 3.1 implies that $\epsilon = \epsilon_v$.

It only remains to prove the uniqueness of a totally balanced game satisfying $\epsilon = \epsilon_v$. This follows from the fact that every totally balanced game is the minimum game of the collection of additive majorants, that is,

$$v(S) = \min\left\{x(S) : x \in P(v)\right\} = \min\left\{x(S) : x \in \epsilon^{-1}(0)\right\}$$

for all $S \subseteq N$, as this shows that v is determined by its least increment function ϵ. □

We end this section by showing how the core of a profit game can be expressed in terms of its indirect function.

Proposition 3.10. *Let $v : 2^N \to \mathbb{R}$ be a balanced profit game. Then*

$$C(v) = \{x \in \mathbb{R}^n : \{-\mathbf{1}_N, \mathbf{0}\} \subseteq \partial\epsilon_v(x)\}.$$

Proof. Suppose $x \in C(v)$. Then $x \in P(v) = \epsilon_v^{-1}(0)$ and hence

$$\epsilon_v(y) - \epsilon_v(x) = \epsilon_v(y) \geq 0 \text{ for all } y \in \mathbb{R}^n.$$

This gives $\mathbf{0} \in \partial\epsilon_v(x)$. Moreover, $x(N) - y(N) = v(N) - y(N) \leq z(N) - y(N)$ for all $y \in \mathbb{R}^n$ and $z \in P(v)$. Since

$$\epsilon_v(y) = \min\{z(N) - y(N) : z \in P(v),\ z \geq y\}$$

for all $y \in \mathbb{R}^n$, we obtain

$$x(N) - y(N) \leq \epsilon_v(y) \text{ for all } y \in \mathbb{R}^n.$$

Thus $\epsilon_v(y) - \epsilon_v(x) \geq \langle -\mathbf{1}_N, y - x\rangle$ for all $y \in \mathbb{R}^n$ and hence $-\mathbf{1}_N \in \partial\epsilon_v(x)$.

In order to prove the reverse inclusion, suppose $\mathbf{0} \in \partial\epsilon_v(x)$. Then, by Theorem 3.9,

$$\epsilon_v(x) = \min\{\epsilon_v(y) : y \in \mathbb{R}^n\} = 0.$$

Hence $x \in \epsilon_v^{-1}(0) = P(v)$. Now, if $-\mathbf{1}_N \in \partial\epsilon_v(x)$ then, taking $y \in C(v) \subset P(v) = \epsilon_v^{-1}(0)$, we have $0 = \epsilon_v(y) \geq x(N) - y(N) \geq v(N) - y(N) = 0$. We thus obtain $x(N) = y(N) = v(N)$, so that $x \in C(v)$. □

References

[1] Edmonds, J. (1970): Submodular functions, matroids and certain polyhedra, in R. K. Guy, H. Hanani, N. Sauer and J. Schönheim, eds., *Combinatorial Structures and Their Applications,* Gordon and Beach, New York, 69–87.

[2] Fujishige, S. (1991): *Submodular Functions and Optimization,* North-Holland, Amsterdam.

[3] Martínez-Legaz, J. E. (1996): Dual representation of cooperative games based on Fenchel-Moreau conjugation, *Optimization* 36, 291–319.

[4] Martínez-Legaz, J. E. (1998): A duality theory for totally balanced games, in *Proceedings of the IV Catalan Days of Applied Mathematics*, Universidad Rovira Virgili, Tarragona, 151–161.

[5] Martínez-Legaz, J. E. (1999): A New Characterization of Totally Balanced Games, in M. H. Wooders, ed., *Topics in Mathematical Economics and Game Theory. Essays in Honor of Robert J. Aumann*, American Mathematical Society, Providence, 83-88.

[6] Peleg, B. and P. Sudhölter (2003): *Introduction to the Theory of Cooperative Games*, Kluwer Academic Publishers, Boston.

[7] Rockafellar, R. T. (1970): *Convex Analysis*, Princeton University Press, Princeton.

[8] Shapley, L. S. (1971): Cores of convex games, *Int. J. Game Theory* 1, 11–26.

[9] Shapley, L. S. and M. Shubik (1969): On market games, *J. Econom. Theory* 1, 9–25.