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PERIODIC ORBITS NEAR EQUILIBRIA VIA AVERAGING THEORY OF SECOND ORDER

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Abstract. Lyapunov, Weinstein and Moser obtained remarkable theorems giving sufficient conditions for the existence of periodic orbits emanating from an equilibrium point of a differential system with a first integral. Using averaging theory of first order we established in [1] a similar result for a differential system *without* assuming the existence of a first integral. Now, using averaging theory of second order, we extend our result to the case when the first order average is identically zero. Our result can be interpreted as a kind of special Hopf bifurcation.

1. Introduction

Consider a system of ordinary differential equations

$$\dot{x} = f(x), \qquad x = (x_1, \dots, x_m) \tag{1}$$

near an equilibrium point which we assume to be the origin $x = 0$. The variables x_k, for $k = 1, \dots, m$, are real, and the dot refers to differentiation with respect to the independent variable t. A special role in the theory is played by the Hamiltonian systems

$$\dot{x}_k = H_{x_{n+k}}, \quad \dot{x}_{n+k} = -H_{x_k}, \qquad k = 1, \dots, n, \tag{2}$$

where H_{x_l} denotes the partial derivative of the Hamiltonian $H(x_1, \dots, x_{2n})$ with respect to the variable x_l.

For the equilibrium point $x = 0$, we consider the linear variational equation

$$\dot{x} = Ax, \qquad A = f_x(0), \tag{3}$$

where $f_x(0)$ denotes the Jacobian matrix of the function f evaluated at $x = 0$. Clearly, every pair of conjugated purely imaginary eigenvalues of A gives rise to periodic solutions of (3). We consider the classical problem of finding periodic solutions near $x = 0$ for the nonlinear system (1). As it is well known, for this purpose the presence of purely imaginary eigenvalues is necessary but not sufficient. In 1907 Lyapunov [3] established the existence of a one-parameter family of periodic solutions under two assumptions. Namely, he assumed the existence of a first integral and a nonresonance condition on the purely imaginary eigenvalues of A (see Theorem 9.2.1 of [5]).

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Theorem 1 (Lyapunov's Theorem). *Assume that the function f is of class $\mathcal{C}^1$ and that a first integral H of system* (1) *is of class $\mathcal{C}^2$ near $x = 0$, with $H_x(0) = 0$ (where H_x denotes the gradient of H) and positive definite Hessian $H_{xx}(0)$, such that the equilibrium at the origin has eigenvalues $\pm\omega i$, $\lambda_3, \ldots, \lambda_m$, where $\omega i \neq 0$ is purely imaginary. If λ_k/ω is not an integer for $k = 3, \ldots, m$, then there exists a one-parameter family of periodic orbits emanating from the equilibrium point.*

In 1973 Weinstein [9, 10] showed that the additional nonresonance condition is not necessary for Hamiltonian systems. His result can be stated as follows:

Theorem 2 (Weinstein's Theorem). *If the Hamiltonian H is of class $\mathcal{C}^2$ near $x = 0$, $H_x(0) = 0$, and the Hessian $H_{xx}(0)$ is positive definite, then for any sufficiently small ε the energy surface $H(x) = H(0) + \varepsilon^2$ contains at least n periodic solutions of system* (2) *whose periods are close to those of the linear system* (3).

In 1976 Moser [6] established a similar result for system (1) assuming the existence of a first integral $H(x)$ with $H_x(0) = 0$ and positive definite Hessian $H_{xx}(0)$ without requiring the system to be Hamiltonian.

Theorem 3 (Moser's Theorem). *If the function f is of class $\mathcal{C}^1$ and a first integral H of system* (1) *is of class $\mathcal{C}^2$ near $x = 0$, with $H_x(0) = 0$ and positive definite Hessian $H_{xx}(0)$, then for any sufficiently small ε the energy surface $H(x) = H(0) + \varepsilon^2$ contains at least one periodic solution of system* (1) *whose period is close to one of the linear system* (3).

Our goal is to obtain similar results to those obtained by Lyapunov, Weinstein and Moser for system (1) but now *without* assuming the existence of a first integral. We consider vector fields $f = f_\varepsilon$ depending on a real parameter ε such that when $\varepsilon = 0$ the origin is an equilibrium point of system (1) with eigenvalues $\pm\omega i \neq 0$ and 0 with multiplicity $m-2$. For such systems we provide sufficient conditions so that periodic orbits bifurcate from the origin when $\varepsilon \neq 0$ is sufficiently small. In [1] we used averaging theory of first order to establish a similar result for a differential system without assuming the existence of a first integral. In the present paper, using averaging theory of second order, we extend the results in [1] to the case when the first order average is identically zero. We note that our results can be interpreted as a kind of special Hopf bifurcation.

