

## THE SET OF PERIODS FOR THE MORSE–SMALE DIFFEOMORPHISMS ON $\mathbb{T}^2$

JUAN LUIS GARCÍA GUIRAO<sup>1</sup> AND JAUME LLIBRE<sup>2</sup>

ABSTRACT. In the present paper, by using Lefschetz zeta function, we study the set of periods of the Morse–Smale diffeomorphisms defined on the two–dimensional torus for every homotopy class.

### 1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In dynamical systems and, particularly in the study of the iteration of self–maps defined on a given compact manifold (i.e. in discrete dynamical systems), the periodic behavior plays an important role. These last thirty years there was a growing number of results showing that certain simple hypotheses force qualitative and quantitative properties (like the set of periods) of a system. Perhaps the best known result in this direction is the paper entitle “Period three implies chaos” for the interval continuous self–maps, see [12].

One of the most useful results for proving the existence of fixed points, or more generally of periodic points for a continuous self–map  $f$  of a compact manifold, is the Lefschetz Fixed Point Theorem and its improvements, see for instance [2, 6, 9, 13, 14, 15]. For studying the periodic points of  $f$  it is convenient to use the Lefschetz zeta function  $\mathcal{Z}_f(t)$  of  $f$ , which is a generating function of the Lefschetz numbers of all iterates of  $f$ . In Section 2 we provide a precise definition of all these notions.

Here we restrict our attention to the important class of discrete smooth dynamical systems defined by the *Morse–Smale diffeomorphisms*. In order to define them first we recall some basic definitions and notations.

Let  $\text{Diff}(\mathbb{M})$  be the space of  $C^1$  diffeomorphisms on a compact manifold  $\mathbb{M}$ . The set  $\text{Diff}(\mathbb{M})$  is a topological space endowed with the topology of

---

*Key words and phrases.* Morse–Smale diffeomorphism, Lefschetz number, zeta function, set of periods.

2000 Mathematics Subject Classification: 58F20.

The first author was supported in part by MCYT grant numbers MTM2005-03860, MTM2005-06098-C02-01, Fundación Séneca, grant number 00684-FI-04, and PAI06-0114 (JCCM). The second author was supported in part by MCYT grant number MTM2005-06098-C02-01 and by CICYT grant number 2005SGR 00550.

the supremum with respect to  $f$  and its differential  $Df$ . All the diffeomorphisms of this paper will be  $C^1$  diffeomorphisms.

We denote by  $f^m$  the  $m$ -th iterate of  $f \in \text{Diff}(\mathbb{M})$ . A point  $x \in \mathbb{M}$  is a *nonwandering point* of  $f$  provided that for any neighborhood  $\mathcal{U}$  of  $x$  there exists a positive integer  $m$  such that  $f^m(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$ . The set of nonwandering points of  $f$  is denoted by  $\Omega(f)$ .

Suppose that  $x \in \mathbb{M}$ . If  $f(x) = x$  and the derivative of  $f$  at  $x$ , denoted by  $Df_x$ , has spectrum disjoint from the unit circle, then  $x$  is called a *hyperbolic fixed point*. If all the eigenvalues of  $Df_x$  lie inside the unit circle, then  $x$  is called a *sink*. When all the eigenvalues have modulus greater than one,  $x$  is called a *source*. Otherwise  $x$  is called a *saddle*.

Suppose that  $y \in \mathbb{M}$ . If  $f^p(y) = y$ , then  $y$  is a *periodic point* of  $f$  of period  $p$  if  $f^j(y) \neq y$  for all  $0 \leq j < p$ . This  $y$  is a *hyperbolic periodic point* if  $y$  is a hyperbolic fixed point of  $f^p$ . The set  $\{y, f(y), \dots, f^{p-1}(y)\}$  is called the *periodic orbit* of the periodic point  $y$ .

Assume  $d$  is the metric on  $\mathbb{M}$  induced by the norm of the supremum, and  $p$  is a hyperbolic fixed point of  $f$ . The *stable manifold* of  $x$  is

$$W^s(x) = \{y \in \mathbb{M} : d(x, f^m(y)) \rightarrow 0 \text{ as } m \rightarrow \infty\},$$

and the *unstable manifold* of  $p$  is

$$W^u(x) = \{y \in \mathbb{M} : d(x, f^{-m}(y)) \rightarrow 0 \text{ as } m \rightarrow \infty\}.$$

For a hyperbolic periodic point  $x$  of period  $p$ , the stable and unstable manifolds are defined to be the stable and unstable manifolds of  $x$  under  $f^p$ .

A diffeomorphism  $f : \mathbb{M} \rightarrow \mathbb{M}$  is *Morse-Smale* if

- (1)  $\Omega(f)$  is finite,
- (2) all periodic points are hyperbolic,
- (3) for each  $x, y \in \Omega(f)$ ,  $W^s(x)$  and  $W^u(y)$  have a transversal intersection.

The first condition implies that  $\Omega(f)$  is the set of all periodic points of  $f$ .

