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Planar real polynomial differential systems of degree $n > 3$ having a weak focus of high order ‡

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Abstract. We construct planar polynomial differential systems of even (respectively odd) degree $n > 3$, of the form linear plus a nonlinear homogeneous part of degree n having a weak focus of order $n^2 - 1$ (respectively $\frac{n^2-1}{2}$) at the origin. As far as we know this provides the highest order known until now for a weak focus of a polynomial differential system of arbitrary degree n.

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1. Introduction and statement of the main result

For every $\alpha \in \mathbb{R}$ we consider a real homogeneous polynomial $f_\alpha(x, y)$ of degree $n - 1$ and the following real polynomial differential system

$$\frac{dx}{dt} = \dot{x} = -y(1 - f_\alpha(x, y)), \quad \frac{dy}{dt} = \dot{y} = x(1 - f_\alpha(x, y)),$$

which has the algebraic curve $\{f_\alpha = 1\}$ of singular points, and an isolated singularity at the origin, i.e. $f_\alpha(0, 0) \neq 1$. We perturb this system as follows

$$\begin{array}{rcl} \dot{x} &=& -y(1 - f_\alpha(x, y)) + P(x, y), \\ \dot{y} &=& x(1 - f_\alpha(x, y)) + Q(x, y), \end{array} \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are homogeneous polynomials of degree $n > 3$ with small real coefficients.

It is well–know that system (1) always has either a center or a weak focus at the origin (i.e. a monodromic singularity), see for instance [1, 12]. To distinguish a center from a focus is a classical difficult problem in the qualitative theory of ordinary differential equations in the plane, called the *center–focus problem*. This problem goes back to the 19th century, see for instance [4, 5, 7, 9, 11] and until now it has been object of an intensive research, see [10, 12, 14].

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A natural question is to know what is the maximal order of a weak focus of a polynomial differential system of degree $n > 1$. The answer to this question is only know for degree 2, i.e. the maximal order a weak focus of a polynomial differential system of degree 2 is 3, see [3]. For polynomial differential system of degree 3 it is known that such a maximal order must be larger than or equal to 11, see [15].

Suppose that we have the following analytic system defined in a neighborhood of the origin

$$\dot{x} = -y + \sum_{i=2}^{\infty} p_i(x,y), \quad \dot{y} = x + \sum_{i=2}^{\infty} q_i(x,y),$$

where p_i and q_i are homogeneous polynomials of degree $i \in \mathbb{N} \cup \{0\}$. We choose one–sided analytic transversal at the origin with a local analytic parameter h and represent the return map by an expansion $r(h) = h + \sum_{i=0}^{\infty} v_i h^i$. Observe that the stability of the singularity at the origin is clearly given by the sign of the first non–zero v_i, and if all the v_i are zero then the origin is a *center*, because all the orbits in a neighborhood are closed except the singular point. If the displacement function $\delta(h) = r(h) - h$ is not flat (i.e. there exists i such that its ith derivative $\delta^{(i)}(0) \neq 0$) we have a *weak focus*. We say that the origin is a *weak focus of order k* if $v_i = 0$ for each $i \leq 2k$ but $v_{2k+1} \neq 0$. Moreover, in Chapter 4 of [12] the author has been studied the cyclicity of this type of singularities and give a proof of the next interesting property: at most k limit cycles can bifurcate from a weak focus of order k under perturbation of the coefficients of $\sum_{i=2}^{\infty} p_i(x,y)$ and $\sum_{i=2}^{\infty} q_i(x,y)$. For more details about the definitions and statements of this paragraph see, for instance [12, 10].

We recall that a number $\alpha \in \mathbb{R}$ is $\mathbb{Q}-$*transcendental* if such α is not a root of a polynomial with coefficients in $\mathbb{Q}$.

The problem of determining the highest possible order of a weak focus is also one of the interesting challenges in this field. As far as we know the weak focus with the largest order for a polynomial differential system of even degree n is $n^2 - n$, this result is due to Bai and Liu [2]. Our main result is to provide a polynomial differential system of even degree n (respectively odd) having a weak focus of order $n^2 - 1$ (respectively $\frac{n^2-1}{2}$) at the origin. Our result improve all the previous known results for $n > 3$ with n even or odd.

Theorem 1 *Let $\alpha \in \mathbb{R}$ be $\mathbb{Q}-$transcendental.*

(a) For every $n = 2m > 2$ there exists $n+1$ real numbers $(\varepsilon_0, \ldots, \varepsilon_n) = (\varepsilon_0(\alpha), \ldots, \varepsilon_n(\alpha))$ such that the system

$$\begin{aligned} \dot{x} &= -y(1 - x^{n-1} - \alpha y^{n-1}) + \sum_{j=0}^{m} \varepsilon_{2j} x^{2j} y^{n-2j}, \\ \dot{y} &= x(1 - x^{n-1} - \alpha y^{n-1}) + \sum_{j=0}^{m-1} \varepsilon_{2j+1} x^{2j} y^{n-2j}, \end{aligned} \tag{2}$$

has a weak focus of order $n^2 - 1$ at the singular point located at the origin.

(b) For every $n = 2m+1 > 3$ there is $n+1$ real numbers $(\varepsilon_0, \ldots, \varepsilon_n) = (\varepsilon_0(\alpha), \ldots, \varepsilon_n(\alpha))$ such that the system

$$\begin{aligned} \dot{x} &= -y(1 - y^{n-1} - \alpha x^{n-2} y) + \sum_{j=0}^{m} \varepsilon_{2j} x^{2j} y^{n-2j}, \\ \dot{y} &= x(1 - y^{n-1} - \alpha x^{n-2} y) + \sum_{j=0}^{m} \varepsilon_{2j+1} x^{2j} y^{n-2j}, \end{aligned} \tag{3}$$

has a weak focus of order $\dfrac{n^2-1}{2}$ at the singular point located at the origin.

In fact for $n = 2$ and from Bautin [3] it follows that the maximum order of a weak focus of system (2) can be 3, and for $n = 3$ and from Vulpe and Sibirskiĭ[13] the maximum order of a weak focus of system (3) can be 5, see also [8].

The paper is organized as follows. In Section 2 we prove Theorem 1 under some additional assumptions. In Section 3 we present some auxiliary results in order to conclude the proof of Theorem 1 without the additional assumptions.

2. Proof of Theorem 1 under additional assumptions

In this section we prove Theorem 1 assuming that some determinants are not zero.

2.1. First case: $n = 2m > 2$

Consider $\varepsilon = (\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{R}^{n+1}$. Let $P_\varepsilon(x, y)$ (respectively $Q_\varepsilon(x, y)$) denote the real polynomial $\sum_{j=0}^{m} \varepsilon_{2j} x^{2j} y^{n-2j}$ (respectively $\sum_{j=0}^{m-1} \varepsilon_{2j+1} x^{2j} y^{n-2j}$).

By means of the polar change of variables $x = r\cos\theta$ and $y = r\sin\theta$ system (2) takes the form

$$\begin{aligned} \dot{r} &= r^n[\cos\theta P_\varepsilon(\cos\theta, \sin\theta) + \sin\theta Q_\varepsilon(\cos\theta, \sin\theta)], \\ \dot{\theta} &= f(r,\theta) + r^{n-1}[\cos\theta Q_\varepsilon(\cos\theta, \sin\theta) - \sin\theta P_\varepsilon(\cos\theta, \sin\theta)], \end{aligned}$$

where $f(r,\theta) = 1 - r^{n-1}(\cos^{n-1}\theta + \alpha \sin^{n-1}\theta)$. Moreover it is easy to check

$$\cos\theta P_\varepsilon(\cos\theta, \sin\theta) + \sin\theta Q_\varepsilon(\cos\theta, \sin\theta) = \sum_{k=0}^{n} \varepsilon_k g_k(\theta),$$

where

$$\begin{aligned} g_{2s}(\theta) &= \cos^{2s+1}\theta \sin^{n-2s}\theta, \quad \text{for all} \quad 0 \le s \le m, \\ g_{2t+1}(\theta) &= \cos^{2t}\theta \sin^{n-2t+1}\theta, \quad \text{for all} \quad 0 \le t \le m-1, \end{aligned} \tag{4}$$

and also that

$$\frac{dr}{d\theta} = \frac{r^n \displaystyle\sum_{k=0}^{n} \varepsilon_k g_k(\theta)}{f(r,\theta) + r^{n-1}[\cos\theta Q_\varepsilon(\cos\theta, \sin\theta) - \sin\theta P_\varepsilon(\cos\theta, \sin\theta)]}$$

satisfies

$$\frac{dr}{d\theta} = \frac{r^n \displaystyle\sum_{k=0}^{n} \varepsilon_k g_k(\theta)}{f(r,\theta)} + O(\varepsilon^2), \tag{5}$$

where $O(\varepsilon^2)$ means $O(\varepsilon_i\varepsilon_j)$ for all i, j.

We denote by $r(\theta, h, \alpha, \varepsilon)$ the analytic solution of (5) such that $r(0, h, \alpha, \varepsilon) = h$ for $h > 0$ sufficiently small. We expand $r(\theta, h, \alpha, \varepsilon)$ in power series of the variable h as

$$r(\theta, h, \alpha, \varepsilon) = h + \sum_{l=1}^{\infty} v_l(\theta, \alpha, \varepsilon) h^l.$$

Therefore from (5) by using that $r(2\pi, h, \alpha, \varepsilon) - h$ is the displacement function associated to system (2) in a neighborhood of the origin, we obtain that

$$r(2\pi, h, \alpha, \varepsilon) - h = F(h, \alpha, \varepsilon) + O(\varepsilon^2), \tag{6}$$

where

$$F(h, \alpha, \varepsilon) = \int_0^{2\pi} h^n \frac{\displaystyle\sum_{k=0}^{n} \varepsilon_k g_k(\theta)}{1 - h^{n-1}(\cos^{n-1}\theta + \alpha\ \sin^{n-1}\theta)} d\theta,$$

and $h > 0$ is small enough (in fact it is sufficient to take $|h| < \min_j\{|\varepsilon_j| : \varepsilon_j \neq 0\}$). In this context,

$$F(h, \alpha, \varepsilon) = \sum_{k=0}^{n} \varepsilon_k \sum_{j=0}^{\infty} h^{j(n-1)+n} \int_0^{2\pi} g_k(\theta)(\cos^{n-1}\theta + \alpha \sin^{n-1}\theta)^j d\theta.$$

Thus for each $h > 0$ small enough we obtain that

$$F(h, \alpha, \varepsilon) = \sum_{k=0}^{n} \varepsilon_k \sum_{i=1}^{\infty} h^{2i(n-1)+1} \int_0^{2\pi} g_k(\theta)(\cos^{n-1}\theta + \alpha \sin^{n-1}\theta)^{2i-1} d\theta,$$

and so

$$F(h, \alpha, \varepsilon) = \sum_{i=1}^{\infty} h^{2i(n-1)+1} \sum_{k=0}^{n} \varepsilon_k \int_0^{2\pi} g_k(\theta)(\cos^{n-1}\theta + \alpha \sin^{n-1}\theta)^{2i-1} d\theta, \tag{7}$$

because n is an even number and

$$\int_0^{2\pi} g_k(\theta)(\cos^{n-1}\theta + \alpha \sin^{n-1}\theta)^{2i} d\theta = 0,$$

for all $i \in \mathbb{N} \cup \{0\}$ as it is shown in Corollary 5 of Section 3.

