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ON THE BIFURCATIONS OF EQUILIBRIA OF A LOCALLY–COURNOT ECONOMIC MODEL

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Abstract. Guirao and Rubio [6] introduces an economic model, which generalizes the classical duopoly of Cournot type, where the competitors are located around a circle or a line and each firm competes "à la Cournot" with its right and left neighboring. For the case of having three and four players we describe completely the bifurcations of equilibria in terms of the production costs of each firm and we study the stability of them. Moreover, for the case of four players we provide some information on two–periodic orbits.

1. Introduction and statement of the main result

Cournot duopoly was introduced by Agustin Cournot [2] who is consider on of the forerunner of the modern microeconimics. The process consist of two firms which produce an identical good and which compete for the market. In each step of the process the firms decide the amount of product which introduce in the market and for making this decission both firms know the amount of product introduced in the market in the previous step by the rival firm. This economic process is mathematically modeled by the following two–dimensional discrete dynamical system

$$F(x, y) = (g(y), f(x)) \tag{1}$$

where f, g are continuous self–maps defined on a compact interval which can be considered, without loss of generality, by normalization $[0, 1]$. The maps f and g are called the *reaction functions* and determine the decisions maked by the firms.

Note that if firm A put on the market at the beginning of the game α_0 product and firm B put β_0, in the next step of the game firm A will produce $g(\beta_0)$, i.e. an amount of product which directly depends on the production level of the firm B in the previous step, on the other hand firm B will produce $f(\alpha_0)$ and so on. Therefore, all the process is governed by the dynamics of the discrete system (1) which strongly depends on the dynamics of the one–dimensional interval maps f and g.

Duopoly is an intermediate situation between monopoly and perfect competition, and analytically is a more complicated case. The reason for this is that oligopolist must consider not only the behaviors of the costumers, but also those of the competitors and their reactions, thus this model has been studied in the literature from different points of view, see for instance [1], [3] [4], [7], [8] [9], [10], [11] or [12].

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While dynamic properties of duopolies have been extensively studied, adjustment dynamics in Cournot processes with more than two players has received much less attention as a consequence of the difficulties which appear for studing discrete dynamical systems with dimension higher than two. The direct generalization of the Cournot duopoly situation is the Cournot oligopoly, i.e. consider n firms which produce and identical good and in each step of the process any firm knows the amoung of product generated by the $n-1$ rival firms in the previous step. Now, the systems which models the situation is given by

$$F(x_1, x_2, ..., x_n) = (f_1(x_2, x_3, ..., x_n), f_2(x_1, x_3, ..., x_n), ..., f_n(x_1, x_2, ..., x_{n-1})) \tag{2}$$

where $f_i : [0,1]^{n-1} \to [0,1]$ is a continuous map. We note that the reaction function f_i depends on $n-1$ variables of indices $j \in \{1, 2, ..., n\}$, $j \neq i$.

To study the dynamics for a system like (2) is quate complicated by the ignorance of the topological dynamics of n–dimensional discrete dynamical systems with $n > 2$ (e.g., note that for these type of systems are not characterized the possible ω–limit sets of the orbits).

Thus, if we want to have some chance of describing dynamics we need to simplify the system with the cost of lossing information by the players on the production level of the rivals.
In [5] is introduced the following model called Cournot–like system:

Definition 1. *A continuous map ϕ from $[0,1]^n$ into itself is Cournot–like if it is of the form:*

$$\phi(x_1, x_2, ..., x_n) = (\phi_{\sigma(1)}(x_{\sigma(1)}), ..., \phi_{\sigma(n-1)}(x_{\sigma(n-1)}), \phi_{\sigma(n)}(x_{\sigma(n)}),$$

where $\phi_i : [0,1] \to [0,1]$ is continuous, $i \in \{1, 2, ..., n\}$ and σ is a cyclic permutation of the set $\{1, 2, ..., n\}$.

In the economic situation models by these type of systems the level of information is quite limited because any player firm only has information on the production level of one of the other firms in the previous step of the process. For these type of systems, see [5], there is a characterization of the dynamical simplicty.