Two diffeomorphisms  $f, g \in \text{Diff}(\mathbb{M})$  are  $C^1$  *equivalent* if and only if there exists a  $C^1$  diffeomorphism  $h : \mathbb{M} \rightarrow \mathbb{M}$  such that  $h \circ f = g \circ h$ . A diffeomorphism  $f$  is *structurally stable* provided that there exists a neighborhood  $\mathcal{U}$  of  $f$  in  $\text{Diff}(\mathbb{M})$  such that each  $g \in \mathcal{U}$  is topologically equivalent to  $f$ .

The class of Morse–Smale diffeomorphisms is structurally stable inside the class of all diffeomorphisms (see [17]), so to understand the dynamics of this class is a relevant problem.

During the last quarter of the XX century several papers were published analyzing the relationships between the dynamics of the Morse–Smale diffeomorphisms and the topology of the manifold where they are defined, see for instance [7, 8, 16, 19].

Morse–Smale diffeomorphisms have a relatively simple orbit structure. In fact, they have a finite set of periodic orbits, and this structure is preserved under small  $C^1$  perturbations. Franks [7] linked the periodic behavior of a Morse–Smale diffeomorphism to its action on the homology. For a given manifold and a homotopy (or isotopy) class of maps on this manifold, this result provides a necessary condition for the Morse–Smale dynamics which can occur in that homotopy (or isotopy) class. Narasimhan [16] showed for a compact surface that the homotopy class of the identity can exhibit a given Morse–Smale dynamics provided it satisfies Franks’ condition [7] and two other necessary properties. Essentially, the homotopy class of the identity on a compact surface can admit any periodic behavior consistent with the Lefschetz zeta function. Certainly, the hypothesis of working in the homology class of the identity is a restriction.

Batterson [3] for the orientation–preserving Morse–Smale diffeomorphisms on the two–dimensional torus  $\mathbb{T}^2$  extended the Narasimhan’s result, providing necessary and sufficient conditions on the set of periodic orbits for any homotopy class. For obtaining this result, he used the fact that the induced homology homomorphisms are *quasi-unipotent* (i.e., every eigenvalue of each homology homomorphism is a root of the unity) for a homotopy class admitting a Morse–Smale diffeomorphism, see Shub [18].

For orientation–reversing diffeomorphisms there are additional obstructions to the ones given by the Lefschetz zeta function. Blanchard and Franks [5] have shown that if an orientation–reversing homeomorphism on  $\mathbb{S}^2$  has periodic orbits having two distinct odd periods, then the topological entropy of this homeomorphism is positive. This implies that the orientation–reversing Morse–Smale diffeomorphisms on  $\mathbb{S}^2$  cannot have more than one odd periods. In [5] it was conjectured: “If  $f$  is an orientation–reversing homeomorphism of a compact oriented surface of genus  $g$ , and if  $f$  has orbits with  $g+2$  distinct odd periods, then the topological entropy of  $f$  is positive”. This result has been proved by Handel [10]. Thus an orientation–reversing Morse–Smale diffeomorphism on  $\mathbb{T}^2$

has orbits with at most two different odd periods. With this new information Batterson [4] provided necessary and sufficient conditions on the set of periodic orbits that can be exhibited by an orientation-reversing Morse–Smale diffeomorphisms on  $\mathbb{T}^2$ .

In this setting a natural question is the characterization of the set of periods of a Morse–Smale diffeomorphism defined on a compact manifold. Our aim in the present paper is to answer this question for the orientation-preserving and the orientation-reversing Morse–Smale diffeomorphisms on  $\mathbb{T}^2$ . Key tools for doing that are the results obtained by Franks [9] on the Lefschetz zeta function for  $C^1$  maps having only hyperbolic periodic points, and the previous results from Batterson on Morse–Smale diffeomorphisms on  $\mathbb{T}^2$ .

We introduce the following seven integer matrices

$$A_1 = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & r \\ 0 & -1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, A_4 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix},$$

$$A_5 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, A_6 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, A_7 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$

where  $r \in \{0, 1, 2, \dots\}$ . Any induced homology homomorphism  $f_{*1}$  on the first rational homology group of a Morse–Smale diffeomorphism  $f$  on  $\mathbb{T}^2$  is a matrix of  $\text{Gl}(2, \mathbb{Z})$  similar to one of the previous seven matrices, see Lemma 3 of [3] or Proposition 11. When  $f_{*1}$  is similar to the matrix  $A$  we write  $f_{*1} \approx A$ . We note that an orientation-reversing Morse–Smale diffeomorphisms on  $\mathbb{T}^2$  induces an homology homomorphism on the first rational homology group which is similar either to the matrix  $A_3$ , or to the matrix  $A_4$ . The other five matrices are induced by the orientation-preserving Morse–Smale diffeomorphisms of  $\mathbb{T}^2$ .

Our main results are the following.