From (6) and (7) it follows that

$$r(\theta, h, \alpha, \varepsilon) = h + \sum_{j=1}^{\infty} v_{j(n-1)+n}(\theta, \alpha, \varepsilon) h^{j(n-1)+n}. \tag{8}$$

Moreover by using (8) and (6) we have that

$$V_{2i(n-1)+1}(\alpha, \varepsilon) := v_{2i(n-1)+1}(2\pi, \alpha, \varepsilon) = \sum_{k=0}^{n} a_{i,k}(\alpha)\varepsilon_k + O(\varepsilon^2),$$

where

$$a_{i,k}(\alpha) = \int_0^{2\pi} g_k(\theta)(\cos^{n-1}\theta + \alpha \sin^{n-1}\theta)^{2i-1} d\theta, \tag{9}$$

and $g_k(\theta)$ denote the functions given in (4).

We consider the $(n+1)\times(n+1)$ matrix $(a_{i,k}(\alpha))$ where $1\le i\le n+1$ and $0\le k\le n$. If for some $\widetilde{\alpha}$ we can prove that

$$\det(a_{i,k}(\widetilde{\alpha}))\neq 0, \tag{10}$$

clearly there exists $(\widetilde{\alpha},\widetilde{\varepsilon})\in\mathbb{R}\times\mathbb{R}^{n+1}$ such that

$$\sum_{k=0}^{n} a_{i,k}(\widetilde{\alpha})\widetilde{\varepsilon}_k = -O(\widetilde{\varepsilon}^2) \quad 1\le i\le n,$$

and

$$\sum_{k=0}^{n} a_{i,k}(\widetilde{\alpha})\widetilde{\varepsilon}_k = 1-O(\widetilde{\varepsilon}^2) \quad i=n+1.$$

Therefore

$$\begin{array}{lcll} V_{2i(n-1)+1}(\widetilde{\alpha},\widetilde{\varepsilon}) & = & 0, & 1\le i\le n,\\ V_{2(n+1)(n-1)+1}(\widetilde{\alpha},\widetilde{\varepsilon}) & = & 1, & i=n+1. \end{array}$$

Hence as $2(n+1)(n-1)+1=2(n^2-1)+1$ statement (a) of Theorem 1 is done if (10) holds.

Remark 2 *If $n = 4$, by direct computation we obtain the explicit value of $\frac{1}{\pi^5}\det(a_{i,k}(\widetilde{\alpha}))$. More precisely this polynomial in α is equal to*

$$\frac{100442349\alpha^5(2261\alpha^8+9044\alpha^6-48642\alpha^4+9044\alpha^2+2261)(\alpha^2-1)^4}{590295810358705651712}.$$

2.2. Second case: $n=2m+1>3$.

Given any $\varepsilon=(\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_n)\in\mathbb{R}^{n+1}$, let $\tilde{P}_\varepsilon(x,y)$ (respectively $\tilde{Q}_\varepsilon(x,y)$) denote the polynomial $\sum_{j=0}^{m}\varepsilon_{2j}x^{2j}y^{n-2j}$ (respectively $\sum_{j=0}^{m}\varepsilon_{2j+1}x^{2j}y^{n-2j}$). By using $x=r\cos\theta$ and $y=r\sin\theta$ system (3) takes the form

$$\begin{array}{lcl} \dot{r} & = & r^n[\cos\theta\tilde{P}_\varepsilon(\cos\theta,\sin\theta)+\sin\theta\tilde{Q}_\varepsilon(\cos\theta,\sin\theta)],\\ \dot{\theta} & = & g(r,\theta)+r^{n-1}[\cos\theta\tilde{Q}_\varepsilon(\cos\theta,\sin\theta)-\sin\theta\tilde{P}_\varepsilon(\cos\theta,\sin\theta)], \end{array}$$

where $g(r,\theta)=1-r^{n-1}(\sin^{n-1}\theta+\alpha\sin\theta\cos^{n-2}\theta)$.

It is easy to check

$$\cos\theta\tilde{P}_\varepsilon(\cos\theta,\sin\theta)+\sin\theta\tilde{Q}_\varepsilon(\cos\theta,\sin\theta)=\sum_{k=0}^{n}\varepsilon_k f_k(\theta),$$

where

$$\begin{array}{lcll} f_{2s}(\theta) & = & \cos^{2s+1}\theta\sin^{n-2s}\theta, & \text{for all} \quad 0\le s\le m,\\ f_{2t+1}(\theta) & = & \cos^{2t}\theta\sin^{n-2t+1}\theta, & \text{for all} \quad 0\le t\le m, \end{array} \tag{11}$$

and also that

$$\frac{dr}{d\theta}=\frac{r^n\sum_{k=0}^{n}\varepsilon_k f_k(\theta)}{g(r,\theta)+r^{n-1}[\cos\theta\tilde{Q}_\varepsilon(\cos\theta,\sin\theta)-\sin\theta\tilde{P}_\varepsilon(\cos\theta,\sin\theta)]}$$

satisfies

$$\frac{dr}{d\theta} = \frac{r^n \displaystyle\sum_{k=0}^{n} \varepsilon_k f_k(\theta)}{g(r,\theta)} + O(\varepsilon^2). \tag{12}$$

We define the analytic solution $r(\theta, h, \alpha, \varepsilon)$ of (12) such that $r(0, h, \alpha, \varepsilon) = h$ for $h > 0$ sufficiently small. Therefore, from (12) by using that $r(2\pi, h, \alpha, \varepsilon) - h$ is the displacement function associated to system (3) we obtain that

$$r(2\pi, h, \alpha, \varepsilon) - h = F(h, \alpha, \varepsilon) + O(\varepsilon^2), \tag{13}$$

where

$$F(h, \alpha, \varepsilon) = \int_0^{2\pi} h^n \frac{\displaystyle\sum_{k=0}^{n} \varepsilon_k f_k(\theta)}{1 - h^{n-1}(\sin^{n-1}\theta + \alpha \sin\theta \cos^{n-2}\theta)} d\theta,$$

and $h > 0$ is small enough (as before it is sufficient to take $|h| < \min_j\{|\varepsilon_k| : \varepsilon_k \neq 0\}$). Furthermore

$$F(h, \alpha, \varepsilon) = \sum_{k=0}^{n} \varepsilon_k \sum_{j=0}^{\infty} h^{j(n-1)+n} \int_0^{2\pi} f_k(\theta)(\sin^{n-1}\theta + \alpha \sin\theta \cos^{n-2}\theta)^j d\theta.$$

Then

$$F(h, \alpha, \varepsilon) = \sum_{i=0}^{\infty} h^{(n-1)i+n} \sum_{k=0}^{n} \varepsilon_k \int_0^{2\pi} f_k(\theta)(\sin^{n-1}\theta + \alpha \sin\theta \cos^{n-2}\theta)^i d\theta.$$

This expansion of $F(h, \alpha, \varepsilon)$ and (13) implies that

$$r(\theta, h, \alpha, \varepsilon) = h + \sum_{j=1}^{\infty} v_{j(n-1)+n}(\theta, \alpha, \varepsilon) h^{j(n-1)+n}. \tag{14}$$

Moreover from (14) we obtain that

$$V_{i(n-1)+n}(\alpha, \varepsilon) := v_{i(n-1)+n}(2\pi, \alpha, \varepsilon) = \sum_{k=0}^{n} b_{i,k}(\alpha)\varepsilon_k + O(\varepsilon^2),$$

where

$$b_{i,k}(\alpha) = \int_0^{2\pi} f_k(\theta)(\sin^{n-1}\theta + \alpha \sin\theta \cos^{n-2}\theta)^i d\theta, \tag{15}$$

and the functions $f_k(\theta)$ are as in (11).

We consider the $(n+1) \times (n+1)$ matrix $(b_{i,k}(\alpha))$ where $0 \leq i, k \leq n$. If for some $\widetilde{\alpha}$ we can prove that

$$\det(b_{i,k}(\widetilde{\alpha})) \neq 0, \tag{16}$$

clearly there exists $(\widetilde{\alpha}, \widetilde{\varepsilon}) \in \mathbb{R} \times \mathbb{R}^{n+1}$ such that

$$\sum_{k=0}^{n} b_{i,k}(\widetilde{\alpha})\widetilde{\varepsilon}_k = -O(\widetilde{\varepsilon}^2) \quad 0 \leq i \leq n-1,$$

and

$$\sum_{k=0}^{n} b_{i,k}(\widetilde{\alpha})\widetilde{\varepsilon}_k = 1 - O(\widetilde{\varepsilon}^2) \quad i = n.$$

Therefore,

$$\begin{array}{rclll} V_{i(n-1)+n}(\widetilde{\alpha},\widetilde{\varepsilon}) & = & 0, & & 0\le i\le n-1, \\ V_{n(n-1)+n}(\widetilde{\alpha},\widetilde{\varepsilon}) & = & 1, & & i=n. \end{array}$$

Thus, as $n(n-1)+n = 2\left(\dfrac{n^2-1}{2}\right)+1$, statement (b) of Theorem 1 is done if (16) holds.

Remark 3 *If $n=5$, by direct computations we obtain the following equality*

$$\frac{1}{\pi^6}\det(b_{i,k}(\alpha)) = -\frac{5\alpha^5(\alpha^2+7)(\alpha^2-8)^4}{288230376151711744}.$$

3. Auxiliary results

In this section we shall prove (10) and (16).

Lemma 4 *For all p and q belong to $\mathbb{N}\cup\{0\}$, we denote by $I[p \mid q] = \int_0^{2\pi}\cos^p\theta\sin^q\theta d\theta$.*

(a) For all $p,q\in\mathbb{N}\cup\{0\}$ the numbers $I[2p+1\mid q]$ and $I[p\mid 2q+1]$ are zero.

(b) If $p,q\in\mathbb{N}\cup\{0\}$ then, $\frac{1}{\pi}I[2p\mid 2q]$ is a rational number.

Proof. Since the integrant function of $I[2p+1\mid q]$ and $I[p\mid 2q+1]$ is odd, statement (a) follows. In order to show statement (b) we first prove a reduced form of this second result. We claim that for each $m\in\mathbb{N}$,

$$\frac{1}{\pi}I[2m\mid 0] = \frac{1}{\pi}\int_0^{2\pi}\cos^{2m}\theta d\theta\in\mathbb{Q}. \tag{17}$$

In fact this claim follows from the indefinite integral 2.513-3 of [6] (see also 2.512-2) which say that

$$\int\cos^{2m}x dx = \frac{1}{2^{2m}}\left(\begin{array}{c}2m\\ m\end{array}\right)x + \frac{1}{2^{2m-1}}\sum_{j=0}^{m-1}\left(\begin{array}{c}2m\\ j\end{array}\right)\frac{\sin((2m-2j)x)}{2m-2j}.$$

This proves (17).