Our results are stated in Section 3, and are proved in the following sections. The proofs use averaging theory of second order. We refer to Section 2 for a summary of this theory.

2. Averaging theory of second order

The next theorem provides a second order approximation for the limit cycles of a periodic system when its average vanishes at first order. For a statement see [4], and for a proof see Theorem 3.5.1 of Sanders and Verhulst [7], or [2].

Consider functions $f, g\colon [0,\infty)\times\Omega \to \mathbb{R}^n$ and $R\colon [0,\infty)\times\Omega\times(0,\varepsilon_0] \to \mathbb{R}^n$, where Ω is an open subset of $\mathbb{R}^n$, such that f, g and R are T-periodic in the first variable. We set

$$f^1(t,x) = \frac{\partial f}{\partial x}(t,x)y^1(t,x), \quad \text{where} \quad y^1(t,x) = \int_0^t f(s,x)\,ds, \tag{4}$$

and we consider the averages of f, f^1 and g, defined respectively by

$$f^0(x) = \frac{1}{T}\int_0^T f(t,x)\,dt, \quad f^{10}(x) = \frac{1}{T}\int_0^T f^1(t,x)\,dt,$$

$$g^0(x) = \frac{1}{T}\int_0^T g^1(t,x)\,dt.$$

We also consider the two initial value problems

$$\dot{x} = \varepsilon f(t,x) + \varepsilon^2 g(t,x) + \varepsilon^3 R(t,x,\varepsilon), \quad x(0) = x_0, \tag{5}$$

and

$$\dot{y} = \varepsilon f^0(y) + \varepsilon^2\big(f^{10}(y) + g^0(y)\big), \quad y(0) = x_0. \tag{6}$$

Theorem 4. *Assume that: (i) $f^0 = 0$; (ii) $\partial f/\partial x$, g and R are Lipschitz in x, and all these functions are continuous on their domain of definition; (iii) $R(t,x,\varepsilon)$ is bounded by a constant uniformly on $[0, L/\varepsilon)\times\Omega\times(0,\varepsilon_0]$; and (iv) the solution $y(t)$ belongs to Ω in the interval of time $[0,1/\varepsilon]$. Then the following statements hold.*

(a) *At time scale $1/\varepsilon$ we have*

$$x(t) = y(t) + \varepsilon y^1(t, y(t)) + O(\varepsilon^2).$$

(b) *If p is an equilibrium point of the averaged system (6) with*

$$\det\big(f_y^{10} + g_y^0\big)(p) \neq 0, \tag{7}$$

then there exists a limit cycle $\phi(t,\varepsilon)$ of period T for system (5) that is close to p, such that $\phi(0,\varepsilon)\to p$ as $\varepsilon\to 0$.

(c) *The stability or instability of the limit cycle $\phi(t,\varepsilon)$ is given respectively by the stability or instability of the equilibrium point p of system (6).*

3. Statement of the results

We formulate in this section our results for equation (1).

3.1. **Standing assumptions.**

We assume throughout the paper that the function f is of class $\mathcal{C}^3$ near $x = 0$, with

$$f(0) = (0, 0, \varepsilon^2\lambda_3 + \varepsilon^3\mu_3, \ldots, \varepsilon^2\lambda_m + \varepsilon^3\mu_m),$$

and

$$f_x(0) = \begin{pmatrix} 0 & -\omega & 0 & \cdots & 0 \\ \omega & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

and that the second and third derivatives $f_{xx}(0)$ and $f_{xxx}(0)$ are independent of ε, where $\omega, \lambda_3, \mu_3, \ldots, \lambda_m, \mu_m$ are real constants such that $\omega \neq 0$ and $\mu_3^2 + \cdots + \mu_m^2 \neq 0$. Under these standing assumptions, and introducing the notation

$$f_{ij}^l = f_{x_i x_j}^l(0) \quad \text{and} \quad f_{ijk}^l = f_{x_i x_j x_k}^l(0) \text{ for } l = 1, 2, \ldots, m,$$