From our point of view Cournot–like models not represent a truthful economic situation since it is very difficult to explain the fact that each player firm only can has information on other firm having a complete ignorance on the rest of player behaviour. For that reason Guirao and Rubio [6] introduces a new model where the information level is higher than in Cournot–like ones and where there is more chance for describing dynamical properties. See next section for a concret description of this model. The aim of the present paper is, for dimensions 3 and 4, to describe completely the bifurcations of equilibria of this new model in terms of the production costs of each firm and we study the stability of them. Moreover, for the case of four players we provide some information on two–periodic orbits. The statement of our the main results is:

Theorem 2. *Let (F_3, Ω_3) be the discrete dynamical system introduced in (3). Then:*

- (i) *if c_1, c_2 and c_3 are not equal, there exists a unique equilibrium equal to $(0,0,0)$ which is strongly unstable;*
- (ii) *if $c_i = c$, $i = 1, 2, 3$, there exists another equilibrium, apart from the stated in* (i), *of the form $(\frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c})$ which is locally an attractor.*

Theorem 3. *Let* (F_4, Ω_4) *be the discrete dynamical system introduced in* (4)*. Then:*

(i) *if* c_1*,* c_2*,* c_3 *and* c_4 *are arbitrary positive constant, the point* $(0,0,0,0)$ *is a strongly unstable equilibrium;*

(ii) *if* $c_i = c$*,* $i = 1,2,3,4$*, the point* $(\frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c})$ *is a stable equilibrium;*

(iii) *if* $c_1 = c_3$ *and* $c_2 \neq c_4$ *the point* $\left(\alpha^2, \sqrt{\frac{2\alpha^2}{c_2}} - 2\alpha^2, \alpha^2, \sqrt{\frac{2\alpha^2}{c_4}} - 2\alpha^2\right)$*, where* α *is the unique root of the cubic polynomial*

$$-\sqrt{2c_2c_4}(\sqrt{c_2}+\sqrt{c_4})+2(c_1c_2+c_1c_4+2c_2c_4+2c_1\sqrt{c_2c_4})X$$
$$-6c_1\sqrt{2c_2c_4}(\sqrt{c_2}+\sqrt{c_4})X^2+9c_1c_2c_4X^3,$$

is an equilibrium. Its stability depends of the values of c_2 *and* c_4*. In the case* $c_i = c$*,* $i = 1,2,3,4$*, the equilibrium coincides with the presented one in* (ii)*;*

(iv) *if* $c_2 = c_4$ *and* $c_1 \neq c_3$*, the equilibria are symmetric to the ones of case* (iii) *interchanging* z *with* w*,* x *with* y*,* c_2 *with* c_1*, and* c_4 *with* c_3*.*

Moreover in the case $c_i = c$*,* $i = 1,2,3,4$*, we have a two–periodic orbit of the system of the form* (x,y,z,w) *where* $x = z = \frac{-8\, 3^{\frac{1}{3}} + (9+7\sqrt{33})^{\frac{2}{3}}}{3^{\frac{2}{3}}(9+7\sqrt{33})^{\frac{1}{3}}c}$ *and* $y = w = 0$*.*

2. The model

Let $N = \{1,2,...,n\}$ be the set of players (i.e., rival firms which produces an identical good) and assume that are physically located around a circle or a line. We assume that the firms compete "à la Cournot" in a local way, i.e., each firm $i \in N$ compete with its closest neighboring in the right and left direction. Let $B_i^\alpha \subset N$ be the neighboring located to distance equal to α of the firm i in the right and left direction. If we denote by $(x_1,...,x_n)$ the production of the firms in some moment and by $(c_1,c_2,...,c_n)$ their production costs, the best response function for the firm i will have the form

$$\phi_i(x_{B_i^\alpha}) = \sqrt{\frac{\sum_{k\in B_i^\alpha} x_k}{c_i}} - \sum_{k\in B_i^\alpha} x_k.$$

We consider that B_i^α is composed by the left and right neighboring, i.e., $\alpha = 1$. In this case, we note that if $n = 2$, we have the classical situation of the Cournot duopoly.