Since

$$\begin{array}{rcl} \dfrac{1}{\pi}I[2p\mid 2q] & = & \dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\cos^{2p}\theta(1-\cos^2\theta)^q d\theta, \\ & = & \dfrac{1}{\pi}\displaystyle\sum_{j=0}^{q}(-1)^j\left(\begin{array}{c}q\\ j\end{array}\right)I[2p+2j\mid 0], \end{array}$$

by using (17) we conclude the proof of statement (b). $\square$

Corollary 5 *Suppose that $n=2m$ and $g_k(\theta)$ satisfy (4). For every $\alpha\in\mathbb{R}$ the numbers*

$$\ell_k(\alpha,2i,n) := \int_0^{2\pi} g_k(\theta)(\cos^{n-1}\theta+\alpha\sin^{n-1}\theta)^{2i}d\theta$$

are zero, for any $i\in\mathbb{N}\cup\{0\}$ and $0\le k\le n$.

Proof. It is easy to see that

$$(\cos^{n-1}\theta + \alpha\sin^{n-1}\theta)^{2i} = \sum_{j=0}^{2i} \left(\begin{array}{c} 2i \\ j \end{array} \right) (\cos^{2m-1}\theta)^{2i-j}(\sin^{2m-1}\theta)^j\alpha^j.$$

Thus from (4) we obtain that

$$\ell_{2s} = \sum_{j=0}^{2i} \left(\begin{array}{c} 2i \\ j \end{array} \right) I[(2m-1)(2i-j)+2s+1 \mid (2m-1)j+n-2s]\alpha^j,$$

for all $0 \le s \le m$ and

$$\ell_{2t+1} = \sum_{j=0}^{2i} \left(\begin{array}{c} 2i \\ j \end{array} \right) I[(2m-1)(2i-j)+2t \mid (2m-1)j+n-2t+1]\alpha^j,$$

for all $0 \le t \le m-1$. Therefore, by Lemma 4 we conclude that $\ell_k(\alpha, 2i, n) = 0$. □

Remark 6 *If $n = 2m+1$ is odd there exists some $\alpha \in \mathbb{R}$ for which (11) does not imply a similar result of Corollary 5, because*

$$\ell_{2t+1}(\alpha, 2i, n) = \int_0^{2\pi} f_{2t+1}(\theta)(\sin^{n-1}\theta + \alpha\sin\theta\cos^{n-2}\theta)^{2i}d\theta \neq 0,$$

when $i \in \mathbb{N}$.

Before to present the next lemma we recall the formula 14.134 of [6] which claims: for each finite set $\{x_1, x_2, \ldots, x_N\}$ the N^{th}–order *Vandermonde's determinant* is

$$\det \left(\begin{array}{ccccc} 1 & x_1 & (x_1)^2 & \cdots & (x_1)^{N-1} \\ 1 & x_2 & (x_2)^2 & \cdots & (x_2)^{N-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & (x_N)^2 & \cdots & (x_N)^{N-1} \end{array} \right) = \prod_{1\le i<j\le N} (x_j - x_i), \qquad (18)$$

where the right–hand side is the continued product of all the differences that can be formed from the $\frac{1}{2}N(N-1)$ pairs of numbers taken from $x_1, x_2, \ldots, x_N$ with the order of the differences taken in the reverse order of the subindices that are involved.

We denote by $\triangle(x_1, x_2, \ldots, x_N)$ the determinant given in (18).

Lemma 7 *If $N = 2m > 2$ and for all $j = 2, 3, \ldots, N$, the differences $x_j - x_{j-1} = \ell > 0$ are a positive constant, then*

$$\triangle(x_2, x_3, \ldots, x_{2m}) = \triangle(x_1, x_2, \ldots, x_{2m-1})$$

and for every $1 < s \le m$ the $(2m-1)^{th}$–order Vandermonde's determinant

$$\triangle(x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_{m+1}, \ldots, x_{2m})$$

is equal to

$$\triangle(x_1, \ldots, x_{m-1}, \ldots, x_{2m-s}, x_{2m-s+2}, \ldots, x_{2m}).$$

Proof. When $m = 2$ we will show that

$$\triangle(x_2, x_3, x_4) = \triangle(x_1, x_2, x_3) \qquad (19a)$$

$$\triangle(x_1, x_3, x_4) = \triangle(x_1, x_2, x_4) \qquad (19b)$$

as long as $x_4 - x_3 = x_3 - x_2 = x_2 - x_1 = \ell > 0$.

From (18) is not difficult to check that $\triangle(x_1, x_2, x_3)$ and $\triangle(x_2, x_3, x_4)$ are equal to $\ell(2\ell^2)$. Thus, (19a) holds. In order to prove (19b) we consider again the right–hand side of (18), so $\triangle(x_1, x_3, x_4)$ is equal to the product of the differences

$$\begin{array}{ll} x_3 - x_1, & x_4 - x_1, \\ & x_4 - x_3. \end{array}$$

In a similar way we obtain that $\triangle(x_1, x_2, x_4)$ is equal to the product of the differences

$$\begin{array}{ll} x_4 - x_2, & x_4 - x_1, \\ & x_2 - x_1. \end{array}$$

Therefore by using that $x_i - x_j = (i - j)\ell$ we obtain (19b) and conclude the proof when $m = 2$.

In the general case (18) implies that both $(2m-1)^{th}$−order determinants $\triangle(x_2, x_3, \ldots, x_{2m})$ and $\triangle(x_1, x_2, \ldots, x_{2m-1})$ are equal to $\prod_{p=1}^{2m-2} p\,!\ell^p$, because $x_i - x_j = (i-j)\ell$. This proves the first part of the lemma. In order to conclude we consider $1 < s \le m$, thus (18) shows that

$$\triangle(x_1, \ldots, x_{s-1}, x_{s+1}, \ldots, x_{m+1}, \ldots, x_{2m})$$

is equal to the product of the following $(2m-1)(m-1)$ differences

$$\begin{array}{cccccc} x_2 - x_1, & \cdots & x_{s-1} - x_1, & x_{s+1} - x_1, & \cdots & x_{2m} - x_1, \\ & \cdots & x_{s-1} - x_2, & x_{s+1} - x_2, & \cdots & x_{2m} - x_2, \\ & & \vdots & \vdots & \ddots & \vdots \\ & & x_{s-1} - x_{s-2}, & x_{s+1} - x_{s-2}, & \cdots & x_{2m} - x_{s-2}, \\ & & & x_{s+1} - x_{s-1}, & \cdots & x_{2m} - x_{s-1}, \\ & & & & \ddots & \vdots \\ & & & & & x_{2m} - x_{2m-2}, \\ & & & & & x_{2m} - x_{2m-1}, \end{array}$$

where for each $j = 2, 3, \ldots, 2m$ all the differences $(x_j - x_i)$ with $i = 1, 2, \ldots, j-1$ defines the $j - th$ column. In a similar way, by using again (18) we have that

$$\triangle(x_1, \ldots, x_{m-1}, \ldots, x_{2m-s}, x_{2m-s+2}, \ldots, x_{2m}).$$

is equal to the product of

$$\begin{array}{cccccc} x_{2m} - x_{2m-1}, & \cdots & x_{2m} - x_{2m-s+2}, & x_{2m} - x_{2m-s}, & \cdots & x_{2m} - x_1, \\ & \cdots & x_{2m-1} - x_{2m-s+2}, & x_{2m-1} - x_{2m-s}, & \cdots & x_{2m-1} - x_1, \\ & & \vdots & \vdots & \ddots & \vdots \\ & & x_{2m-s+3} - x_{2m-s+2}, & x_{2m-s+3} - x_{2m-s}, & \cdots & x_{2m-s+3} - x_1, \\ & & & x_{2m-s+2} - x_{2m-s}, & \cdots & x_{2m-s+2} - x_1, \\ & & & & \ddots & \vdots \\ & & & & & x_3 - x_1, \\ & & & & & x_2 - x_1, \end{array}$$

but in this case, for each $j = 2m, 2m-1, \ldots, 2$ all the differences $x_j - x_i$ with $i = 1, 2, \ldots, j-1$ defines the $j - th$ row.

Since $x_i - x_j = (i - j)\ell = [(2m - j) - (2m - i)]\ell = x_{2m-j} - x_{2m-i}$ a direct computation give the lemma because both tables are the same. $\square$

3.1. Even case

This subsection is devoted to prove (10) when $n = 2m > 2$ and consequently the proof of statement (a) of Theorem 1 will be done without any additional assumption. We assume that $m \geq 3$, because the case $m = 2$ was shown in Remark 2.

At the end of this subsection we present a proposition whose proof needs some preparatory lemmas. In this context, for every $i \in \mathbb{N}$ and $0 \leq k \leq n = 2m$, $a_{i,k}(\alpha)$ is as in (9). The Newton binomial implies that

$$a_{i,2s}(\alpha) = \sum_{j=0}^{2i-1} \binom{2i-1}{j} I[(n-1)(2i-1-j) + 2s + 1 \mid (n-1)j + n - 2s]\alpha^j, \quad (20)$$

for all $0 \leq s \leq m$, and also that

$$a_{i,2t+1}(\alpha) = \sum_{j=0}^{2i-1} \binom{2i-1}{j} I[(n-1)(2i-1-j) + 2t \mid (n-1)j + n - 2t + 1]\alpha^j, \quad (21)$$

for all $0 \leq t \leq m-1$.

For each $1 \leq i \leq n+1$, Lemma 4 implies that $a_{i,2s}(\alpha)$ (resp. $a_{i,2t+1}(\alpha)$) is a polynomial in α which is make up by monomials of even (resp. odd) degree. In particular, $\frac{d}{d\alpha} a_{i,2s}(\alpha)|_{\alpha=0}$ and $a_{i,2t+1}(0)$ are zero.

Lemma 8 *Set $n = 2m > 4$. For every $1 \leq i \leq n+1$ and $0 \leq k \leq n$ consider $a_{i,k}(\alpha)$ as in (9). We define the $(2m+1) \times (2m+1)$ matrix $(\overline{a}_{i,k})$ by the following rules: $\overline{a}_{i,k} = a_{i,k}(0)$ if k is even; $\overline{a}_{i,k} = \frac{d}{d\alpha} a_{i,k}(\alpha)|_{\alpha=0}$ if $k \neq 2m-1$ is odd; $\overline{a}_{1,2m-1} = 0$ and*

$$\overline{a}_{i,2m-1} = \binom{2i-1}{3} I[(2m-1)(2i-3) - 1 \mid 6m] +$$

$$-\frac{2m+1}{2m-1} \binom{2i-1}{2} I[(2m-1)(2i-3) + 1 \mid 6m-2],$$

for all $2 \leq i \leq 2m+1$. Then the coefficient of α^{m+2} of the polynomial $\det(a_{i,k}(\alpha))$, in the variable α, is $\det(\overline{a}_{i,k})$.