**Theorem 1.** *Let  $f$  be an orientation-preserving Morse–Smale diffeomorphism on  $\mathbb{T}^2$ , and let  $f_{*1}$  be the induced homology homomorphism on its first rational homology group. We denote by  $S$  any finite set of positive integers, eventually  $S$  can be empty. Then*

$$\text{Per}(f) = \begin{cases} S & \text{if } f_{*1} \approx A_1, \\ S \cup \{1\} & \text{if } f_{*1} \approx A_2, \\ S \cup \{1, 2\} & \text{if } f_{*1} \approx A_5, \\ S \cup \{1, 3\} & \text{if } f_{*1} \approx A_6, \\ S \cup \{1, 2, 3\} & \text{if } f_{*1} \approx A_7. \end{cases}$$

Conversely, any set of the form  $S$ ,  $S \cup \{1\}$ ,  $S \cup \{1, 2\}$ ,  $S \cup \{1, 3\}$  and  $S \cup \{1, 2, 3\}$  is realizable as the set of periods for some orientation-preserving Morse–Smale diffeomorphism  $f$  on  $\mathbb{T}^2$  having  $f_{*1} \approx A_i$  for  $i$  equal to 1, 2, 5, 6 and 7, respectively.

**Theorem 2.** *Let  $f$  be an orientation-reversing Morse–Smale diffeomorphism on  $\mathbb{T}^2$ , and let  $f_{*1}$  be the induced homology homomorphism on its first rational homology group. We denote by  $S$  any finite set of even positive integers, eventually  $S$  can be empty. Let  $p$  and  $q$  be two arbitrary distinct odd positive integers. Then*

$$\text{Per}(f) = \begin{cases} \text{either} & S, \\ \text{or} & S \cup \{p\}, \\ \text{or} & S \cup \{p, q\}, \end{cases}$$

if  $f_{*1} \approx A_i$  for  $i = 3, 4$ . Conversely, any set of the form  $S$ ,  $S \cup \{p\}$  and  $S \cup \{p, q\}$  is realizable as the set of periods for some orientation-reversing Morse–Smale diffeomorphism  $f$  on  $\mathbb{T}^2$  having  $f_{*1} \approx A_i$  for  $i = 3, 4$ .

When two Morse–Smale diffeomorphisms  $f$  and  $g$  on  $\mathbb{T}^2$  are *homotopic*, we shall write  $g \simeq f$ . We define the *minimal set of periods* of the Morse–Smale diffeomorphism  $f$  to be the set

$$\text{MPer}_{\text{ms}}(f) = \bigcap_{g \simeq f} \text{Per}(g).$$

From Theorems 1 and 2 it follows immediately the next result.

**Corollary 3.** *Let  $f$  be a Morse–Smale diffeomorphism on  $\mathbb{T}^2$ , and let  $f_{*1}$  be the induced homology homomorphism on its first rational homology group. Then*

$$\text{MPer}_{\text{ms}}(f) = \begin{cases} \emptyset & \text{if } f_{*1} \approx A_i \text{ for } i = 1, 3, 4, \\ \{1\} & \text{if } f_{*1} \approx A_2, \\ \{1, 2\} & \text{if } f_{*1} \approx A_5, \\ \{1, 3\} & \text{if } f_{*1} \approx A_6, \\ \{1, 2, 3\} & \text{if } f_{*1} \approx A_7. \end{cases}$$

When two continuous self-maps  $f$  and  $g$  on  $\mathbb{T}^2$  are *homotopic*, we shall write  $g \simeq f$ . We define the *minimal set of periods* of  $f$  to be the set

$$\text{MPer}(f) = \bigcap_{g \simeq f} \text{Per}(g).$$

Using the results of [1] we obtain the following result.

**Proposition 4.** *Let  $f$  be a continuous self-map on  $\mathbb{T}^2$ , and let  $f_{*1}$  be the induced homology homomorphism on its first rational homology group. Then*

$$\text{MPer}(f) = \begin{cases} \emptyset & \text{if } f_{*1} \approx A_i \text{ for } i = 1, 3, 4, \\ \{1\} & \text{if } f_{*1} \approx A_i \text{ for } i = 2, 6, \\ \{1, 2\} & \text{if } f_{*1} \approx A_5, \\ \{1, 2, 3\} & \text{if } f_{*1} \approx A_7. \end{cases}$$

We remark that in six of the homotopy classes of Morse–Smale diffeomorphisms  $\text{MPer}_{\text{ms}}(f) = \text{MPer}(f)$ , but in the class for which  $f_{*1} \approx A_6$  we have

$$\text{MPer}_{\text{ms}}(f) = \{1, 3\} \supsetneq \text{MPer}(f) = \{1\}.$$

The rest of the paper is structured as follows. The Lefschetz zeta function is defined in Section 2. In Section 3 we present the basic results of Franks and Batterson that we shall use in the proofs of our results. Proposition 4 is proved in Section 4. Finally, Theorems 1 and 2 are proved in Section 5.

## 2. LEFSCHETZ ZETA FUNCTION

We will study the set of periods of the Morse–Smale diffeomorphisms on  $\mathbb{T}^2$  using the Lefschetz fixed point theory. The key work of Lefschetz in 1920's was to relate the homology class of a given map with an earlier work on the indices of Brouwer on the self-maps of compact manifolds. These two notions provide equivalent definitions for the Lefschetz numbers, and from their comparison, one obtains information about the existence of fixed points.

