This is a preprint of: "On the polynomial differential systems having polynomial first integrals", Belen García, Jaume Llibre, Jesús Suárez Pérez del Río, *Bull. Sci. Math.*, vol. 136, 309–316, 2012.
DOI: [10.1016/j.bulsci.2011.11.003]

ON THE POLYNOMIAL DIFFERENTIAL SYSTEMS HAVING POLYNOMIAL FIRST INTEGRALS

BELEN GARCIA[1], JAUME LLIBRE[2] AND JESÚS S. PÉREZ DEL RÍO[1]

Abstract. We consider the class of complex planar polynomial differential systems having a polynomial first integral. Inside this class the systems having minimal polynomial first integrals without critical remarkable values are the Hamiltonian ones. Here we mainly study the subclass of polynomial differential systems such that their minimal polynomial first integrals have a unique critical remarkable value.

In particular we characterize all the Liénard polynomial differential systems $\dot{x} = y$, $\dot{y} = -f(x)y - g(x)$, with $f(x)$ and $g(x)$ complex polynomials in the variable x, having a minimal polynomial first integral with a unique critical remarkable value.

1. Introduction and statement of the main results

The nonlinear ordinary differential equations or simple the differential systems appear in many branches of applied mathematics, physics, and in general in applied sciences. In general the differential systems cannot be solved explicitly, so the qualitative information provided by the theory of dynamical systems is the best that one can expect to obtain.

For a planar differential system the existence of a first integral determines completely its phase portrait, i.e. the description of the domain of definition of the differential system as union of all the orbits or trajectories of the system. To provide the phase portrait of a differential system is the main objective of the qualitative theory of the differential systems. Thus for planar differential systems one of the main questions is: How to recognize if a given planar differential system has a first integral?

In this paper we study the existence of polynomial first integrals in planar polynomial differential systems. More precisely, we want to study the polynomial first integrals of the differential systems

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

1991 *Mathematics Subject Classification.* 34C05, 34A34, 34C14.

Key words and phrases. Polynomial differential system, Liénard differential system, polynomial first integral, critical remarkable value.

The first and third authors are partially supported by a MEC/FEDER grant number MTM2008-06065. The second author is partially supported by a MEC/FEDER grant number MTM2008-03437 and by a CICYT grant number 2005SGR 00550.

where P and Q are complex polynomials in the variables x and y, and where the dot denotes derivative with respect to the variable t that can be consider real or complex.

The search of first integrals is a classical tool for classifying all trajectories of a planar differential system (1). Polynomial first integrals are a particular case of the Darboux first integrals. In 1878 Darboux [7] showed how the first integrals of planar polynomial systems possessing sufficient invariant algebraic curves can be constructed. The best improvements to Darboux's results for planar polynomial systems are due to Poincaré [14] in 1897, to Jouanolou [10] in 1979, to Prelle and Singer [15] in 1983, and to Singer [16] in 1992. But the results of the Darboux theory of integrability provide sufficient conditions for finding in general Liouvillian first integrals, and in particular rational first integrals, but do not provide neither sufficient nor necessary conditions for the existence of polynomial first integrals. See also the recent result of Coutinho and Menasché Schechter [6] providing sufficient conditions for the non–existence of invariant algebraic curves.

As usual $\mathbb{C}[x, y]$ denotes the ring of all complex polynomials in the variables x and y. We say that $H \in \mathbb{C}[x, y] \setminus \mathbb{C}$ is a *polynomial first integral* of system (1) on $\mathbb{C}^2$ if $H(x(t), y(t))$ is constant for all values of t such that $(x(t), y(t))$ is defined on $\mathbb{C}^2$. Obviously, H is a first integral of system (1) if and only if

$$P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} = 0 \tag{2}$$

in $\mathbb{C}^2$.