Proof. From (21), we have that $a_{i,2t+1}(0) = 0$ and

$$\frac{d}{d\alpha} a_{i,2t+1}(\alpha)|_{\alpha=0} = (2i-1) I[(2m-1)(2i-2) + 2t \mid 4m - 2t],$$

as long as $0 \leq t \leq m-1$ and $1 \leq i \leq 2m+1$. In a similar way, (20) implies that

$$a_{i,2s}(0) = I[(2m-1)(2i-1) + 2s + 1 \mid 2m - 2s],$$

for all $0 \leq s \leq m$ and $1 \leq i \leq 2m+1$.

By using $t = m-1$ and $s = 0$ we obtain that

$$\frac{d}{d\alpha} a_{i,2m-1}(\alpha)|_{\alpha=0} = \frac{2m+1}{2m-1} a_{i,0}(0). \quad (22)$$

This is a direct consequence of the indefinite integral 2.510 of [6] i.e.

$$\int \cos^p \theta \sin^q \theta d\theta = -\frac{\cos^{p+1}\theta \sin^{q-1}\theta}{p+1} + \frac{q-1}{p+1} \int \cos^{p+2}\theta \sin^{q-2}\theta d\theta. \quad (23)$$

From (22) is not difficult to check that

$$a_{i,2m-1}(\alpha) - \alpha \frac{2m+1}{2m-1} a_{i,0}(\alpha) = \alpha^3 \overline{a}_{i,2m-1} + \text{'higher order terms'}.$$

Since α is a common factor of each even column, by using elementary column operations we have that $\det(a_{i,k}(\alpha))$ is divided by $\alpha^{m-1}\alpha^3$. Moreover the coefficient of α^{m+2} in the polynomial $\det(a_{i,k}(\alpha))$ is $\det(\overline{a}_{i,k})$. This proves the lemma. $\square$

Observe that in the polynomial $\det(a_{i,k}(\alpha))$ given in Lemma 8 the coefficient for every α^j is zero when $j < m+2$.

Lemma 9 *Let $\overline{a}_{i,k}$ be as in Lemma 8. If $1 \le i \le 2m+1$ for all $0 \le k \le 2m$ the numbers $\overline{a}_{i,k}$ are divided by $\overline{a}_{i,1} \neq 0$ (a common factor of the ith row). Therefore there exists a $(2m+1)\times(2m+1)$ matrix $(c_{i,k})$ such that $\det(\overline{a}_{i,k}) = \det(c_{i,k}) \prod_{i=1}^{2m+1} \overline{a}_{i,1}$, where for each $1 \le i \le 2m+1$ we have that $\overline{a}_{i,1}(c_{i,0}, 1, c_{i,2}, \dots, c_{i,2m}) = (\overline{a}_{i,0}, \overline{a}_{i,1}, \overline{a}_{i,2}, \dots, \overline{a}_{i,2m})$.*

Proof. We will use again the indefinite integral 2.510 of [6] but this time written as

$$\int \cos^p\theta \sin^q\theta d\theta = \frac{\cos^{p-1}\theta \sin^{q+1}\theta}{q+1} + \frac{p-1}{q+1}\int \cos^{p-2}\theta \sin^{q+2}\theta d\theta. \tag{24}$$

Consider $k = 2t+1$ and $t \neq m-1$, the definition of $\overline{a}_{i,j}$ given in Lemma 8 and (21) with $j=0$ imply that

$$\overline{a}_{i,2t+1} = (2i-1)I[(2m-1)(2i-2)+2t \mid 4m-2t].$$

So from (24) it follows that

$$\overline{a}_{i,2t+1} = \frac{(2m-1)(2i-2)+2t-1}{4m-2t+1}\overline{a}_{i,2(t-1)+1},$$

for each $1 \le t \le m-2$. By using this $t > 0$ times, we have that

$$\overline{a}_{i,2t+1} = \overline{a}_{i,1}\prod_{j=0}^{t-1}\frac{(2m-1)(2i-2)+2t-2j-1}{4m-2t+2j+1},$$

for all $1 \le t \le m-2$. Therefore

$$c_{i,2t+1} = \prod_{j=0}^{t-1}\frac{(2m-1)(2i-2)+2t-2j-1}{4m-2t+2j+1}. \tag{25}$$

as long as $1 \le t \le m-2$ and $1 \le i \le 2m+1$. Hence we obtain the definitions of $m-1$ columns of $(c_{i,k})$, because $c_{i,1} = 1$, for all $1 \le i \le 2m+1$.

In a similar way we can consider all the even columns. As $\overline{a}_{i,2s} = a_{i,2s}(0)$ the equation (20) (with $j=0$) implies that $\overline{a}_{i,2s} = I[(2m-1)(2i-1)+2s+1 \mid 2m-2s]$. So from (24) we have that

$$\overline{a}_{i,2s} = \frac{(2m-1)(2i-1)+2s}{2m-2s+1}\ \overline{a}_{i,2(s-1)},$$

for every $1 \le s \le m$. This implies that for each $1 \le s \le m$ and $1 \le i \le 2m+1$

$$\overline{a}_{i,2s} = \overline{a}_{i,0}\prod_{j=1}^{s}\frac{(2m-1)(2i-1)+2s-2j+2}{2m-2s+2j-1}, \tag{26}$$

where

$$\overline{a}_{i,0} = I[(2m-1)(2i-1)+1 \mid 2m].$$

Furthermore as

$$\overline{a}_{i,1} = (2i-1)I[(2m-1)(2i-1)+1-2m \mid 2m+2m], \tag{27}$$

from (24) it follows that

$$\overline{a}_{i,0} = \overline{a}_{i,1}\frac{1}{2i-1}\prod_{j=0}^{m-1}\frac{(2m-1)(2i-1)-2j}{2m+2j+1}.$$

Thus expanding the first factor in this product

$$\overline{a}_{i,0} = \overline{a}_{i,1}\frac{2m-1}{2m+1}\prod_{j=1}^{m-1}\frac{(2m-1)(2i-1)-2j}{2m+2j+1}. \tag{28}$$

This shows that

$$c_{i,0} = \frac{2m-1}{2m+1}\prod_{j=1}^{m-1}\frac{(2m-1)(2i-1)-2j}{2m+2j+1}, \tag{29}$$

for all $1 \le i \le 2m+1$.

By using (26) equation (28) shows that

$$c_{i,2s} = \frac{2m-1}{2m+1}\prod_{j=1}^{m-1}\frac{(2m-1)(2i-1)-2j}{2m+2j+1}\prod_{j=1}^{s}\frac{(2m-1)(2i-1)+2s-2j+2}{2m-2s+2j-1}, \tag{30}$$

for every $1 \le s \le m$ and $1 \le i \le 2m+1$. Therefore if k is even the definition of the column $(c_{i,k})$ follows.

Now we consider the last odd column. Since $\overline{a}_{1,2m-1} = 0$, we consider $i \ge 2$, thus by using (23) in the definition of $\overline{a}_{i,2m-1}$ given in Lemma 8 we have that

$$\overline{a}_{i,2m-1} = \frac{-4}{6m-3}(2i-1)(i-1)I[(2m-1)(2i-3)+1 \mid 6m-2],$$

for all $2 \le i \le 2m+1$. Again from (23) and the form of $\overline{a}_{i,1}$ (just as in (27)) we obtain that $\overline{a}_{i,2m-1}$ is equal to the product of $C := I[(2m-1)(2i-1)+1-2m \mid 4m]$ with

$$\frac{-4}{6m-3}(2i-1)(i-1)\prod_{j=0}^{m-2}\frac{6m-2-2j-1}{(2m-1)(2i-3)+1+2j+1}.$$

But $\overline{a}_{i,1} = (2i-1)C$, so for all $2 \le i \le 2m+1$ we obtain that

$$\overline{a}_{i,2m-1} = -\overline{a}_{i,1}\frac{4(i-1)}{6m-3}\prod_{j=0}^{m-2}\frac{6m-2j-3}{(2m-1)(2i-3)+2j+2}.$$

Therefore

$$c_{i,2m-1} = \frac{-4(i-1)}{6m-3}\prod_{j=0}^{m-2}\frac{6m-2j-3}{(2m-1)(2i-3)+2j+2}. \tag{31}$$

This conclude the proof because $\overline{a}_{1,2m-1} = 0 =: c_{1,2m-1}$. □

Remark 10 *For all $m \ge 2$, (28) implies that there is no a constant $\ell \ne 0$ such that for all $1 \le i \le 2m+1$, $\overline{a}_{i,0} = \ell\overline{a}_{i,1}$ with ℓ independent of the subindex i. Furthermore for each $1 \le i \le 2m+1$ the numbers $\overline{a}_{i,1}$ are different from zero, because*

$$\overline{a}_{i,1} = (2i-1)\prod_{j=0}^{2m-1}\frac{4m-2j-1}{(4m-2)(i-1)+2j+1}I[(4m-2)i+2 \mid 0]$$

(this last equality can be obtained from (27) by using (23)).

For each $0 \leq k \leq 2m-1$ the kth column $(c_{i,k})$ of the matrix of Lemma 9 has a common factor, for instance from (29) it follows

$$c_{i,0} = \left(\frac{2m-1}{2m+1} \prod_{j=1}^{m-1} \frac{1}{2m+2j+1} \right) \prod_{j=1}^{m-1} [(2m-1)(2i-1)-2j].$$

Therefore from (29), (25), (30) and (31) we obtain:

Remark 11 *Let* $(\overline{c}_{i,k})$ *be the* $(2m+1)\times(2m+1)$ *matrix defined by the following rules* $\overline{c}_{1,2m-1} = 0$ *and for all* $2 \leq i \leq 2m+1$,

$$\overline{c}_{i,2m-1} = (i-1) \prod_{j=0}^{m-2} \frac{1}{(2m-1)(2i-3)+2j+2}.$$

Moreover, for $1 \leq i \leq 2m+1$ *we take*

$$\overline{c}_{i,k} = \begin{cases} \displaystyle\prod_{j=0}^{t-1} [(2m-1)(2i-2)+2t-2j-1] & k = 2t+1 \quad \text{and} \quad 1 \leq t \leq m-2, \\ \displaystyle\prod_{j=1}^{m-1} [(2m-1)(2i-1)-2j] & \text{k=0}, \\ 1 & \text{k=1}, \end{cases}$$

and,

$$\overline{c}_{i,2s} = \prod_{j=1}^{m-1} [(2m-1)(2i-1)-2j] \prod_{j=1}^{s} [(2m-1)(2i-1)+2s-2j+2]\,;$$

when $1 \leq s \leq m$. *We conclude that there exist a constant* $\ell \neq 0$ *such that* $\det(c_{i,k}) = \det(\overline{c}_{i,k})\ell$, *where* $(c_{i,k})$ *is defined in Lemma 9 and also in (29), (25), (30) and (31).*

Observe that for every $0 \leq t \leq m-2$ the $\overline{c}_{i,2t+1}$ are polynomials in i with integer coefficients whose degree is t.