Given a continuous map  $f : \mathbb{M} \rightarrow \mathbb{M}$  on a compact  $n$ -dimensional manifold, its *Lefschetz number*  $L(f)$  is defined as

$$L(f) = \sum_{k=0}^n (-1)^k \text{trace}(f_{*k}),$$

where  $f_{*k} : H_k(\mathbb{M}, \mathbb{Q}) \rightarrow H_k(\mathbb{M}, \mathbb{Q})$  is the induced homomorphism by  $f$  on the  $k$ -th rational homology group of  $\mathbb{M}$ . For  $k = 0, \dots, n$ , the  $H_k(\mathbb{M}, \mathbb{Q})$  is a finite dimensional vector space over  $\mathbb{Q}$ , and the  $f_{*k}$  is a linear map given by a matrix with integer entries. One of the main results connecting the algebraic topology with the fixed point theory is the Lefschetz Fixed Point Theorem:

**Theorem 5.** *Let  $f : \mathbb{M} \rightarrow \mathbb{M}$  be a continuous map on a compact manifold and  $L(f)$  be its Lefschetz number. If  $L(f) \neq 0$  then  $f$  has a fixed point.*

For a proof of Theorem 5 see [6].

Our interest is to describe the set of periods of  $f$ . To this purpose, it is useful to have information on the whole sequence  $\{L(f^m)\}_{m=0}^{\infty}$  of the Lefschetz numbers of all the iterates of  $f$ . Thus we define the Lefschetz zeta function of  $f$  as

$$\mathcal{Z}_f(t) = \exp\left(\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m\right).$$

This function generates the sequence of all Lefschetz numbers, and it may be computed independently through [9]

$$(1) \quad \mathcal{Z}_f(t) = \prod_{k=0}^n \det(I_{n_k} - t f_{*k})^{(-1)^{k+1}},$$

where  $n = \dim \mathbb{M}$ ,  $n_k = \dim H_k(\mathbb{M}, \mathbb{Q})$ ,  $I_{n_k}$  is the  $n_k \times n_k$  identity matrix, and we take  $\det(I_{n_k} - t f_{*k}) = 1$  if  $n_k = 0$ .

By simple calculations using (1) we obtain the Lefschetz zeta functions for the different homotopy classes of Morse–Smale diffeomorphisms on  $\mathbb{T}^2$ :

**Proposition 6.** *Let  $f$  be a Morse–Smale diffeomorphism on  $\mathbb{T}^2$  such that its homology homomorphism  $f_{*1}$  is the matrix  $A_i$ . Then*

$$\mathcal{Z}_f(t) = \begin{cases} 1 & \text{if } f_{*1} \approx A_i \text{ with } i = 1, 3, 4, \\ \frac{(1+t)^2}{(1-t)^2} & \text{if } f_{*1} \approx A_2, \\ \frac{1+t^2}{(1-t)^2} & \text{if } f_{*1} \approx A_5, \\ \frac{1-t^3}{(1-t)^3} & \text{if } f_{*1} \approx A_6, \\ \frac{1+t^3}{(1-t)(1-t^2)} & \text{if } f_{*1} \approx A_7. \end{cases}$$

### 3. FRANKS AND BATTERSON RESULTS

If  $\gamma$  is a hyperbolic periodic orbit of period  $p$ , then for each  $x \in \gamma$  let  $E_x^u$  denotes the subspace of  $T_x \mathbb{M}$  generated by the eigenvectors of  $Df_x^p$  corresponding to eigenvalues whose moduli are greater than one. Let  $E_x^s$  be the subspace of  $T_x \mathbb{M}$  generated by the remaining eigenvectors. Define the orientation type  $\Delta$  of  $\gamma$  to be  $+1$  if  $Df_x^p : E_x^u \rightarrow E_x^u$  preserves orientation, and  $-1$  if it reverses orientation. Note that  $\Delta$  is well defined. The *index*  $u$  of  $\gamma$  is the dimension of  $E_x^u$  for some  $x \in \gamma$ . Finally, we define the triple  $(p, u, \Delta)$  associated to  $\gamma$ .

For a diffeomorphism having all its periodic orbits hyperbolic, its *periodic data* is defined to be the collection consisting of the triples  $(p, u, \Delta)$ , where the same triple can occur more than once provided it corresponds to different periodic orbits. Franks [7] proved the following result.

**Theorem 7.** *Let  $f$  be a  $C^1$  map on a compact manifold having finitely many periodic orbits all of which are hyperbolic, and let  $\Sigma$  be the period data of  $f$ . Then the Lefschetz zeta function of  $f$  satisfies*

$$\mathcal{Z}_f(t) = \prod_{(p,u,\Delta) \in \Sigma} (1 - \Delta t^p)^{(-1)^{u+1}}.$$

Using this notion of periodic data Batterson in [3] and [4] characterized the orientation-preserving and the orientation-reversing Morse–Smale diffeomorphisms on  $\mathbb{T}^2$ , respectively. His results can be stated as follow.