If we suppose that the number of players is equal to three, the model is governed by the three–dimensional discrete dynamical system given by

$$F_3(x,y,z) = \left(\sqrt{\frac{y+z}{c_1}} - (y+z), \sqrt{\frac{x+z}{c_2}} - (x+z), \sqrt{\frac{x+y}{c_3}} - (x+y)\right) \tag{3}$$

defined in $\Omega_3 = \{(x,y,z) \in \mathbb{R}^3 : x+y > 0, x+z > 0, y+z > 0\}$.

In the same way, for the case of having four firms, the model is defined by a four–dimensional discrete system $F_4(x,y,z,w)$ of the form (x,y,z,w) goes to

(4)
$$\left(\sqrt{\frac{y+w}{c_1}} - (y+w), \sqrt{\frac{x+z}{c_2}} - (x+z), \sqrt{\frac{y+w}{c_3}} - (y+w), \sqrt{\frac{x+z}{c_4}} - (x+z)\right)$$

defined in $\Omega_4 = \{(x,y,z,w) \in \mathbb{R}^4 : y+w > 0, x+z > 0\}$.

The objective of the next section is to study the equilibria of systems (3) and (4) in terms of production costs $c_i's$. In the case $n = 4$ we give some information on the two–periodic orbits.

3. Proof of Theorem 2

In this section we state the equilibria and their bifurcations for the system (3) depending on $c_i's$.

Proof of Theorem 2. We consider the map F_3 given by

$$(x, y, z) \to \left(\sqrt{\frac{y+z}{c_1}} - (y+z), \sqrt{\frac{x+z}{c_2}} - (x+z), \sqrt{\frac{x+y}{c_3}} - (x+y)\right).$$

A point (x, y, z) will be an equilibrium point if and only if is held simultaneously

- $k_1 = \sqrt{\frac{y+z}{c_1}} - (x+y+z) = 0,$
- $k_2 = \sqrt{\frac{x+z}{c_2}} - (x+y+z) = 0,$
- $k_3 = \sqrt{\frac{x+y}{c_3}} - (x+y+z) = 0.$

Clearly, one solution of the system, for any value of $c_i's$, is $(0, 0, 0)$. Now, consider the equivalent system $e_1 = e_2 = e_3 = 0$ where

- $e_1 = k_1 - k_2$
- $e_2 = k_1 - k_3$
- $e_3 = k_2 - k_3$

It is easy to note that the case $c_3 = c_1 + c_2$ does not provide any fixed point different from the origin. If c_3 is different from $c_1 + c_2$, the solution of the system $e_1 = e_2 = e_3 = 0$ is

$$x = -\frac{2c_1 z}{c_1 + c_2 - c_3} + \sqrt{\frac{2z}{c_1 + c_2 - c_3}},$$

$$y = -\frac{2c_2 z}{c_1 + c_2 - c_3} + \sqrt{\frac{2z}{c_1 + c_2 - c_3}},$$

$$z = -\frac{2c_3 z}{c_1 + c_2 - c_3} + \sqrt{\frac{2z}{c_1 + c_2 - c_3}}.$$

This system has a solution different from $z = 0$ in the case $c_i = c$, $i = 1, 2, 3$. Thus, the solution has the form

$$\left(-2z + \sqrt{\frac{2z}{c}}, -2z + \sqrt{\frac{2z}{c}}, -2z + \sqrt{\frac{2z}{c}}\right),$$

now for the equilibrium condition

$$-2z + \sqrt{\frac{2z}{c}} = z,$$

and therefore the equilibrium point has the form $\left(\frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c}\right)$.

In short, we obtain that $(0, 0, 0)$ is always an equilibrium point, and when $c_1 = c_2 = c_3$ then $\left(\frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c}\right)$ is another equilibrium point.