Lemma 12 *We define the* $(2m+1)\times(2m+1)$ *matrix* $(e_{i,k})$ *as follows: for every* $1 \leq i \leq 2m+1$ *we take*

$$e_{i,k} = \begin{cases} i^t & k = 2t+1 \quad \text{and} \quad 1 \leq t \leq m-2, \\ i^{m+s-1} & k = 2s \quad \text{and} \quad 0 \leq s \leq m, \\ \overline{c}_{i,2m-1} & k = 2m-1, \\ 1 & \text{k=1}. \end{cases}$$

Then there exists a constant $\ell \neq 0$ *such that* $\det(\overline{c}_{i,k}) = \det(e_{i,k})\ell$ *where the matrix* $(\overline{c}_{i,k})$ *is given in Remark 11.*

Proof. The basic idea is make elementary column operations in the matrix $(\overline{c}_{i,k})$ of the last remark. From these operations we will obtain the $(2m+1)\times(2m+1)$ matrix $(C_{i,k})$ whose determinant remains $\det(\overline{c}_{i,k})$, but except the $(2m-1)$th column, the ith column of $(C_{i,k})$ is given by monomials in i whose coefficients are integer. We proceed giving the details. First we define $C_{i,2m-1} := \overline{c}_{i,2m-1} =: e_{i,2m-1}$. Next consider the 3th column given by $t = 1$, thus $\overline{c}_{i,3} = (4m-2)i + (3-4m)$. By using that $\overline{c}_{i,1} = 1 =: e_{i,1}$ we obtain that $(4m-2)i = \overline{c}_{i,3} - (3-4m)\overline{c}_{i,1}$. Therefore we define

$$C_{i,3} := (4m-2)e_{i,3} \quad \text{where} \quad e_{i,3} := i.$$

If $t = 2$ then $\overline{c}_{i,5} = (16m^2 - 16m + 4)i^2 + (48m - 32m^2 - 16)i + (16m^2 - 32m + 15)$. As $e_{i,1} = 1$ and $e_{i,3} = i$ we get $(16m^2 - 16m + 4)i^2 = \overline{c}_{i,5} - (48m - 32m^2 - 16)e_{i,3} - (16m^2 - 32m + 15)e_{i,1}$. Hence we define

$$C_{i,5} := (16m^2 - 16m + 4)e_{i,5} \quad \text{where} \quad e_{i,5} = i^2.$$

In a similar way, by using induction over t, we can define the remain odd–columns and obtain that $C_{i,2t+1} = (\ell_t)i^t$ where $\ell_t \neq 0$ is a constant independent of i. So we conclude that $e_{i,2t+1} = i^t$ if $0 \leq t \leq m-2$ and obtain all the odd–columns.

If we take $s = 0$, $\overline{c}_{i,0} = \beta_{m-1}i^{m-1} + \beta_{m-2}i^{m-2} + \cdots + \beta_1 i + \beta_0$, so by using the columns $\{e_{i,1} = 1, e_{i,3} = i, \ldots, e_{i,2m-3} = i^{m-2}\}$ we obtain that

$$C_{i,0} = \beta_{m-1}e_{i,0} \quad \text{where} \quad e_{i,0} = i^{m-1}.$$

If $s = 1$ then $\overline{c}_{i,2} = \alpha_m i^m + \alpha_{m-1}i^{m-1} + \cdots + \alpha_0$. By using the columns $\{e_{i,1} = 1, e_{i,3} = i, \ldots, e_{i,2m-3} = i^{m-2}e_{i,0} = i^{m-1}\}$ we have that

$$C_{i,2} = \alpha_m e_{i,2} \quad \text{where} \quad e_{i,2} = i^m.$$

By induction on s we obtain that $e_{i,2s}$ is i^{m+s-1}. Hence the lemma is proved. □

Lemma 13 *We define the* $(2m+1)\times(2m+1)$ *matrix* $(\overline{e}_{i,k})$ *as follows: for all* $k \neq 1$, $\overline{e}_{1,k} = 0$ *and* $\overline{e}_{1,1} = 1$; *and for every* $2 \leq i \leq 2m+1$ *we take*

$$\overline{e}_{i,k} = \begin{cases} i^{t-1} & k = 2t+1 \quad \text{and} \quad 1 \leq t \leq m-2, \\ i^{m+s-2} & k = 2s \quad \text{and} \quad 0 \leq s \leq m, \\ \displaystyle\prod_{j=0}^{m-2} \frac{1}{(2m-1)(2i-3)+2j+2} & \text{k=2m-1.} \\ \dfrac{1}{i-1} & \text{k=1,} \end{cases}$$

Then $\det(e_{i,k}) = \det(\overline{e}_{i,k}) \displaystyle\prod_{i=2}^{2m+1}(i-1)$, *where the matrix* $(e_{i,k})$ *is defined in Lemma 12.*

Proof. As in the proof of Lemma 12 the basic idea is to make elementary column operations. Thus for every $k \notin \{1, 2m-1\}$ we consider $e_{i,k} - e_{i,1} = e_{i,k} - 1$. Since, $e_{1,2m-1} = 0$ after these operations with $(e_{i,k})$ the first row will be $(0, 1, 0, 0, \ldots, 0)$.

For each $2 \leq i \leq 2m+1$ the new $i-$row can be divided by $i-1$, so we have

$$(e_{i,0} - 1, 1, \ldots, e_{i,2m-1}, e_{i,2m} - 1)$$

is equal to

$$(i-1)\left(\frac{e_{i,0}-1}{i-1}, \frac{1}{i-1}, \ldots, \frac{e_{i,2m}-1}{i-1}\right) =: (i-1)(E_{i,0}, E_{i,1}, \ldots, E_{i,2m}).$$

In particular for all $2 \leq i \leq 2m+1$ since $E_{i,1} = \dfrac{1}{i-1} =: \overline{e}_{i,1}$ we obtain that

$$E_{i,2m-1} = \prod_{j=0}^{m-1} \frac{1}{(2m-1)(2i-3)+2j+2} =: \overline{e}_{i,2m-1}.$$

Moreover if we define the first row $\overline{e}_{1,k} = E_{1,k}$ as $(0,1,0,0,\ldots,0)$ we obtain that $\det(e_{i,k}) = \det(E_{i,k}) \prod_{i=2}^{2m+1} (i-1)$.

If $t=1$, $2t+1=3$ so in the position (i,3) of the new matrix $(E_{i,k})$ we found 1. Thus we get $\overline{e}_{i,3} = 1 = i^{t-1}$. In similar way for $t=2$ in the position (i,5) we found $\dfrac{i^2-1}{i-1} = i+1$. By using elementary column operations with $\overline{e}_{i,3} = 1$ we obtain that $\overline{e}_{i,5} = i = i^{t-1}$. Proceeding by induction over t we get that for all $2 \leq i \leq 2m+1$ and $1 \leq t \leq m-2$, $\overline{e}_{i,2t+1} = i^{t-1}$. This conclude the proof for all the odd columns, because the first row $\overline{e}_{1,k}$ is $(0,1,0,\ldots,0)$ and for all $i \geq 2$, $\overline{e}_{i,2m-1}$ was defined as $E_{i,2m-1}$.

If $s=0$ in the position $(i,0)$ of $(E_{i,k})$ we found $\dfrac{i^{m-1}-1}{i-1} = i^{m-2}+i^{m-3}+\ldots+1$. By using the columns $\{\overline{e}_{i,3} = 1, \overline{e}_{i,5} = i, \ldots, \overline{e}_{i,2m-3} = i^{m-3}\}$ we can get $\overline{e}_{i,0} = i^{m-2}$. In similar way for $s=1$ in the position (i,2) we found $\dfrac{i^m-1}{i-1} = i^{m-1}+i^{m-2}+\ldots+1$. From the columns $\{\overline{e}_{i,3} = 1, \ldots, \overline{e}_{i,2m-3} = i^{m-3}, \overline{e}_{i,0} = i^{m-2}\}$ we can get $\overline{e}_{i,2} = i^{m-1}$. By induction over s we prove that for all $2 \leq i \leq 2m+1$ and $0 \leq s \leq m$, $\overline{e}_{i,2s} = i^{m+s-2}$. Hence we obtain the definitions of all the even columns of $(\overline{e}_{i,k})$. This concludes the proof of the lemma, because $\det(\overline{e}_{i,k}) = \det(E_{i,k})$. $\square$

Notice that $(0,1,0,0\ldots,0)$ is the first row of the matrix $(\overline{e}_{i,k})$ of Lemma 13. Therefore the following remark is easy to check.

Remark 14 *Set* $h(i) = \prod_{j=0}^{m-2} [(2m-1)(2i-3)+2j+2]$. *If we consider the* $2m \times 2m$ *matrix* $(g_{i,k})$ *with* $1 \leq i \leq 2m$ *and* $0 \leq k \leq 2m-1$ *given by*

$$\begin{array}{rcll} g_{i,k} & = & (i+1)^k, & k \neq 2m-1, \\ g_{i,k} & = & \dfrac{1}{h(i+1)}, & k = 2m-1. \end{array}$$

Then $|\det(g_{i,k})| = |\det(\overline{e}_{i,k})|$ *where* $(\overline{e}_{i,k})$ *is defined in Lemma 13.*

Proposition 15 *For every* $1 \leq i \leq n+1$ *and* $0 \leq k \leq n = 2m$, *consider that* $a_{i,k}(\alpha)$ *is given by (9). If* $\alpha \in \mathbb{R}$ *is* $\mathbb{Q}-$*transcendental, then the* $(n+1)\times(n+1)$ *matrix* $(a_{i,k}(\alpha))$ *is non–singular, i.e.* $\det(a_{i,k}(\alpha)) \neq 0$.

Proof. From Remark 2, this proposition is true if $m=2$, thus we can suppose that $m \geq 3$. Lemma 4 implies that $\dfrac{1}{\pi^{n+1}}\det(a_{i,k}(\alpha))$ will be a polynomial in α with rational coefficients. Then in order to conclude we shall prove that some coefficient of such polynomial will be different from zero. Therefore, by Lemmas 8, 9, 12 and 13, we shall have established this proposition if we prove that $\det(g_{i,k}) \neq 0$ where $(g_{i,k})$ is the matrix given in Remark 14.