**Theorem 8.** *Let  $g \in \text{Diff}(\mathbb{T}^2)$  and suppose that  $g$  preserves the orientation. There exists a Morse–Smale diffeomorphism  $f$  on  $\mathbb{T}^2$  isotopic to  $g$  and with periodic data  $\{(p_i, u_i, \Delta_i)\}_{i=1}^n$  if and only if*

- (a)  $u_i = 0$  and  $u_j = 2$  for some  $i$  and  $j$ ,
- (b)  $\mathcal{Z}_g(t) = \prod_{i=1}^n (1 - \Delta_i t^{p_i})^{(-1)^{u_i+1}}$ ,
- (c)  $\Delta_i = +1$  for each  $i$  with  $u_i \in \{0, 2\}$ .

If  $p$  is an odd positive integer, then the collection  $\{(p, 0, +1), (p, 2, -1), (2p, 1, +1)\}$  is a *minimal data collection*. The minimal data collection associated to  $p$  contains the fewest numbers of periodic orbits possible in an orientation-reversing Morse–Smale diffeomorphism on  $\mathbb{T}^2$ .

**Theorem 9.** *Let  $g \in \text{Diff}(\mathbb{T}^2)$  and suppose that  $g$  reverses the orientation. There exists a Morse–Smale diffeomorphism  $f$  on  $\mathbb{T}^2$  isotopic to  $g$  and with periodic data  $\{(p_i, u_i, \Delta_i)\}_{i=1}^n$  if and only if*

- (a)  $u_i = 0$  and  $u_j = 2$  for some  $i$  and  $j$ ,
- (b)  $\mathcal{Z}_g(t) = 1 = \prod_{i=1}^n (1 - \Delta_i t^{p_i})^{(-1)^{u_i+1}}$ ,
- (c)  $\Delta_i = +1$  if  $u_i = 0$ , and  $\Delta_i = (-1)^{p_i}$  if  $u_i = 2$ ,
- (d)  $\{p_i\}_{i=1}^n$  contains at most two different odd numbers,
- (e) if  $g_{*1} \approx A_3$  then  $\{(p_i, u_i, \Delta_i)\}_{i=1}^n$  is not a minimal data collection.

#### 4. PROOF OF PROPOSITION 4

From now on let  $f$  be a Morse–Smale diffeomorphism defined on  $\mathbb{T}^2$ . Let  $f_{*1} : H_1(\mathbb{T}^2, \mathbb{Q}) \rightarrow H_1(\mathbb{T}^2, \mathbb{Q})$  denote the first induced homology homomorphism, corresponding to some  $2 \times 2$  matrix  $A$  of integers. Let  $d$  be  $\det(f_{*1})$  and  $t$  be the trace( $f_{*1}$ ).



The seven matrices of Proposition 11 represent exactly the different homology homomorphism  $f_{*1}$  of the homotopy classes of Morse–Smale diffeomorphisms on  $\mathbb{T}^2$ . We associate to each matrix its pair (trace, determinant) =  $(t, d)$ . Thus, we have  $A_1 \leftrightarrow (2, 1)$ ,  $A_2 \leftrightarrow (-2, 1)$ ,  $A_3 \leftrightarrow (0, -1)$ ,  $A_4 \leftrightarrow (0, -1)$ ,  $A_5 \leftrightarrow (0, 1)$ ,  $A_6 \leftrightarrow (-1, 1)$ ,  $A_7 \leftrightarrow (1, 1)$ .

The minimal set of periods for continuous self-maps of the two-dimensional torus are described in [1]. The set of periods are characterized up to homotopy by the following four subsets:

- (i) If  $t - d = 1$ , then  $\text{MPer}(f) = \emptyset$ ,
- (ii) If  $t \neq 0$  and  $t + d = -1$ , then  $\text{MPer}(f) = \{n : n \text{ is odd}\}$ ,
- (iii) If  $t + d = 0$  or  $t + d = -2$ , then  $\text{MPer}(f) = \mathbb{N} \setminus \{2\}$ ,
- (iv) If  $(t, d)$  does not satisfied (1)–(3), then  $\text{MPer}(f) = \mathbb{N}$ ,

and the following nine exceptional cases for  $(t, d)$  given in

$$\{(-2, 1), (-2, 2), (-1, 0), (-1, 1), (-1, 2), (0, 0), (0, 1), (0, 2), (1, 1)\}.$$

The minimal set of periods for these exceptional cases are described in Table 1.

$(t, d)$	$\text{MPer}(f)$
$(-2, 1)$	$\{1\}$
$(-2, 2)$	$\mathbb{N} \setminus \{2, 3\}$
$(-1, 0)$	$\{1\}$
$(-1, 1)$	$\{1\}$
$(-1, 2)$	$\mathbb{N} \setminus \{3\}$
$(0, 0)$	$\{1\}$
$(0, 1)$	$\{1, 2\}$
$(0, 2)$	$\mathbb{N} \setminus \{4\}$
$(1, 1)$	$\{1, 2, 3\}$

TABLE 1. The set  $\text{MPer}(f)$  in the nine exceptional cases.