For studing the stability of the equilibria we compute the matrix M composed by the partial derivatives of the reaction functions of F. Indeed,

$$M = \begin{pmatrix} \frac{\partial F_3^1}{\partial x} & \frac{\partial F_3^1}{\partial y} & \frac{\partial F_3^1}{\partial z} \\ \frac{\partial F_3^2}{\partial x} & \frac{\partial F_3^2}{\partial y} & \frac{\partial F_3^2}{\partial z} \\ \frac{\partial F_3^3}{\partial x} & \frac{\partial F_3^3}{\partial y} & \frac{\partial F_3^3}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 & -1 + \frac{1}{2c_1\sqrt{\frac{y+z}{c_1}}} & -1 + \frac{1}{2c_1\sqrt{\frac{y+z}{c_1}}} \\ -1 + \frac{1}{2c_2\sqrt{\frac{y+z}{c_2}}} & 0 & -1 + \frac{1}{2c_2\sqrt{\frac{y+z}{c_2}}} \\ -1 + \frac{1}{2c_3\sqrt{\frac{y+z}{c_3}}} & -1 + \frac{1}{2c_3\sqrt{\frac{y+z}{c_3}}} & 0 \end{pmatrix}.$$

In this setting we conclude that the origin is strongly unstable in all directions because the partial derivatives which appear in the matrix M tend to infinity when $(x, y, z) \to 0$. Let assume that the costs $c_i's$ are equal to c, then the characteristical polynomial of the matrix M evaluated at the equilibrium point $\left(\frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c}\right)$ is equal to $P_\lambda = \frac{1}{32}(-1 + 6\lambda - 32\lambda^3)$. Note now that the roots of P_λ are equal to $\frac{1}{4}$ double and $-\frac{1}{2}$. Thus, the equilibrium point $\left(\frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c}\right)$ is locally an attractor because the absolute value of the eigenvalues at this fixed point are smaller than 1, ending the proof. ∎

Remark 4. *The dynamics on the straight line $x = y = z$ when $c_1 = c_2 = c_3 = c$, is equivalent to the dynamics of an one–dimensional map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sqrt{\frac{2x}{c}} - 2x$. On the line $x = y = z$ the fixed point $x = \frac{2}{9c}$ is an attractor. The point $x = 0$ is strongly unstable.*

4. Proof of Theorem 3

The equations of the equilibria points of the system (Ω_4, F_4) where F_4 is the transformation such that every (x, y, z, w) goes to

$$\left(\sqrt{\frac{y+w}{c_1}} - (y+w), \sqrt{\frac{x+z}{c_2}} - (x+z), \sqrt{\frac{y+w}{c_3}} - (y+w), \sqrt{\frac{x+z}{c_4}} - (x+z)\right)$$

are the following:

- $k_1 = \sqrt{\frac{y+w}{c_1}} - (x + y + w) = 0,$
- $k_2 = \sqrt{\frac{x+z}{c_2}} - (x + y + z) = 0,$
- $k_3 = \sqrt{\frac{y+w}{c_3}} - (y + w + z) = 0,$
- $k_4 = \sqrt{\frac{x+z}{c_4}} - (x + z + w) = 0.$

On the other hand the matrix M composed by the partial derivatives of the reaction functions of F_4 is

$$M = \begin{pmatrix} 0 & -1 + \frac{1}{2c_1\sqrt{\frac{w+y}{c_1}}} & 0 & -1 + \frac{1}{2c_1\sqrt{\frac{w+y}{c_1}}} \\ -1 + \frac{1}{2c_2\sqrt{\frac{x+z}{c_2}}} & 0 & -1 + \frac{1}{2c_2\sqrt{\frac{x+z}{c_2}}} & 0 \\ 0 & -1 + \frac{1}{2c_3\sqrt{\frac{w+y}{c_3}}} & 0 & -1 + \frac{1}{2c_3\sqrt{\frac{w+y}{c_3}}} \\ -1 + \frac{1}{2c_4\sqrt{\frac{x+z}{c_4}}} & 0 & -1 + \frac{1}{2c_4\sqrt{\frac{x+z}{c_4}}} & 0 \end{pmatrix}$$

where entry $a_{i,j}$ correspond to $\frac{\partial F_4^i}{\partial \alpha_j}$ with $\alpha_1 = x$, $\alpha_2 = y$, $\alpha_3 = z$ and $\alpha_4 = w$.