we can rewrite system (1) in the form

$$\begin{aligned}
\dot{x}_1 &= -\omega x_2 + \sum_{i=1}^m \sum_{j=i}^m \delta_{ij} f_{ij}^1 x_i x_j + \sum_{i=1}^m \sum_{j=i}^m \sum_{k=j}^m \delta_{ijk} f_{ijk}^1 x_i x_j x_k \\
&\quad + O_4(x_1, \ldots, x_m, \varepsilon), \\
\dot{x}_2 &= \omega x_1 + \sum_{i=1}^m \sum_{j=i}^m \delta_{ij} f_{ij}^2 x_i x_j + \sum_{i=1}^m \sum_{j=i}^m \sum_{k=j}^m \delta_{ijk} f_{ijk}^2 x_i x_j x_k \\
&\quad + O_4(x_1, \ldots, x_m, \varepsilon), \\
\dot{x}_k &= \varepsilon^2 \lambda_k + \varepsilon^3 \mu_k + \sum_{i=1}^m \sum_{j=i}^m \delta_{ij} f_{ij}^k x_i x_j + \sum_{i=1}^m \sum_{j=i}^m \sum_{k=j}^m \delta_{ijk} f_{ijk}^k x_i x_j x_k \\
&\quad + O_4(x_1, \ldots, x_m, \varepsilon),
\end{aligned} \tag{8}$$

for $k = 3, \ldots, m$, where $\delta_{ij} = 1$ for $i \neq j$, $\delta_{ii} = 1/2$, $\delta_{iii} = 1/6$, $\delta_{ijk} = 1/2$ for $i = j < k$ or $i < j = k$, and $\delta_{ijk} = 1$ for $i < j < k$. Moreover, each $O_4(x_1, \ldots, x_m, \varepsilon)$ denotes a term of order 4 in $x_1, \ldots, x_m$ and ε.

Now we introduce coordinates $(\rho, \theta, y_3, \ldots, y_m)$ in $\mathbb{R}^m$ satisfying $x_1 = \varepsilon\rho\cos\theta$, $x_2 = \varepsilon\rho\sin\theta$, and $x_k = \varepsilon y_k$ for $k = 3, \ldots, m$. Our main aim is to apply the averaging theory of second order when the averaged system vanishes at first order (see Section 2). In this case one is not able to apply the averaging theory of first order to the system, or more precisely to the reduced system in $\mathbb{R}^{m-1}$ using the coordinates $\rho, y_3, \ldots, y_m$ in the region $\dot{\theta} \neq 0$, taking the variable θ as the new time. It is shown in [1] that the averaged system vanishes at first order if and only if the following conditions hold:

(H1) $\lambda_j = 0$ for $j = 3, \ldots, m$;
(H2) $f_{1j}^1 = -f_{2j}^2$ for $j = 3, \ldots, m$;
(H3) $f_{11}^k = -f_{22}^k$ for $j = 3, \ldots, m$;
(H4) $f_{jl}^k = 0$ for $j = 3, \ldots, m$ and $l = j, \ldots, m$.

Thus, these will be standing assumptions in the paper.

3.2. **Main results.** The following result gives explicitly the averaged system of second order for an arbitrary dimension m.

Theorem 5. *In coordinates $(\rho, y_3, \ldots, y_m)$ and in the region $\dot\theta \neq 0$, the averaged system of second order of the differential system* (8) *is given by*