Since Morse–Smale diffeomorphisms have a finite number of periodic points, the pairs  $(t, d)$  for the seven matrices must be either case (i), or in one of the nine exceptional cases. In fact, the matrices  $A_i$ 's for  $i \in \{1, 3, 4\}$  satisfy (i), and the matrices  $A_i$ 's for  $i \in \{2, 5, 6, 7\}$  belong to the exceptional cases.

From these previous results on the set  $\text{MPer}(f)$  it follows Proposition 4.

For the seven matrices of Proposition 11 we have the following result, taking into account that the Morse–Smale diffeomorphisms  $f : \mathbb{T}^2 \rightarrow \mathbb{T}^2$

with  $f_{*1} \approx A_i$  is a subset of all the continuous self-maps of  $\mathbb{T}^2$  with  $f_{*1} \approx A_i$

**Proposition 10.** *Let  $f$  be a Morse–Smale diffeomorphism on  $\mathbb{T}^2$  such that its homology homomorphism  $f_{*1}$  is the matrix  $A_i$ . Then*

$$\text{MPer}_{\text{ms}}(f) \supset \text{MPer}(f) = \begin{cases} \emptyset & \text{if } f_{*1} \approx A_i \text{ with } i = 1, 3, 4, \\ \{1\} & \text{if } f_{*1} \approx A_i \text{ with } i = 2, 6, \\ \{1, 2\} & \text{if } f_{*1} \approx A_5, \\ \{1, 2, 3\} & \text{if } f_{*1} \approx A_7. \end{cases}$$

Note that Corollary 3 improves Proposition 10.

## 5. PROOFS OF THEOREMS 1 AND 2

From now on let  $f$  be a Morse–Smale diffeomorphism defined on  $\mathbb{T}^2$ . It is well known that the induced homology homomorphisms of a homotopy class of Morse–Smale diffeomorphisms are quasi-unipotent, see [18]. The following result is proved in Lemma 3 of [3].

**Proposition 11.** *Let  $A \in \text{Gl}(2, \mathbb{Z})$  and suppose that both eigenvalues of  $A$  are roots of unity. The  $A$  is similar over the integer to exactly one of the matrices  $A_i$  for  $i \in \{1, \dots, 7\}$ .*

The seven matrices of Proposition 11 represent exactly the different homology homomorphism  $f_{*1}$  of the homotopy classes of Morse–Smale diffeomorphisms on  $\mathbb{T}^2$ .

*Proof of Theorem 1:* Let  $f$  be an orientation-preserving Morse–Smale diffeomorphism on  $\mathbb{T}^2$ , and let  $f_{*1}$  be the induced homology homomorphism on its first rational homology group.

First, we assume that  $f_{*1} \approx A_1$ . Clearly, since a Morse–Smale diffeomorphism has finitely many periodic orbits,  $\text{Per}(f)$  is equal to a finite set of positive integers eventually empty.

Now, we prove the converse, i.e. for a given set  $S$  empty or formed by finitely many positive integers we shall see that there exists an orientation-preserving Morse–Smale diffeomorphism  $f$  such that  $f_{*1} \approx A_1$  and  $\text{Per}(f) = S$ .

If  $S = \emptyset$  then we consider the diffeomorphism  $f$  on  $\mathbb{T}^2$  defined by the linear map  $A_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and the identification of  $\mathbb{T}^2$  with  $\mathbb{R}^2/\mathbb{Z}^2$ . Then from the results of [1] and [11], it follows that  $\text{Per}(f) = \emptyset$ .

Now we assume that  $S$  is not the empty set. We work with the Morse–Smale diffeomorphisms  $f$  such that  $f_{*1} \approx A_1$ , given by Theorem 8(a).

Then we have a such  $f$  with at least one sink and one source corresponding to the triples  $(p_1, 0, +1)$  and  $(p_2, 2, +1)$ , respectively; eventually  $p_1$  and  $p_2$  can coincide. So, by Proposition 6 and Theorem 8(b), the Lefschetz zeta function of  $f$ ,  $\mathcal{Z}_f(t)$ , is equal to 1 and must have at least the two factors

$$\frac{(1 - t^{p_1})(1 - t^{p_2})}{(1 - t^{p_1})(1 - t^{p_2})}.$$

Taking into account Theorem 8(b,c) and that a Morse–Smale diffeomorphism must have finitely many periodic orbits, a finite number of factors of the form

$$\frac{1 - t^p}{1 - t^p},$$

with  $p$  an arbitrary positive integer can appear in  $\mathcal{Z}_f(t) = 1$ . Therefore, by Theorem 8(b), in this case any finite set of positive integers can be the set of periods of  $f$ . Hence, Theorem 1 is proved when  $f_{*1} \approx A_1$ .