For any $1 \leq i \leq 2m$ and $0 \leq k \leq 2m-1$ let $M_{i,k}(g)$ denote the $(2m-1)\times(2m-1)$ sub–matrix of $g = (g_{i,k})$ obtained by deleting the row i and the column k of $(g_{i,k})$. The Laplace expansion says that for any $0 \leq k \leq 2m-1$

$$\det(g_{i,k}) = \sum_{i=1}^{2m} (-1)^{i+k+1} g_{i,k} \, \det(M_{i,k}(g)).$$

In particular, when $k = 2m - 1$ we obtain that

$$\det(g_{i,k}) = \sum_{i=1}^{m} \frac{(-1)^i}{h(i+1)} \det(M_{i,k}(g)) + \sum_{i=m+1}^{2m} \frac{(-1)^i}{h(i+1)} \det(M_{i,k}(g)).$$

Therefore

$$\det(g_{i,k}) = \sum_{s=1}^{m} \left[\frac{(-1)^s}{h(s+1)} \det(M_{s,2m-1}(g)) + \frac{(-1)^{s+1}}{h(2m+2-s)} \det(M_{2m+1-s,2m-1}(g)) \right]. \quad (32)$$

By using that for all $1 \le i \le 2m$, $M_{i,2m-1}(g)$ is a Vandermonde matrix, as in (18), is not difficult to check that

$$\det(M_{1,2m-1}(g)) = \det(M_{2m,2m-1}(g)) = \prod_{j=1}^{2m-2} j!.$$

More precisely, from Lemma 7 we can show that for every $1 \le s \le m$

$$\det(M_{s,2m-1}(g)) = \det(M_{2m+1-s,2m-1}(g)) > 0.$$

Thus from (32) we have that

$$\det(g_{i,k}) = \sum_{s=1}^{m} \det(M_{s,2m-1}(g)) \left[\frac{(-1)^s}{h(s+1)} + \frac{(-1)^{s+1}}{h(2m+2-s)} \right]. \quad (33)$$

We claim that the nonzero numbers

$$A_i = \frac{(-1)^i}{h(i+1)} + \frac{(-1)^{i+1}}{h(2m+2-i)}$$

satisfy

$$\begin{array}{lll} A_t + A_{t+1} < 0 & \text{if} \quad 1 \le t \le m-1 & \text{is odd,} \\ A_t < 0 & \text{if} \quad 1 \le t \le m & \text{is odd.} \end{array} \quad (34)$$

Since $h(i) = \prod_{j=0}^{m-2} [(2m-1)(2i-3) + 2j + 2] > 0$ if $1 < i$, we have that

$$\frac{1}{h(2)} > \frac{1}{h(3)} > \cdots > \frac{1}{h(2m)} > \frac{1}{h(2m+1)} > 0.$$

This shows (34) because for every $1 \le i \le m-1$

$$A_i + A_{i+1} = (-1)^i \left[\left(\frac{1}{h(i+1)} - \frac{1}{h(i+2)} \right) + \left(\frac{1}{h(2m-i+1)} - \frac{1}{h(2m-i+2)} \right) \right].$$

In order to conclude we remark that from (33) we obtain $\det(g_{i,k}) \le \ell \sum_{i=1}^{m} A_i$, where $\ell = \max\{\det(M_{s,2m-1}(g)) : 1 \le s \le 2m\}$. Then by using (34) we have:

$$\det(g_{i,k}) \le \ell \sum_{j=1}^{\tilde{k}} (A_{2j-1} + A_{2j}) < 0, \quad \text{if } m = 2\tilde{k} > 2, \quad \text{and}$$

$$\det(g_{i,k}) \le \ell \big[A_{2\tilde{k}+1} + \sum_{j=1}^{\tilde{k}} (A_{2j-1} + A_{2j}) \big] < 0, \quad \text{if } m = 2\tilde{k}+1 > 2.$$

This conclude the proof of the proposition. □

3.2. Odd case

This subsection is devoted to prove that (16) is true if $n = 2m + 1$ and conclude the proof of Theorem 1 without any additional assumptions. We assume that $m \geq 3$, because in the case $m = 2$, it was proved in Remark 3.

For every $0 \leq i, k \leq n = 2m + 1$, $b_{i,k}(\alpha)$ is given by (15). Note that Lemma 4 shows that for all $0 \leq s \leq m$, $b_{0,2s}(\alpha) = 0$. The Newton binomial implies that if $1 \leq i \leq n = 2m + 1$ and $0 \leq s \leq m$ then

$$b_{i,2s}(\alpha) = \sum_{j=0}^{i} \left(\begin{array}{c} i \\ j \end{array} \right) I[(n-2)j + 2s + 1 \mid (n-1)(i-j) + j + n - 2s]\alpha^j. \quad (35)$$

Moreover if $0 \leq t \leq m$, $b_{0,2t+1}(\alpha) = I[2t \mid n + 1 - 2t]$ and if $1 \leq i \leq n$ and $0 \leq t \leq m$ then

$$b_{i,2t+1}(\alpha) = \sum_{j=0}^{i} \left(\begin{array}{c} i \\ j \end{array} \right) I[(n-2)j + 2t \mid (n-1)(i-j) + j + n - 2t + 1]\alpha^j. \quad (36)$$

Lemma 16 *Let $n = 2m + 1 > 5$. For every $0 \leq i, k \leq n$, $b_{i,k}(\alpha)$ is given by (15). We define the $(2m + 2) \times (2m + 2)$ matrix $(\bar{b}_{i,k})$ as follows: $\bar{b}_{i,k} = b_{i,k}(0)$ if k is odd; $\bar{b}_{i,k} = \frac{d}{d\alpha} b_{i,k}(\alpha)|_{\alpha=0}$ if $k \neq 0$ is even; $\bar{b}_{0,0} = \bar{b}_{1,0} = 0$, $\bar{b}_{2,0} = \frac{2}{3m} I[6m - 2 \mid 4]$ and*

$$\bar{b}_{i,0} = \left(\begin{array}{c} i \\ 3 \end{array} \right) I[6m - 2 \mid 2mi - 4m + 4] - \frac{2m-1}{2m} \left(\begin{array}{c} i \\ 2 \end{array} \right) I[6m - 4 \mid 2mi - 4m + 6] + \frac{3}{2m} \left(\begin{array}{c} i \\ 2 \end{array} \right) I[6m - 2 \mid 2mi - 4m + 4],$$

for all $3 \leq i \leq 2m + 1$. Then the coefficient of α^{m+3} in the polynomial $\det(b_{i,k}(\alpha))$ in the variable α is $\det(\bar{b}_{i,k})$.

Proof. Lemma 4 implies that for each $0 \leq i \leq 2m + 1$ and $0 \leq s \leq m$ the number $b_{i,2s}(0) = 0$. Thus by definition of $b_{i,2s}(\alpha)$, equation (35) gives

$$\frac{d}{d\alpha} b_{i,2s}(\alpha)|_{\alpha=0} = iI[2(m + s) \mid 2mi + 2(1 - s)]. \quad (37)$$

Moreover from (36) for each $0 \leq i \leq 2m + 1$ we have that

$$b_{i,2m-1}(0) = I[2m - 2 \mid 2mi + 4] \quad \text{and} \quad b_{i,2m+1}(0) = I[2m \mid 2mi + 2].$$

Therefore by using (23) we obtain that

$$\frac{d}{d\alpha} b_{i,0}(\alpha)|_{\alpha=0} = (\frac{2m-1}{2m}) b_{i,2m-1}(0) - (\frac{3}{2m}) b_{i,2m+1}(0). \quad (38)$$

for all $0 \leq i \leq 2m + 1$.

We claim that the polynomial

$$B_{i,0}(\alpha) := b_{i,0}(\alpha) - \alpha \left[\frac{2m-1}{2m} b_{i,2m-1}(\alpha) - \frac{3}{2m} b_{i,2m+1}(\alpha) \right] = \alpha^3 \bar{b}_{i,0} + \text{h.o.t.}$$

where the column $(\bar{b}_{i,0})$ is as in Lemma 16, and as usual h.o.t. means "higher order terms".

If $i = 0$ the claim follows directly form (23), because $\bar{b}_{0,0} = 0$. If $i = 1$ note that $b_{1,0}(\alpha)$ has degree one and both polynomials $b_{i,2m-1}(\alpha)$ joint to $b_{i,2m+1}(\alpha)$ are constants, thus we obtain the claim from (38) by using that $\bar{b}_{1,0} = 0$. If $i = 2$, also $b_{2,0}(\alpha)$ has degree one, so $b_{2,0}(\alpha) = [\frac{d}{d\alpha} b_{2,0}(\alpha)|_{\alpha=0}]\alpha$; moreover form (36) we have that

$$b_{2,2m-1}(\alpha) = b_{2,2m-1}(0) + I[6m-4 \mid 6]\alpha^2$$

and

$$b_{2,2m+1}(\alpha) = b_{2,2m+1}(0) + I[6m-2 \mid 4]\alpha^2;$$

then (38) implies that $B_{2,0}(\alpha) = \alpha^3(\frac{3}{2m} I[6m-2 \mid 4] - \frac{2m-1}{2m} I[6m-4 \mid 6])$; but the equation (23) shows that $I[6m-4 \mid 6] = \frac{5}{6m-3} I[6m-2 \mid 4]$; then $B_{2,0}(\alpha)$ is equal to $\alpha^3 \frac{2}{3m} I[6m-2 \mid 4]$; therefore $B_{2,0}(\alpha) = \alpha^3 \bar{b}_{2,0}$. If $i = 3$ in (35) and (36) respectively, we have that

$$b_{3,0}(\alpha) = \left[\frac{d}{d\alpha} b_{3,0}(\alpha)|_{\alpha=0}\right]\alpha + I[6m-2 \mid 2m+4]\alpha^3,$$

$$b_{3,2m-1}(\alpha) = b_{3,2m-1}(0) + 3I[6m-4 \mid 2m+6]\alpha^2,$$

and

$$b_{3,2m+1}(\alpha) = b_{3,2m+1}(0) + 3I[6m-2 \mid 2m+4]\alpha^2;$$

thus (38) implies that $B_{3,0}(\alpha)$ is equal to

$$\alpha^3 \left(I[6m-2 \mid 2m+4] - \frac{2m-1}{m} 3I[6m-4 \mid 2m+6] + \frac{3}{2m} 3I[6m-2 \mid 2m+4] \right).$$

Therefore $B_{3,0}(\alpha) = \alpha^3 \bar{b}_{3,0}$. If $i > 3$ we have that

$$b_{i,0}(\alpha) = \left[\frac{d}{d\alpha} b_{i,0}(\alpha)|_{\alpha=0}\right]\alpha + \binom{i}{3} I[6m-2 \mid 2mi-4m+4]\alpha^3 + \cdots,$$

$$b_{i,2m-1}(\alpha) = b_{i,2m-1}(0) + \binom{i}{2} I[6m-4 \mid 2mi-4m+6]\alpha^2 + \cdots,$$

and

$$b_{i,2m+1}(\alpha) = b_{i,2m+1}(0) + \binom{i}{2} I[6m-2 \mid 2mi-4m+4]\alpha^2 + \cdots;$$

from this we obtain that $B_{i,0}(\alpha) = \alpha^3 \bar{b}_{i,0} + \cdots$. We conclude the proof of the claim.