Polynomial first integrals for the following 3–dimensional quadratic polynomial differential system of Lotka–Volterra kind

$$x' = x(Cy + z), \quad y' = y(x + Az), \quad z' = z(Bx + y),$$

have been characterized by Moulin–Ollagnier [13] and Labrunie [12]. Cairó and Llibre [3] classify the polynomial first integrals for the 2–dimensional quadratic polynomial differential system of Lotka–Volterra kind

$$x' = x(a_1 + b_{11}x + b_{12}y), \qquad y' = y(a_2 + b_{21}x + b_{22}y).$$

In fact, both results on Lotka–Volterra systems are related, see the relationship between both systems in [2].

Recently in [4] the authors obtain all quadratic polynomial differential systems having a polynomial first integral and do the topological classification of the phase portraits of such quadratic systems in [9]. This classification has been improved using invariant theory in the 12–dimensional parameter space of all quadratic polynomial differential systems, see [1].

From now on we write the polynomial differential system (1) as follows

$$\dot{x} = P(x, y) = \sum_{j=0}^{k} p_j(x)y^j, \qquad \dot{y} = Q(x, y) = \sum_{i=0}^{l} q_i(x)y^i, \tag{3}$$

and *we assume in all this paper* that

(i) $p_k(x) \neq 0$,
(ii) $q_l(x) \neq 0$, and
(iii) P and Q coprime.

Note that we always can assume conditions (i) and (ii). If P and Q are not coprime in the ring of all polynomials $\mathbb{C}[x, y]$, and R is their greatest common divisor, then doing a rescaling by R of the independent variable we get the polynomial differential system $\dot{x} = \tilde{P} = P/R$ and $\dot{y} = \tilde{Q} = Q/R$, for which $\tilde{P}$ and $\tilde{Q}$ are coprime. In short the three conditions (i), (ii) and (iii) are essentially working conditions that simplify the statement of the results and their proofs.

Our first result is the next one proved in section 2.

Theorem 1. *The following statements hold.*

(a) *If $k > l - 1$ and $H = H(x, y)$ is a polynomial first integral of system* (3) *then, except by the multiplication for a nonzero constant, we have that* $H = y^s + \sum_{i=0}^{s-1} H_i(x) y^i$ *with $s > 0$.*
(b) *If $k = l - 1$ and system* (3) *has a polynomial first integral, then the degree of the polynomial $p_k(x)$ is equal to the degree of the polynomial $q_l(x)$ plus one.*
(c) *If $k < l - 1$, then system* (3) *has no polynomial first integrals.*

A polynomial first integral H of system (3) is called *minimal* if for any other polynomial first integral $\tilde{H}$ of (3) we have that the degree of H is smaller than or equal to the degree of $\tilde{H}$.

Now we introduce the concept of remarkable value due to Poincaré (see [14]). Poincaré used these values in order to study the polynomial differential systems having a rational first integral, and in particular a polynomial first integral.

Let H be a minimal polynomial first integral of the differential system (3). We say that $c \in \mathbb{C}$ is a *remarkable value* of H if the polynomial $H + c$ is not irreducible in $\mathbb{C}[x, y]$, i.e. if there exist values $p_1, \ldots, p_q \in \mathbb{N}$ such that $H + c = u_1^{p_1} \ldots u_q^{p_q}$, where u_i are irreducible polynomials in $\mathbb{C}[x, y]$ called *remarkable factors* associated to c with *exponent* p_i. Furthermore if there exists i such that $p_i > 1$, then the remarkable value is called *critical* and the corresponding factor u_i is called *critical remarkable factor*. Note that every curve $u_i = 0$ is an invariant algebraic curve of system (3), see section 2 for the definition. In [5] the authors have proved that the number of remarkable values of a minimal polynomial first integral of a polynomial differential system is finite. For additional information on the remarkable values see [8].

System (3) is called *Hamiltonian* if there exists a polynomial $H = H(x, y)$ such that $P = \partial H/\partial y$ and $Q = -\partial H/\partial x$. Clearly H is a first integral of this system. Javier Chavarriga proved that a polynomial differential system having a polynomial first integral without critical remarkable values is Hamiltonian, see the proof in [8]. *Our main interest is the study of the polynomial differential systems* (3) *having minimal polynomial first integrals with a unique critical remarkable value.* We note that the minimal polynomial first integrals having a unique critical remarkable value can have other non–critical remarkable values.