Suppose now that  $f_{*1} \approx A_2$ . Then, by Proposition 10 we have that  $1 \in \text{Per}(f)$ . Since a Morse–Smale diffeomorphism has finitely many periodic orbits,  $\text{Per}(f)$  is equal to  $S \cup \{1\}$ , where  $S$  is empty or a finite set of positive integers.

We prove the converse, i.e. for a given  $S \cup \{1\}$  where  $S$  is empty or formed by an arbitrary finite number of positive integers we shall see that there exists an orientation–preserving Morse–Smale diffeomorphism  $f$  such that  $f_{*1} \approx A_2$  and  $\text{Per}(f) = S \cup \{1\}$ . So, by Proposition 6, the Lefschetz zeta function of  $f$ ,  $\mathcal{Z}_f(t)$ , is equal to  $(1 + t)^2/(1 - t)^2$ . By Theorem 8(b,c) and since a Morse–Smale diffeomorphism has finitely many periodic orbits, a finite number of factors of the form  $(1 - t^p)/(1 - t^p)$ , with  $p$  an arbitrary positive integer can appear in  $\mathcal{Z}_f(t)$ . In short, by Theorem 8(b), in this case the set of periods contains the number 1 and eventually an arbitrary finite set of positive integers. Hence, Theorem 1 follows for  $f_{*1} \approx A_2$ .

The cases  $f_{*1} \approx A_5$  or  $f_{*1} \approx A_7$  follow using the same arguments than in the case  $f_{*1} \approx A_2$ .

Suppose now that  $f_{*1} \approx A_6$ . Then, by Proposition 10 we have that  $1 \in \text{Per}(f)$ . Moreover, by Proposition 6 and Theorem 7 we get that  $\{1, 3\} \in \text{Per}(f)$ . Since a Morse–Smale diffeomorphism has finitely many periodic orbits,  $\text{Per}(f)$  is equal to  $S \cup \{1, 3\}$ , where  $S$  is empty or a finite set of positive integers.

We prove the converse, i.e. for a given  $S \cup \{1, 3\}$  where  $S$  is empty or formed by an arbitrary finite number of positive integers we shall see that there exists an orientation–preserving Morse–Smale diffeomorphism

$f$  such that  $f_{*1} \approx A_6$  and  $\text{Per}(f) = S \cup \{1, 3\}$ . So, by Proposition 6, the Lefschetz zeta function of  $f$ ,  $\mathcal{Z}_f(t)$ , is equal to  $(1 - t^3)/(1 - t)^3$ . By Theorem 8(b,c) and since a Morse–Smale diffeomorphism has finitely many periodic orbits, a finite number of factors of the form  $(1 - t^p)/(1 - t^p)$ , with  $p$  an arbitrary positive integer can appear in  $\mathcal{Z}_f(t)$ . In short, by Theorem 8(b), in this case the set of periods contains the numbers 1 and 3, and eventually an arbitrary finite set of positive integers. Hence, Theorem 1 follows for  $f_{*1} \approx A_6$ . So the proof of Theorem 1 is done. ■

*Proof of Theorem 2:* Let  $f$  be an orientation–reversing Morse–Smale diffeomorphism on  $\mathbb{T}^2$ , and let  $f_{*1}$  be the induced homology homomorphism on its first rational homology group. By Handel’s Theorem (see [10] or the introduction), the set of periods of  $f$  contains at most two odd numbers.

Since a Morse–Smale diffeomorphism has finitely many periodic orbits (eventually the set of periodic orbits can be empty),  $\text{Per}(f)$  is equal to a finite set of positive integers or to the empty set, and since this set of periods contains at most two odd numbers the direct statement of Theorem 2 is proved.

We denote by  $S$  any non–empty finite set of even positive integers, or the empty set. Let  $p$  and  $q$  be two arbitrary distinct odd positive integers. Now, we shall prove the converse when  $f_{*1} \approx A_3$ ; i.e. for a given set  $S$ ,  $S \cup \{p\}$  and  $S \cup \{p, q\}$  we shall see that there exists an orientation–reversing Morse–Smale diffeomorphism  $f$  such that  $f_{*1} \approx A_3$  and  $\text{Per}(f)$  is either  $S$ , or  $S \cup \{p\}$ , or  $S \cup \{p, q\}$ , respectively.

First assume that  $S = \emptyset$ . Then, from the results of [1] and [11], the diffeomorphism  $f$  on  $\mathbb{T}^2$  defined by the linear map  $A_3 : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  and the identification of  $\mathbb{T}^2$  with  $\mathbb{R}^2/\mathbb{Z}^2$ , satisfies that  $\text{Per}(f) = \emptyset$ .