Clearly, by the two previous expressions, always $(0, 0, 0, 0)$ is a fixed point strongly unstable because the derivatives of the linear part of the map at it goes to infinity.

We consider now three different cases:

- Case $c_1 = c_2 = c_3 = c$. In this case the equilibria equations are reduced to
$$-x - 2y + \sqrt{\frac{2y}{c}} = 0, \qquad -2x + \sqrt{\frac{2x}{c}} - y = 0.$$
Solving this system we obtain that there is an additional equilibrium point of the form $(\frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c}, \frac{2}{9c})$ which is locally stable because its eigenvalues are $(0, 0, \frac{-1}{2}, \frac{1}{2})$ if we assume that $c > 0$.
- Case $c_3 = c_1$ and $c_4 \neq c_2$. In this setting equations of equilibria have the form
$$-w - x - y + \sqrt{\frac{x+y}{c_1}} = 0 \text{ where } w = -2x + \sqrt{\frac{2x}{c_4}} \text{ and } y = -2x + \sqrt{\frac{2x}{c_2}}.$$
Computing, we reduce the problem to solve
$$3x - \sqrt{\frac{2x}{c_2}} - \sqrt{\frac{2x}{c_4}} + \sqrt{\frac{-4x + \sqrt{\frac{2x}{c_2}} + \sqrt{\frac{2x}{c_4}}}{c_1}} = 0. \tag{5}$$
We can assume that x is not zero, otherwise we obtain the equilibrium point $(0, 0, 0, 0)$. Dividing equation (5) by $\sqrt{x}$ we obtain
$$3\sqrt{x} - \sqrt{\frac{2}{c_2}} - \sqrt{\frac{2}{c_4}} + \sqrt{\frac{-4 + \sqrt{\frac{2}{c_2 x}} + \sqrt{\frac{2}{c_4 x}}}{c_1}} = 0. \tag{6}$$
Replacing in equation (6) variable $\sqrt{x}$ by X we have
$$3X - \sqrt{\frac{2}{c_2}} - \sqrt{\frac{2}{c_4}} + \sqrt{\frac{-4X + \sqrt{\frac{2}{c_2}} + \sqrt{\frac{2}{c_4}}}{c_1 X}} = 0. \tag{7}$$
Now, we eliminate the squareroot containg the variable X in equation (7) taking squares, but note that doing this we can add fictious solutions. Thus, is
$$-\sqrt{2c_2c_4}(\sqrt{c_2} + \sqrt{c_4}) + 2(c_1c_2 + c_1c_4 + 2c_2c_4 + 2c_1\sqrt{c_2c_4})X \tag{8}$$
$$-6c_1\sqrt{2c_2c_4}(\sqrt{c_2} + \sqrt{c_4})X^2 + 9c_1c_2c_4X^3 = 0, \tag{9}$$
and therefore computing is stated that the point
$$\left(\alpha^2, \sqrt{\frac{2\alpha^2}{c_2}} - 2\alpha^2, \alpha^2, \sqrt{\frac{2\alpha^2}{c_4}} - 2\alpha^2\right),$$
where α is the unique root of the cubic polynomial (8) is an equilibrium of the system different from the origin. Note that the stability of this equilibrium depends on the values of c_2 and c_4 and in the case $c_i = c$, $i = 1, 2, 3, 4$, the equilibrium coincides with the presented one in (ii).
- Case $c_4 = c_2$ and $c_3 \neq c_1$. In fact the equilibria points are symmetric to ones obatined in the previous case interchanging z with w, x with y, c_2 with c_1, and c_4 with c_3.