Our first result on the minimal polynomial first integrals having a unique critical remarkable value is the following one proved in section 2.

Theorem 2. *Let* $H = \sum_{i=0}^{s} H_i(x)y^i$ *be with* $H_s(x) \neq 0$ *a minimal polynomial first integral of system* (3)*. Assume that* H *has a unique critical remarkable value* c *and that*

$$H + c = \prod_{j=1}^{q} u_j^{p_j}, \tag{4}$$

with p_j *positive integers and with some* $p_j > 1$*. Let* n_j *be the degree of the polynomial* u_j *in the variable* y*. Then*

$$k + 1 = \sum_{j=1}^{q} n_j.$$

Now we restrict our attention to the polynomial differential systems (3) having minimal first integrals with a unique critical remarkable value and such that the degree k of the polynomial P in the variable y is one. As we shall see this class of polynomial differential systems contain the relevant class of Liénard polynomial differential systems.

If $k = 1$ in systems (3) then, using Theorem 1, all these systems having a polynomial first integral can be written as

$$\dot{x} = a(x)y + b(x), \qquad \dot{y} = c(x)y^2 + d(x)y + e(x), \tag{5}$$

with $a(x) \neq 0$.

Theorem 3. *Assume that the polynomial differential system* (5) *has a minimal polynomial first integral* H *with a unique critical remarkable value. Then*

$$H = F(x)(A(x)y + B(x))^p(C(x)y + D(x))^q, \tag{6}$$

where A*,* B*,* C*,* D *and* $F3$ *are polynomials in the variable* x*, and* p *and* q *are distinct positive integers, or*

$$H = F(x)(A(x)y^2 + B(x)y + C(x))^p, \tag{7}$$

where A*,* B C *and* F *are polynomials in the variable* x*, and* p *is a positive integer.*

Theorem 3 is proved in section 2.

In the study of dynamical systems and differential equations, a Liénard differential equation is

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{8}$$

where $f(x)$ and $g(x)$ are real C^1 functions. Here the dot denotes differentiation with respect to the time t. These equations appeared in the works of the French physicist Alfred-Marie Liénard when he studied the development of radio and vacuum tubes. Liénard differential equations were intensely studied as they can be used to model oscillating circuits. Under certain additional assumptions Liénard's theorem guarantees the existence of a limit cycle for equation (8), see [11].

Instead of working with the differential equations of second order (8) we shall work with the following equivalent planar differential system with two equations of first order

$$\dot{x} = y, \qquad \dot{y} = -f(x)y - g(x). \tag{9}$$

The Liénard differential systems (9) with $f(x)$ and $g(x)$ complex polynomials in the variable x are called *polynomial Liénard differential systems*. One of our main goals of this paper is to characterize the polynomial Liénard differential systems (8) having a polynomial first integral with a unique critical remarkable value, and to provide an explicit expression of these systems and of their polynomial first integrals. Note that the polynomial Liénard differential systems (9) are a particular subclass of systems (5), and consequently of systems (3).

Theorem 4. *For the complex polynomial Liénard differential system* (9) *the following statements hold.*

(a) *If $g(x) = 0$, then a polynomial first integral of system* (9) *is $H = y + F(x)$ where $F(x) = \int f(x)dx$.*

(b) *If $f(x) = 0$, then a polynomial first integral of system* (9) *is $H = y^2/2 + \int g(x)dx$.*

(c) *Assume that $f(x)g(x) \neq 0$ and let H be a minimal polynomial first integral of system* (9) *having a unique remarkable value.*

Then there exist positive integers p and q, $p \neq q$ such that $g(x) = \frac{q}{q-p} f(x) \left(c + \frac{p}{p-q}F(x)\right)$ with $c \in \mathbb{C}$, and

$$H = \left(y + c + \frac{p}{p-q}F(x)\right)^p \left(y - \frac{q}{p}\left(c + \frac{p}{p-q}F(x)\right)\right)^q,$$

except by the multiplication for a nonzero constant.

Theorem 4 is proved in section 2.