From the claim we can use elementary column operations for obtaining that $\det(b_{i,k}(\alpha))$ is divided by $\alpha^3\alpha^m$, and the coefficient of α^{m+3} in the polynomial $\det(b_{i,k}(\alpha))$ is $\det(\bar{b}_{i,k})$. This proves the lemma. $\square$

Lemma 17 *Let $\bar{b}_{i,k}$ be as in Lemma 16. For each $0 \le i,k \le 2m+1$ the number $\bar{b}_{i,k}$ is divided by $\bar{b}_{i,2m+1} \ne 0$. Therefore there exists a $(2m+2) \times (2m+2)$ matrix $(d_{i,k})$ such that $\det(\bar{b}_{i,k}) = \det(d_{i,k}) \prod_{i=0}^{2m+1} \bar{b}_{i,2m+1}$ where, for each $0 \le i \le 2m+1$ we have that $\bar{b}_{i,2m+1}(d_{i,0}, \ldots, d_{i,2m}, 1) = (\bar{b}_{i,0}, \ldots, \bar{b}_{i,2m}, \bar{b}_{i,2m+1})$.*

Proof. If $0 \le t \le m$ we have that $\bar{b}_{i,2t+1} = I[2t \mid 2mi + 2m + 2 - 2t]$, for all $0 \le i \le 2m+1$ (see (36)); from (23) it follows that for each $t \le m-1$,

$$\bar{b}_{i,2t+1} = \frac{2mi+2m-2t+1}{2t+1} \bar{b}_{i,2(t+1)+1}.$$

Thus proceeding inductively we obtain that

$$\overline{b}_{i,2t+1} = \overline{b}_{i,2m+1} \prod_{j=1}^{m-t} \frac{2mi + 2m - 2t - 2j + 3}{2t + 2j - 1},$$

for all $0 \le t \le m - 1$. Therefore for all $0 \le t \le m - 1$ we have

$$d_{i,2t+1} = \prod_{j=1}^{m-t} \frac{2mi + 2m - 2t - 2j + 3}{2t + 2j - 1}. \tag{39}$$

This proves the lemma when $k = 2t + 1$ and $0 \le t \le m$, because $d_{i,2m+1} = 1$.

In a similar way for all $1 \le s \le m - 1$ we have that

$$\overline{b}_{i,2s} = \frac{2mi - 2s + 1}{2m + 2s + 1} \overline{b}_{i,2(s+1)},$$

because $\overline{b}_{i,2s} = iI[2m + 2s \mid 2mi + 2 - 2s]$ for all $s \ne 0$ (see (37)). Then for each $1 \le s \le m - 1$ and $0 \le i \le 2m + 1$ we get that

$$\overline{b}_{i,2s} = \overline{b}_{i,2m} \prod_{j=1}^{m-s} \frac{2mi - 2s - 2j + 3}{2m + 2s + 2j - 1}. \tag{40}$$

Furthermore as $\overline{b}_{i,2m+1} = I[2m \mid 2mi + 2]$ by using (24) $m > 2$ times we have

$$\overline{b}_{i,2m} = i\, \overline{b}_{i,2m+1} \prod_{j=0}^{m-1} \frac{4m - 2j - 1}{2mi - 2m + 2j + 3}. \tag{41}$$

Therefore (40) and (41) imply that for each $1 \le s \le m$ the lemma follows when we take

$$d_{i,2s} = i \prod_{j=0}^{m-1} \frac{4m - 2j - 1}{2mi - 2m + 2j + 3} \prod_{j=1}^{m-s} \frac{2mi - 2s - 2j + 3}{2m + 2s + 2j - 1}. \tag{42}$$

This gives the proof when $k = 2s$ and $1 \le s \le m$.

Finally we study the first even column $(\overline{b}_{i,0})$ where $0 \le i \le 2m + 1$. As $\overline{b}_{0,0} = \overline{b}_{1,0} = 0$, we only consider $i \ge 2$. Since $\overline{b}_{2,0} = \frac{2}{3m} I[6m - 2 \mid 4]$ from (24) it follows that

$$\overline{b}_{2,0} = \frac{2}{3m} \prod_{j=0}^{2m-2} \frac{6m - 2j - 3}{2j + 5} \overline{b}_{2,2m+1}$$

because (36) says that $\overline{b}_{2,2m+1} = I[2m \mid 4m + 2]$. From this we obtain that

$$d_{2,0} = \frac{2}{3m} \prod_{j=0}^{2m-2} \frac{6m - 2j - 3}{2j + 5} = \frac{2}{3m} \prod_{j=1}^{2m-1} \frac{6m - 2j - 1}{2j + 3}. \tag{43}$$

Now we work with $i > 2$ and we use equation (23) again, which implies that $I[6m - 4 \mid 2mi - 4m + 6] = \dfrac{2mi - 4m + 5}{6m - 3} I[6m - 2 \mid 2mi - 4m + 4]$, then for all $3 \le i \le 2m + 2$ the definition given in Lemma 16 shows that

$$\overline{b}_{i,0} = D\left[\begin{pmatrix} i \\ 3 \end{pmatrix} - \begin{pmatrix} i \\ 2 \end{pmatrix} \left(\frac{mi - 2m - 2}{3m} \right) \right],$$

where $D := I[6m-2 \mid 2mi-4m+4]$. But (36) implies that $\overline{b}_{i,2m+1} = I[2m \mid 2mi+2]$, thus by using (24) in $2m-1$ times is not difficult to show that this D satisfies that $D = \overline{b}_{i,2m+1} \prod_{j=1}^{2m-1} \frac{6m-2j-1}{2mi-4m+2j+3}$. Therefore for all $3 \le i \le 2m+2$ we get

$$d_{i,0} = \left[\binom{i}{3} - \binom{i}{2} \left(\frac{mi-2m-2}{3m}\right) \right] \prod_{j=1}^{2m-1} \frac{6m-2j-1}{2mi-4m+2j+3}. \tag{44}$$

This concludes the proof, because $\overline{b}_{0,0} = \overline{b}_{1,0} = 0$. □

For each $0 \le k \le 2m+1$ it is easily seen that the kth column of the matrix $(d_{i,k})$ has a common factor, for instance since $d_{i,2m} = \frac{\overline{b}_{i,2m}}{\overline{b}_{i,2m+1}}$ from (41) it follows that

$$d_{i,2m} = \prod_{j=0}^{m-1} [4m-2j-1] \left(i \prod_{j=0}^{m-1} \frac{1}{2mi-2m+2j+3} \right).$$

Also, by (43) and (44), $\prod_{j=1}^{2m-1} [6m-2j-1]$ is a common factor of the column $(0, 0, d_{2,0}, \dots, d_{2m+1,0})$. Therefore from (41), (39), (42), (43) and (44) we obtain:

Remark 18 *Let* $(\overline{d}_{i,k})$ *be the* $(2m+2) \times (2m+2)$ *matrix defined with the following rules* $\overline{d}_{0,0} = \overline{d}_{1,0} = 0$, $\overline{d}_{2,0} = \frac{2}{3m} \prod_{j=1}^{2m-1} \frac{1}{2j+3}$ *and for all* $3 \le i \le 2m+2$,

$$\overline{d}_{i,0} = \left[\binom{i}{3} - \binom{i}{2} \left(\frac{mi-2m-2}{3m}\right) \right] \prod_{j=1}^{2m-1} \frac{1}{2mi-4m+2j+3}.$$

Moreover, for each $0 \le i \le 2m+1$, *we take*

$$\overline{d}_{i,k} = \begin{cases} 1 & \text{k=2m+1;} \\ i \prod_{j=0}^{m-1} \frac{1}{2mi-2m+2j+3} & \text{k=2m;} \\ \prod_{j=1}^{m-t} [2mi+2m-2t-2j+3] & k = 2t+1, 0 \le t \le m-1; \\ i \prod_{j=0}^{m-1} \frac{1}{2mi-2m+2j+3} \prod_{j=1}^{m-s} [2mi-2s-2j+3] & k = 2s, 1 \le s \le m-1. \end{cases}$$

Then there is a constant $\ell \neq 0$ *such that* $\det(d_{i,k}) = \det(\overline{d}_{i,k})\ell$, *where* $(d_{i,k})$ *is defined in Lemma 17 and also in (41), (39), (42), (43) and (44).*

For each $0 \le i \le 2m+1$

$$\ell_i = \prod_{j=0}^{m-1} [2mi-2m+2j+3] \tag{45}$$

is different from zero. Thus if for all $0 \le i,k \le 2m+1$ we define $f_{i,k} := \overline{d}_{i,k}\ell_i$, then $\det(f_{i,k}) = \det(\overline{d}_{i,k}) \prod_{i=0}^{2m-1} \ell_i$. Moreover if $\tilde{\mathbf{j}} = m+j$ equation (45) takes the form $\ell_i = \prod_{\tilde{\mathbf{j}}=m}^{2m-1} [2mi - 4m + 2\tilde{\mathbf{j}} + 3]$ which implies that

$$\ell_i \prod_{j=1}^{2m-1} \frac{1}{2mi-4m+2j+3} = \prod_{j=1}^{m-1} \frac{1}{2mi-4m+2j+3}.$$

Therefore since $\ell_2 \prod_{j=1}^{2m-1} \frac{1}{2j+3} = \prod_{j=1}^{m-1} \frac{1}{2j+3}$ it is easily seen that:

Remark 19 *If $(f_{i,k})$ is the $(2m+2)\times(2m+2)$ matrix given by $f_{i,k} := \overline{d}_{i,k}\ell_i$, where ℓ_i is as in (45) and $(\overline{d}_{i,k})$ like in Remark 18, then $det(f_{i,k}) = det(\overline{d}_{i,k}) \prod_{i=0}^{2m-1} \ell_i$. More precisely, $(f_{i,k})$ satisfy that $f_{0,0} = f_{1,0} = 0$, $f_{2,0} = \frac{2}{3m}\prod_{j=1}^{m-1}\frac{1}{2j+3}$, and for all $3 \le i \le 2m+1$ $f_{i,0} = \left[\binom{i}{3} - \binom{i}{2}\left(\frac{mi-2m-2}{3m}\right)\right] \prod_{j=1}^{m-1} \frac{1}{2mi-4m+2j+3}$. Also for each $0 \le i \le 2m+1$,*

$$f_{i,k} = \begin{cases} \ell_i & \text{k=2m+1;} \\ i & \text{k=2m;} \\ \prod_{j=0}^{m-1}[2mi-2m+2j+3]\prod_{j=1}^{m-t}[2mi+2m-2t-2j+3] & k=2t+1, 0 \le t \le m-1; \\ i\prod_{j=1}^{m-s}[2mi-2s-2j+3] & k=2s, 1 \le s \le m-1. \end{cases}$$

Note that for every $0 \le t \le m-1$, $f_{i,2t+1}$ is polynomial in i with integer coefficients whose degree is $2m-t$.

Lemma 20 *If we define the $(2m+2)\times(2m+2)$ matrix $(\overline{f}_{i,k})$ as follows: $\overline{f}_{0,0} = \overline{f}_{1,0} = 0$, and for all $2 \le i \le 2m+1$ we take*

$$\overline{f}_{i,0} = \frac{i(i-1)}{3m} \prod_{j=1}^{m-1} \frac{1}{2mi-4m+2j+3}.$$

Moreover, for each $0 \le i \le 2m+1$ we take

$$\overline{f}_{i,k} = \begin{cases} 1 & \text{k=2m+1,} \\ i^{m-s+1} & k=2s \text{ and } 1 \le s \le m, \\ i^{2m-t} & k=2t+1 \text{ and } 0 \le t \le m-1. \end{cases}$$

Then there exists a constant $\ell \ne 0$ such that $\det(\overline{f}_{i,k}) = \det(f_{i,k})\ell$, where $(f_{i,k})$ is as in Remark 19.