For the case $c_i = c$, $i = 1, 2, 3, 4$, we study the two–periodic orbits of the system (i.e., non–equilibria points with the property $F^2(x, y, z, w) = (x, y, z, w)$.) In general the equations of the two–periodics points are the following:

- $g_1 = w + y + z - \sqrt{\frac{w+y}{c_3}} - \sqrt{\frac{x+z}{c_2}} + \sqrt{\frac{-w-x-y+\sqrt{\frac{w+y}{c_3}}-z+\sqrt{\frac{x+z}{c_2}}}{c_1}} = 0,$

- $g_2 = 2w + y - \sqrt{\frac{w+y}{c_1}} - \sqrt{\frac{w+y}{c_3}} + \sqrt{\frac{-2w-2y+\sqrt{\frac{w+y}{c_1}}+\sqrt{\frac{w+y}{c_3}}}{c_2}} = 0,$
- $g_3 = w + y + x - \sqrt{\frac{w+y}{c_1}} - \sqrt{\frac{x+z}{c_2}} + \sqrt{\frac{-w-x-y+\sqrt{\frac{w+y}{c_1}}-z+\sqrt{\frac{x+z}{c_2}}}{c_3}} = 0,$
- $g_4 = w + 2y - \sqrt{\frac{w+y}{c_1}} - \sqrt{\frac{w+y}{c_3}} + \sqrt{\frac{-2w-2y+\sqrt{\frac{w+y}{c_1}}+\sqrt{\frac{w+y}{c_3}}}{c_4}} = 0.$

For the case $c_i = c$, $i = 1,2,3,4$, is $g_1 - g_3 = 0$ and $g_2 - g_4 = 0$ which means $x = z$ and $w = y$, so introducing this into the system of equations and solving which the adequate computational tool we obtain that the point (x, y, z, w) where $x = z = \frac{-8\,3^{\frac{1}{3}}+(9+7\sqrt{33})^{\frac{2}{3}}}{3^{\frac{2}{3}}(9+7\sqrt{33})^{\frac{1}{3}}c}$ and $y = w = 0$ is a two–periodic orbit of the system, ending the proof. ∎

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References

[1] G.I. Bischi, L. Gardini and C. Mammana, *Multistability and Cyclic Attractors in Duopoly Games*, Chaos, Solitions and Fractals. **11**, (2000), 543-564.

[2] A. Cournot, *Réserches sur les principles mathemátiques de la théorie de la richesse*, Hachette, Paris, 1938.

[3] R.A. Dana and L. Montrucchio, *Dynamical Complexity in Duopoly Games*, J. Econom. Theory. **40**, (1986), 40-56.

[4] W. Govaerts and R. Khoshsiar Ghaziani, *Stable cycles in Cournot duopoly model of Kopel*, J. Comp. Appl. Math. **218**, (2008), 247-258.

[5] J.L.G. Guirao and R.G. Rubio, *Detecting simple dynamics in Cournot–like models*, J. Comp. Appl. Math., to appear.

[6] J.L.G. Guirao and R.G. Rubio, *Extensions of Cournot Duopoly: A dynamical view*, Economic Letters, to appear.

[7] M. Kopel, *Simple and Complex Adjustement Dynamics in Cournot Duopoly Games*, Chaos, Solitions and Fractals, **7**, (1996), 2031-2048.

[8] T. Puu, *Chaos in Duopoly Pricing*, Chaos, Solitions and Fractals, **1**, (1991), 573-581.

[9] T. Puu, *Nonlinear Economic Dynamics*, (1997), Springer-Verlag, Berlin.

[10] D. Rand, *Exotic Phenomena in Games and Duopoly Models*, J. Math. Econom. **5**, (1978), 173-184.

[11] M.S. Santos and J. Vigo–Aguiar, *Error bounds for a numerical solution for dynamic economic models*, Appl. Math. Lett. **9(4)**, (1996), 41–45.

[12] M.S. Santos and J. Vigo–Aguiar, *Analysis of a Numerical Dynamic Programming Algorithm Applied to Economic Models*,Econometrica **60(2)**, (1998), 409–426.

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