We have some numerical evidences that the following open question can have a positive answer.

Open question. *All the polynomial Liénard differential systems* (9) *with* $f(x)g(x) \neq 0$ *and with a polynomial first integral are the ones described in the statement (c) of Theorem 4.*

2. Proof of the results

In this section we prove Theorems 1, 2, 3 and 4. But first we recall two basic definitions that we shall use.

A non–constant function $R = R(x, y)$ is called an *integrating factor* of system (3) if

$$\frac{d(RP)}{dx} + \frac{d(RQ)}{dy} = 0.$$

So the differential system $\dot{x} = RP$, $\dot{y} = RQ$ is Hamiltonian, and consequently there exists a first integral H such that

$$\dot{x} = RP = \frac{\partial H}{\partial y}, \qquad \dot{y} = RQ = -\frac{\partial H}{\partial x}.$$

Then we say that R is *the integrating factor associated to the first integral* H, and vice versa.

Let $u = u(x, y) \in \mathbb{C}[x, y]$, i.e. u is a complex polynomial in the variables x and y. Then we say that the algebraic curve $u = 0$ is *invariant* for the system (3) if

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} = Ku, \tag{10}$$

for some polynomial $K \in \mathbb{C}[x, y]$.

Proof of Theorem 1. We write the polynomial H as a polynomial in the variable y with coefficients polynomials in the variable x, i.e.

$$H = \sum_{i=0}^{s} H_i(x)y^i.$$

We claim that $s > 0$. For proving the claim we suppose that $H = H(x)$. Then, from the definition of first integral (2) we get that $P(x, y)H'(x) = 0$. Here as usual $H'(x)$ denotes the derivative of H with respect to the variable x. Since $P(x, y) \neq 0$ we have that H is constant, in contradiction with the fact that H is a first integral. Consequently the claim is proved.

From the definition of the first integral we get that

$$\left(\sum_{j=0}^{k} p_j(x)y^j\right)\left(\sum_{i=0}^{s} H_i'(x)y^i\right) + \left(\sum_{i=0}^{l} q_i(x)y^i\right)\left(\sum_{i=1}^{s} i\,H_i(x)y^{i-1}\right) = 0. \tag{11}$$

If $k > l - 1$ the degree of (11) in the variable y is $s + k$. The coefficient of y^{s+k} in (11) is $p_k(x)H_s'(x) = 0$. Therefore, since $p_k(x) \neq 0$ we get that $H_s(x)$ is a constant $\alpha \in \mathbb{C}$. So, in what follows we shall work with the first integral H/α instead of H. Hence $H_s(x) = 1$. Then statement (a) is proved.

Assume $k = l - 1$ again the degree of (11) in the variable y is $s + k$. The coefficient of y^{s+k} in (11) is $p_k(x)H'_s(x) + sq_l(x)H_s(x) = 0$. So, clearly in order that this differential equation has a polynomial solution it is necessary that the degree of $p_k(x)$ must be equal to the degree of $q_l(x)$ plus one. Hence statement (b) is proved.

If $k < l-1$ the degree of (11) in the variable y is $s+l-1$. The coefficient of y^{s+l-1} in (11) is $sq_l(x)H_s(x)$. Since $s > 0$, and $q_l(x)$ and $H_s(x)$ are nonzero, we get that system (3) has no polynomial first integral. □

Proposition 5. *Assume that system* (3) *has a minimal polynomial first integral* $H = \sum_{i=0}^{s} H_i(x)y^i$ *with* $s > 0$ *(see Theorem 1). Suppose that* H *has only one critical remarkable value* c*. Let* $u_1, \ldots, u_q$ *be all the distinct remarkable factors of* $H + c$ *with exponents* $p_1, \ldots, p_q$*, respectively; i.e.* $H + c = \prod_{j=1}^{q} u_j^{p_j}$*. Then* $\alpha \prod_{j=1}^{q} u_j^{p_j-1}$ *is the polynomial integrating factor of system* (3) *associated to* H *for some* $\alpha \in \mathbb{C}$*.*

Proof. We have that $\partial H/\partial x = S \prod_{j=1}^{q} u_j^{p_j-1}$ and $\partial H/\partial y = T \prod_{j=1}^{q} u_j^{p_j-1}$, and every u_j does not divide both polynomials S and T. If R is the integrating factor associated to H we have that $\partial H/\partial y = PR$ and $\partial H/\partial x = -QR$. So, since the polynomials P and Q are coprime, we obtain that $R = U \prod_{j=1}^{q} u_j^{p_j-1}$, and for all $j = 1, \ldots, q$ the polynomial u_j does not divide U.