$$
\begin{aligned}
\frac{d\rho}{d\theta} = \varepsilon^2 \Bigg[& \frac{\rho^3}{16\omega^2}\Bigg(f^1_{11}f^1_{12} + f^1_{12}f^1_{22} - f^1_{11}f^2_{11} - f^2_{11}f^2_{12} + f^1_{22}f^2_{22} - f^2_{12}f^2_{22} \\
& - \sum_{j=3}^{m} f^1_{2j}f^j_{22} - \sum_{j=3}^{m} f^2_{1j}f^j_{22} + 2\sum_{j=3}^{m} f^2_{2j}f^j_{12}\Bigg) \\
& + \frac{\rho}{2\omega^2}\sum_{i=3}^{m}\sum_{j=i}^{m}\delta_{ij}\big(f^1_{ij}f^1_{12} + f^1_{ij}f^2_{22} - f^2_{ij}f^1_{11} - f^2_{ij}f^2_{12}\big)y_iy_j \\
& + \frac{\rho}{\omega^2}\sum_{j=3}^{m}\sum_{i=j}^{m}\sum_{l=3}^{m}\delta_{ij}\big(f^2_{ij}f^j_{1l} - f^1_{ij}f^j_{2l}\big)y_iy_l \\
& + \frac{\rho^3}{16\omega}\big(f^1_{111} + f^1_{122} + f^2_{112} + f^2_{222}\big) \\
& + \frac{\rho}{2\omega}\sum_{i=3}^{m}\sum_{j=i}^{m}\big(\delta_{1ij}f^1_{1ij} + \delta_{2ij}f^2_{2ij}\big)y_iy_j\Bigg], \\
\frac{dy_k}{d\theta} = \varepsilon^2\Bigg[& \mu_k - \frac{1}{\omega^2}\sum_{i=3}^{m}\sum_{j=i}^{m}\sum_{l=3}^{m}\delta_{ij}\big(f^2_{ij}f^k_{1l} - f^1_{ij}f^k_{2l}\big)y_iy_jy_l \\
& + \frac{\rho^2}{4\omega^2}\sum_{j=3}^{m}\big(f^1_{2j}f^k_{22} + f^2_{1j}f^k_{22} - 2f^2_{2j}f^k_{12} \\
& - f^2_{11}f^k_{1j} - f^2_{22}f^k_{1j} + f^k_{22}f^k_{2j} + f^1_{22}f^k_{2j}\big)y_j \\
& + \frac{\rho^2}{4\omega}\sum_{j=3}^{m}\big(f^k_{11j} + f^k_{22j}\big)y_j + \frac{1}{\omega}\sum_{i=3}^{m}\sum_{j=i}^{m}\sum_{l=j}^{m}\delta_{ijl}f^k_{ijl}y_iy_jy_l\Bigg],
\end{aligned}
\tag{9}
$$

for $k = 3, \ldots, m$.

Theorem 5 is proved in Section 4.

By Theorem 4, looking for the equilibrium points of system (9) satisfying condition (7) we obtain periodic orbits of system (8). Taking into account that the radial polar coordinate ρ is only well defined when $\rho > 0$, we are only interested in the equilibrium points $(\rho, y_3, \ldots, y_m)$ of system (9) with $\rho > 0$. Therefore, using Bézout's Theorem (see for instance [8]) we obtain the following corollary of Theorem 5.

Corollary 6. *Assume that the differential system* (9) *satisfying (H1)–(H4) has finitely many equilibrium points. Then for any sufficiently small $\varepsilon \neq 0$ there are at most 3^{m-1} periodic orbits bifurcating from the origin of system* (8).

3.3. **The case $m = 3$.** In dimension 3 we can be more precise. Set

$$\begin{aligned}
\Delta_1 &= \frac{1}{16\omega^2}\Big[f^1_{11}f^1_{12} + f^1_{12}f^1_{22} - f^1_{11}f^2_{11} - f^2_{11}f^2_{12} + f^1_{22}f^2_{22} - f^2_{12}f^2_{22} \\
&\quad - f^1_{23}f^3_{22} - f^2_{13}f^3_{22} + 2f^2_{23}f^3_{12} + \omega\big(f^1_{111} + f^1_{122} + f^2_{112} + f^2_{222}\big)\Big], \\
\Delta_2 &= \frac{\rho}{4\omega^2}\Big(f^1_{33}f^1_{12} + f^1_{33}f^2_{22} - f^2_{33}f^1_{11} - f^2_{33}f^2_{12}\Big) + \frac{1}{2}\big(f^2_{33}f^3_{13} - f^1_{33}f^3_{23}\big) \\
&\quad + 16\omega\big(f^1_{133} + f^2_{233}\big), \\
\Delta_3 &= \frac{1}{4\omega^2}\big(f^1_{23}f^3_{22} + f^2_{13}f^3_{22} - 2f^2_{23}f^3_{12} - f^2_{11}f^3_{13} - f^2_{22}f^3_{13} + f^3_{22}f^3_{23} + f^1_{22}f^3_{23}\big) \\
&\quad + \frac{4}{\omega}f^3_{113}, \\
\Delta_4 &= -\frac{1}{2\omega^2}\big(f^2_{33}f^3_{13} - f^1_{33}f^3_{23}\big) + \frac{1}{6\omega}f^3_{333}.
\end{aligned}$$

Then we have the following statement.