Now suppose that  $S \neq \emptyset$ . By Theorem 9(a) we have an  $f$  with  $f_{*1} \approx A_3$  and with at least one sink and one source corresponding to the triples  $(p_1, 0, +1)$  and  $(p_2, 2, +1)$ , respectively; eventually  $p_1$  and  $p_2$  can coincide. So, by Proposition 6 and Theorem 9(c), the Lefschetz zeta function of  $f$ ,  $\mathcal{Z}_f(t)$ , is equal to 1 and must have at least the two factors

$$\frac{(1 - t^{p_1})(1 + (-1)^{p_2} t^{p_2})}{(1 - t^{p_1})(1 + (-1)^{p_2} t^{p_2})}.$$

Taking into account Theorem 9(b,c) and that a Morse–Smale diffeomorphism has finitely many periodic orbits, a finite number of factors of the form

$$\frac{1 - t^{p_3}}{1 - t^{p_3}},$$

with  $p_3$  (either an arbitrary positive even integer, or  $p$ , or  $q$ ) can appear in  $\mathcal{Z}_f(t) = 1$ . Therefore, by Theorem 9(b,e), in this case any of the three sets  $S$ ,  $S \cup \{p\}$  and  $S \cup \{p, q\}$  can be the set of periods of  $f$ . Hence, Theorem 2 is proved when  $f_{*1} \approx A_3$ .

The proof of the converse in the case  $f_{*1} \approx A_4$  is similar. ■

#### REFERENCES

- [1] L. ALSEDÀ, S. BALDWIN, J. LLIBRE, R. SWANSON AND W. SZLENK, *Minimal sets of periods for torus maps via Nielsen numbers*, Pacific J. of Math. **169** (1995), 1–31.
- [2] I.K. BABENKO AND S.A. BAGATYI, *The behavior of the index of periodic points under iterations of a mapping*, Math. USSR Izvestiya **38** (1992), 1–26.
- [3] S. BATTERSON, *The dynamics of Morse–Smale diffeomorphisms on the torus*, Trans. Amer. Math. Soc. **256** (1979), 395–403.
- [4] S. BATTERSON, *Orientation-reversing Morse–Smale diffeomorphisms on the torus*, Trans. Amer. Math. Soc. **264** (1981), 29–37.
- [5] P. BLANCHARD AND J. FRANKS, *The dynamical complexity of orientation reversing homeomorphisms of surfaces*, Invent. Math., **62(2)** (1980/81), 333–339.
- [6] R.F. BROWN, *The Lefschetz fixed point theorem*, Scott, Foresman and Company, Glenview, IL, 1971.
- [7] J. FRANKS, *Some smooth maps with infinitely many hyperbolic periodic points*, Trans. Amer. Math. Soc. **226** (1977), 175–179.
- [8] J. FRANKS AND C. NARASIMHAN, *The periodic behavior of Morse–Smale diffeomorphisms*, Invent. Math. **48** (1978), 279–292.
- [9] J. FRANKS, *Homology and dynamical systems*, CBSM Regional Conf. Ser. in Math., no **49**, Amer Math. Soc., Providence, R.I., 1982.
- [10] M. HANDEL, *The entropy of orientation reversing homeomorphisms of surfaces*, Topology **21** (1982), 291–296.
- [11] B. JIANG AND J. LLIBRE, *Minimal sets of periods for torus maps*, Discrete and Continuous Dynamical Systems **4** (1998), 301–320.
- [12] T.Y. LI AND J. YORKE, *Period three implies chaos*, Amer. Math. Monthly **82** (1975), 985–992.
- [13] J. LLIBRE, *Lefschetz numbers for periodic points*, Contemporary Math. **152** Amer. Math. Soc., Providence, RI, (1993), 215–227.
- [14] J. LLIBRE AND R. SWANSON, *Periodic points for transversal maps on surfaces*, Houston J. of Math. **19** (1993), 395–403.
- [15] T. MATSUOKA, *The number of periodic points of smooth maps*, Erg. Th. & Dyn. Sys. **9** (1989), 153–163.
- [16] C. NARASIMHAN, *The periodic behavior of Morse–Smale diffeomorphisms on compact surfaces*, Trans. Amer. Math. Soc. **248** (1979), 145–169.
- [17] J. PALIS AND S. SMALE, *Structural stability theorems*, Proc. Sympos. Pure Math. **14**, Amer Math. Soc., Providence, R.I., 1970, 223–231.
- [18] M. SHUB, *Morse–Smale diffeomorphisms are unipotent on homology*, Dynamical systems (Proc. Sympos., Univ. Bahia, Salvador, 1971), Academic Press, New Yorke, 1973.
- [19] M. SHUB AND D. SULLIVAN, *Homology theory and dynamical systems*, Topology **14** (1975), 109–132.

<sup>1</sup> DEPARTAMENTO DE MATEMÁTICA APLICADA Y ESTADÍSTICA. UNIVERSIDAD POLITÉCNICA DE CARTAGENA, C/ PASEO ALFONSO XIII, 30203-CARTAGENA (REGIÓN DE MURCIA), SPAIN

*E-mail address:* `juan.garcia@upct.es`

<sup>2</sup>DEPARTAMENT DE MATEMÀTIQUES. UNIVERSITAT AUTÒNOMA DE BARCELONA, BELLATERRA, 08193-BARCELONA (CATALUNYA), SPAIN

*E-mail address:* `jllibre@mat.uab.es`