We claim that U is a constant. Otherwise each irreducible factor w of U in $\mathbb{C}[x, y]$ divides to $\partial H/\partial x$ and $\partial H/\partial y$, so it would exist another critical remarkable value d such that w^2 divides $H + d$, in contradiction with the assumption that we have a unique critical remarkable value. □

Proof of Theorem 2. From Proposition 5 we know that the integrating factor R associated to the first integral H is

$$R = \alpha \prod_{j=1}^{q} u_j^{p_j-1}, \tag{12}$$

with $\alpha \in \mathbb{C}$.

Since R is the integrating factor associated to H we have that

$$\frac{\partial H}{\partial y} = PR.$$

Computing the degrees of the both sides of the previous equality in the variable y we obtain that

$$s - 1 = k + \sum_{j=1}^{q}(p_j - 1)n_j.$$

By (4) we know that

$$\sum_{j=1}^{q} p_j n_j = s.$$

Consequently we get that

$$k+1 = \sum_{j=1}^{q} n_j.$$

Hence the theorem is proved. □

Proof of Theorem 3. Using the notations introduced in Theorem 2 we get that

$$k+1 = \sum_{j=1}^{q} n_j.$$

If we denote r the number of remarkable values that depends on y, since $k = 1$ we have two possibilities, either $r = 2$ and $n_1 = n_2 = 1$, or $r = 1$ and $n_1 = 2$. Hence the corresponding first integrals can be written in the form (6) or (7).

□

Proof of Theorem 4. Statements (a) and (b) of Theorem 4 follows easily.

Now we prove statement (c). Since system (9) is a particular case of (3) with $k = 1$ and $l = 1$, clearly $k > l - 1$. Then, by Theorem 1 we know that the polynomial first integral of system (9) begins with y^s and then, using Theorem 3, can be written as in the form (6) with $F = A = C = 1$ or (7) with $F = A = 1$. In the second case taking in account that the first integral is minimal, we have that $p = 1$ and the first integral has no remarkable critical values. In short, the first integral can be written as

$$H = (y + L(x))^p (y + M(x))^q, \tag{13}$$

where p and q are different positive integers. Substituting H in the definition of the first integral (2) we get that

$$a(x)y^2 + b(x)y + c(x) = 0,$$

where

$$\begin{aligned}
a(x) &= -pf(x) - qf(x) + pL'(x) + qM'(x),\\
b(x) &= -(p+q)g(x) - f(x)(qL(x) + pM(x)) + pM(x)L'(x) + qL(x)M'(x),\\
c(x) &= -g(x)(qL(x) + pM(x)).
\end{aligned}$$

From $c(x) = 0$ we obtain $M(x) = -qL(x)/p$. Then the equation $b(x) = 0$ becomes

$$\frac{(p+q)}{p}\left(p\, g(x) + qL(x)L'(x)\right) = 0.$$

Consequently

$$g(x) = -qL(x)L'(x)/p. \tag{14}$$

Simplifying the equation $a(x) = 0$ we get that

$$\frac{(p+q)}{p}\left(pf(x) + (q-p)L'(x)\right) = 0.$$

Therefore

$$L(x) = c + \frac{p}{p-q}F(x).$$

Substituting $L(x)$ in (13) and (14) the proof of the theorem is completed. □

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Departamento de Matemáticas, Universidad de Oviedo. Avda Calvo Sotelo, s/n., 33007, Oviedo, Spain.

E-mail address: belen.garcia@uniovi.es, jspr@uniovi.es

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
E-mail address: jllibre@mat.uab.cat