Theorem 7. *Under the assumptions of Subsection 3.1, if $\mu_3 \neq 0$,*

$$\Delta_1\Delta_2 < 0, \quad \textit{and} \quad \Delta_2\Delta_3 - \Delta_1\Delta_4 \neq 0,$$

then for any sufficiently small $\varepsilon \neq 0$ system (8) has a periodic solution which is close to the circle of radius

$$\frac{|\omega\mu_3|^{1/3}\sqrt{|\Delta_2|}}{|\Delta_1|^{3/2}|\Delta_2\Delta_3 - \Delta_1\Delta_4|}\,\varepsilon.$$

Theorem 7 is proved in Section 5.

3.4. **An example with $m = 4$.** The following system in $\mathbb{R}^4$ satisfies the standing assumptions of Subsection 3.1, and has three periodic orbits.

Example 1. *The system*

$$\begin{aligned}
\dot{x}_1 &= -\omega x_2 + x_1x_2 + x_4^2 + \frac{1}{\omega}x_1x_3^2 + \frac{1}{\omega}x_1x_3x_4 - \frac{4}{3\omega}x_1^3, \\
\dot{x}_2 &= \omega x_1, \\
\dot{x}_3 &= \varepsilon^3\mu_3 + x_4x_3^2\mu_3 - 2\mu_3x_1^2x_3 - 2\mu_3x_4x_1^2, \\
\dot{x}_4 &= \varepsilon^3\mu_4 + x_2x_3 - 2\mu_4x_1^2x_3 - 2\mu_4x_4x_1^2 + \frac{1}{\omega}(-1 + \mu_4\omega)x_4x_3^2,
\end{aligned} \tag{10}$$

has three limit cycles bifurcating from the origin for $\varepsilon \neq 0$ sufficiently small.

The details of the example are given in Section 6.

4. Proof of Theorem 5

Equivalent form of system (8). Under the standing assumptions in Section 3.1 we can write system (1) as in (8). Furthermore, using the conditions

(H1)–(H4) we can rewrite system (8) in the form

$$
\begin{aligned}
\dot{x}_l = {} & (-1)^l \omega x_{3-l} + \frac{1}{2} f_{11}^l x_1^2 + f_{12}^l x_1 x_2 + (l-2) \sum_{j=3}^m f_{2j}^2 x_1 x_j \\
& + (l-1) \sum_{j=3}^m f_{1j}^2 x_1 x_j + \frac{1}{2} f_{22}^l x_2^2 + \sum_{j=3}^m f_{2j}^l x_2 x_j + \sum_{i=3}^m \sum_{j=i}^m \delta_{ij} f_{ij}^l x_i x_j \\
& + \frac{1}{6} f_{111}^l x_1^3 + \frac{1}{2} f_{112}^l x_1^2 x_2 + \frac{1}{2} \sum_{j=3}^m f_{11j}^l x_1^2 x_j \\
& + \frac{1}{2} f_{122}^l x_1 x_2^2 + \sum_{j=3}^m f_{12j}^l x_1 x_2 x_j + \sum_{j=3}^m \sum_{k=j}^m \delta_{1jk} f_{1jk}^l x_1 x_j x_k \\
& + \frac{1}{6} f_{222}^l x_2^3 + \frac{1}{2} \sum_{j=3}^m f_{22j}^l x_2^2 x_j + \sum_{j=3}^m \sum_{k=j}^m \delta_{2jk} f_{2jk}^l x_2 x_j x_k \\
& + \sum_{i=3}^m \sum_{j=3}^m \sum_{k=j}^m \delta_{ijk} f_{ijk}^l x_i x_j x_k + O_4(x_1, \dots, x_m, \varepsilon),
\end{aligned}
$$

for $l = 1, 2$, and

$$
\begin{aligned}
\dot{x}_k = {} & \varepsilon^3 \mu_k - \frac{1}{2} f_{22}^k (x_1^2 - x_2^2) + f_{12}^k x_1 x_2 + \sum_{j=3}^m f_{1j}^k x_1 x_j \\
& + \sum_{j=3}^m f_{2j}^k x_2 x_j + \frac{1}{6} f_{111}^k x_1^3 + \frac{1}{2} f_{112}^k x_1^2 x_2 \\
& + \frac{1}{2} \sum_{j=3}^m f_{11j}^k x_1^2 x_j + \frac{1}{2} f_{122}^k x_1 x_2^2 + \sum_{j=3}^m f_{12j}^k x_1 x_2 x_j \\
& + \sum_{j=3}^m \sum_{k=j}^m \delta_{1jk} f_{1jk}^k x_1 x_j x_k + \frac{1}{6} f_{222}^k x_2^3 + \frac{1}{2} \sum_{j=3}^m f_{22j}^k x_2^2 x_j \\
& + \sum_{j=3}^m \sum_{k=j}^m \delta_{2jk} f_{2jk}^k x_2 x_j x_k + \sum_{i=3}^m \sum_{j=3}^m \sum_{k=j}^m \delta_{ijk} f_{ijk}^k x_i x_j x_k \\
& + O_4(x_1, \dots, x_m, \varepsilon),
\end{aligned}
$$

for $k = 3, \dots, m$.