Proof. We proceed as in Lemma 12 in order to define the auxiliary $(2m+2)\times(2m+2)$ matrix $(F_{i,k})$ with $\det(F_{i,k}) = \det(f_{i,k})$. So we begin with $F_{i,2m} := i = f_{i,2m}$. If $s = m-1$ we have that $f_{i,2m-2} = 2mi^2 + (3-2m)i$. By using that $f_{i,2m} = i =: \overline{f}_{i,2m}$ we obtain that $2mi^2 = f_{i,2m-2} - (3-2m)f_{i,2m}$. Therefore we consider

$$F_{i,2m-2} = 2m\overline{f}_{i,2m-2} \quad \text{where} \quad \overline{f}_{i,2m-2} = i^2.$$

Moreover if $s = m-2$ then $f_{i,2m-4} = 4m^2i^3 + (16m-8m^2)i^2 + (4m^2-16m+15)i$. As $\overline{f}_{i,2m} = i$ and $\overline{f}_{i,2m-2} = i^2$ we obtain that $4m^2i^3 = \overline{d}_{i,2m-4} - (16m-8m^2)\overline{f}_{i,2m-2} - (4m^2-16m+15)\overline{f}_{i,2m}$. Therefore we define

$$F_{i,2m-4} = 4m^2\overline{f}_{i,2m-4} \quad \text{where} \quad \overline{f}_{i,2m-4} = i^3$$

Proceeding by induction on s we obtain the definition all the even columns of $(F_{i,k})$ and so we conclude that for all $1 \le s \le m$, $\overline{f}_{i,2s}$ is i^{m-s+1}.

In a similar way in the odd columns is not difficult to check that $f_{i,2m+1} = \beta_0 + \beta_m i^m + \beta_{m-1}i^{m-1} + \ldots + \beta_1 i$, so by elementary operations with the columns $\{\overline{f}_{i,2} = i^m, \overline{f}_{i,4} = i^{m-1}, \ldots, \overline{f}_{i,2m} = i\}$ we obtain that

$$F_{i,2m+1} = \beta_0\overline{f}_{i,2m+1} \quad \text{where} \quad \overline{f}_{i,2m+1} = 1,$$

and $\beta_0 \neq 0$ is independent of i. If $t = m-1$ then $f_{i,2m-1} = \alpha_{m+1}i^{m+1} + \alpha_m i^m + \ldots + \alpha_0$. By using the columns $\{\overline{f}_{i,2} = i^m, \overline{f}_{i,4} = i^{m-1}, \ldots, \overline{f}_{i,2m} = i, \overline{f}_{i,2m+1} = 1\}$ we define

$$F_{i,2m-1} = \alpha_{m+1}\overline{f}_{i,2m-1} \quad \text{where} \quad \overline{f}_{i,2m-1} = i^{m+1},$$

and $\alpha_{m+1} \neq 0$ is independent of i. By using induction over t and that $F_{i,0} = f_{i,0}$ we obtain the requested matrix $(F_{i,k})$, and also that for all $0 \le t \le m-2$ $\overline{f}_{i,2t+1}$ is i^{2m-t}. This proves the lemma because $\overline{f}_{i,0} = f_{i,0}$. □

Since the first row is $(0,0,\ldots,0,1)$ from the Laplace expansion, by changing some columns and some elementary row operations it is easy to check:

Remark 21 *Let $(\overline{f}_{i,k})$ be as in Lemma 20. If $(h_{i,k})$ is the $(2m+1)\times(2m+1)$ matrix defined by*

$$\begin{aligned} h_{i,k} &= i^{k-1}, & k &\neq 2m+1, \\ h_{i,k} &= \frac{\overline{f}_{i,0}}{i}, & k &= 2m+1, \end{aligned}$$

for all $1 \le i,k \le 2m+1$. Then $|\det(\overline{f}_{i,k})| = |\det(h_{i,k})| \prod_{i=1}^{2m+1} i.$

Lemma 22 *Let $(h_{i,k})$ be as in Remark 21. If we consider the $2m\times 2m$ matrix $(\overline{h}_{i,k})$ with $2 \le i,k \le 2m+1$ given by*

$$\begin{aligned} \overline{h}_{i,k} &= i^{k-2}, & k &\neq 2m+1, \\ \overline{h}_{i,k} &= \frac{1}{3m}\prod_{j=1}^{m-1}\frac{1}{2mi-4m+2j+3}, & k &= 2m+1. \end{aligned}$$

Then $|\det(h_{i,k})| = |\det(\overline{h}_{i,k})| \prod_{i=2}^{2m+1}(i-1).$

Proof. We proceed doing elementary operations as in Lemma 13. Thus for every $k \notin \{1, 2m+1\}$ we consider $h_{i,k} - h_{i,1} = h_{i,k} - 1$. Then after these operations the first row will be $(1, 0, \ldots, 0)$, because $\overline{f}_{1,0} = 0$.

For each $2 \le i \le 2m+1$, the new i–row can be divided by $i-1$, thus we have $(1, h_{i,2} - 1, \ldots, h_{i,2m} - 1, \overline{f}_{i,0}/i)$, which is equal to

$$(i-1)\left(\frac{1}{i-1}, \frac{h_{i,2}-1}{i-1}, \ldots, \frac{\overline{f}_{i,0}}{i(i-1)}\right) =: (i-1)(H_1, H_2, \ldots H_{2m+1}).$$

Thus $H_{2,2m+1} = \dfrac{1}{3m}\displaystyle\prod_{j=1}^{m-1}\frac{1}{2j+3}$ for all $3 \le i \le 2m+1$ and we obtain that

$$H_{i,2m+1} = \frac{1}{3m}\prod_{j=1}^{m-1}\frac{1}{2mi - 4m + 2j + 3}.$$

Furthermore if we define the first row $(H_{1,k})$ as $(1, 0, \ldots, 0)$ we obtain that $|\det(h_{i,k})| = |\det(H_{i,k})| \displaystyle\prod_{i=2}^{2m+1}(i-1)$.

Since the first row of the $(2m+1)\times(2m+1)$ matrix $(H_{i,k})$ is $(1, 0, \ldots, 0)$ we may define the $2m \times 2m$ matrix $(\overline{h}_{i,k})$ as the sub–matrix of $H_{i,k}$ obtained by deleting the first row and the first column. As $\det(\overline{h}_{i,k}) = \det(H_{i,k})$ we conclude the proof of the lemma. □

Proposition 23 *Let $n = 2m+1 \in \mathbb{N}$. For every $0 \le i, k \le n+1$, $b_{i,k}(\alpha)$ is given by (15). If $\alpha \in \mathbb{R}$ is $\mathbb{Q}$–transcendental, then the $(n+1)\times(n+1)$ matrix $(b_{i,k}(\alpha))$ is non–singular, i.e. $\det(b_{i,k}(\alpha)) \neq 0$.*

Proof. By using Lemmas 16, 17, 22 and 20, we shall have established this proposition if we prove that $\det(\overline{h}_{i,k}) \neq 0$ where $(\overline{h}_{i,k})$ is the $2m \times 2m$ matrix given in Lemma 22.

For any $2 \le i, k \le 2m+1$ let $M_{i,k}(\overline{h})$ denote the $(2m-1)\times(2m-1)$ sub–matrix of $\overline{h} = (\overline{h}_{i,k})$ obtained by deleting the row i and the column k of $(\overline{h}_{i,k})$. It is well–know that

$$\det(\overline{h}_{i,k}) = \sum_{i=2}^{2m+1}(-1)^{i+k}\overline{h}_{i,2m+1}\,\det(M_{i,k}(\overline{h})),$$

for any $2 \le k \le 2m+1$. In particular if $k = 2m+1$ it follows that $\det(\overline{h}_{i,k})$ is equal to

$$\sum_{i=2}^{m+1}(-1)^{i+1}\overline{h}_{i,2m+1}\det(M_{i,2m+1}(\overline{h})) + \sum_{i=m+2}^{2m+1}(-1)^{i+1}\overline{h}_{i,2m+1}\det(M_{i,2m+1}(\overline{h})).$$

Therefore $\det(\overline{h}_{i,k})$ takes the form

$$\sum_{s=2}^{m+1}(-1)^{s+1}\left[\overline{h}_{s,2m+1}\det(M_{s,2m+1}(\overline{h})) - \overline{h}_{2m+3-s,2m+1}\det(M_{2m+3-s,2m+1}(\overline{h}))\right].$$

For every $2 \le i \le 2m+1$, $M_{i,2m+1}(\overline{h})$ is a Vandermonde matrix as in Lemma 7, then

$$\det(M_{s,2m+1}(\overline{h})) = \det(M_{2m+3-s,2m+1}(\overline{h})) > 0.$$

Therefore

$$\det(\overline{h}_{i,k}) = \sum_{s=2}^{m+1} \det(M_{s,2m+1}(\overline{h})) \left[(-1)^{s+1}\overline{h}_{s,2m+1} + (-1)^s \overline{h}_{2m+3-s,2m+1}\right]. \tag{46}$$

In order to conclude we define $\tilde{h}(i) = 3m \prod_{j=1}^{m-1} [2mi - 4m + 2j + 3]$ and for every $2 \leq s \leq m+1$ the nonzero numbers $B_s = \dfrac{(-1)^{s+1}}{\tilde{h}(s)} + \dfrac{(-1)^s}{\tilde{h}(2m+3-s)}$. Thus from (46) we have that

$$\det(\overline{h}_{i,k}) \leq \ell \sum_{s=2}^{m+1} B_s,$$

where $\ell = \max\{\det(M_{s,2m+1} : 2 \leq s \leq 2m+1\}$. Moreover it is not difficult to see that

$$\frac{1}{\tilde{h}(2)} > \frac{1}{\tilde{h}(3)} > \cdots > \frac{1}{\tilde{h}(2m)} > \frac{1}{\tilde{h}(2m+1)} > 0$$

which imply

$$\begin{array}{lll} B_s + B_{s+1} < 0 & \text{if} \quad 2 \leq s \leq m & \text{is even,} \\ B_s < 0 & \text{if} \quad 2 \leq s \leq m+1 & \text{is even,} \end{array} \tag{47}$$

because

$$B_i + B_{i+1} = (-1)^i \left[\left(\frac{1}{\tilde{h}(i+1)} - \frac{1}{\tilde{h}(i)} \right) + \left(\frac{1}{\tilde{h}(2m-i+3)} - \frac{1}{\tilde{h}(2m-i+2)} \right) \right]$$

for $2 \leq i \leq m+1$. Therefore, from (47) and (46) where $2 \leq s \leq m+1$ we have

$$\det(\overline{h}_{i,k}) \leq \ell \sum_{j=1}^{\tilde{k}} (B_{2j} + B_{2j+1}) < 0 \quad \text{if } m = 2\tilde{k} > 2, \text{ and}$$

$$\det(\overline{h}_{i,k}) \leq \ell[B_{2\tilde{k}+2} + \sum_{j=1}^{\tilde{k}} (B_{2j} + B_{2j+1})] < 0 \quad \text{if } m = 2\tilde{k}+1 > 2.$$

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