Introduction of new variables. Since we want to apply averaging theory, we introduce the change of coordinates

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \quad x_k = x_k, \quad k = 3, \dots, m,$$

after which we can rewrite system (8) in the form

$$
\begin{aligned}
\dot r &= T_{11}(\theta, r, x_3, \ldots, x_m) + T_{12}(\theta, r, x_3, \ldots, x_m)\\
&\quad + H_1(\theta, r, x_3, \ldots, x_m) + O_4(r, x_3, \ldots, x_m),\\
\dot\theta &= \omega + T_2(\theta, r, x_3, \ldots, x_m) + H_2(\theta, r, x_3, \ldots, x_m)/r,\\
\dot x_k &= \varepsilon^3\mu_k + T_{k1}(\theta, r, x_3, \ldots, x_m) + T_{k2}(\theta, r, x_3, \ldots, x_m)\\
&\quad + H_k(\theta, r, x_3, \ldots, x_m) + O_4(r, x_3, \ldots, x_m)
\end{aligned}
\tag{11}
$$

for $k = 3, \ldots, m$, where

$$
\begin{aligned}
T_{11} &= r^2\Big[\frac12 f^1_{11}\cos^3\theta + \Big(f^1_{12} + \frac12 f^2_{11}\Big)\cos^2\theta\sin\theta\\
&\quad + \Big(\frac12 f^1_{22} + f^2_{12}\Big)\cos\theta\sin^2\theta + \frac12 f^2_{22}\sin^3\theta\Big]\\
&\quad + r\sum_{j=3}^m \big[f^2_{2j}(\sin^2\theta - \cos^2\theta) + \big(f^1_{2j} + f^2_{1j}\big)\cos\theta\sin\theta\big]x_j\\
&\quad + \sum_{i=3}^m\sum_{j=i}^m \delta_{ij}\big(f^1_{ij}\cos\theta + f^2_{ij}\sin\theta\big)x_ix_j,
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
T_{12} &= r^3\Big[\frac16 f^1_{111}\cos^4\theta + \frac12(f^1_{122} + f^2_{112})\cos^2\theta\sin^2\theta + \frac16 f^2_{222}\sin^4\theta\Big]\\
&\quad + r\sum_{i=3}^m\sum_{j=i}^m(f^1_{1ij}\cos^2\theta + f^1_{2ij}\sin^2\theta)x_ix_j,
\end{aligned}
$$

and

$$
\begin{aligned}
T_2 &= r\Big[\frac12 f^2_{11}\cos^3\theta + \Big(f^2_{12} - \frac12 f^1_{11}\Big)\cos^2\theta\sin\theta\\
&\quad + \Big(\frac12 f^2_{22} - f^1_{12}\Big)\cos\theta\sin^2\theta - \frac12 f^1_{22}\sin^3\theta\Big]\\
&\quad + \sum_{j=3}^m \big(f^2_{1j}\cos^2\theta - f^1_{2j}\sin^2\theta + 2f^2_{2j}\cos\theta\sin\theta\big)x_j\\
&\quad + \frac1r\sum_{i=3}^m\sum_{j=i}^m \delta_{ij}\big(f^2_{ij}\cos\theta - f^1_{ij}\sin\theta\big)x_ix_j,
\end{aligned}
$$

and where for $k = 3, \ldots, m$,

$$
\begin{aligned}
T_{k1} &= r^2\Big[-\frac12 f^k_{22}(\cos^2\theta - \sin^2\theta) + f^k_{12}\cos\theta\sin\theta\Big]\\
&\quad + r\sum_{j=3}^m \big(f^k_{1j}\cos\theta + f^k_{2j}\sin\theta\big)x_j,
\end{aligned}
\tag{13}
$$

and

$$
T_{k2} = \frac{r^2}{2}\sum_{j=3}^m \big(f^k_{11j}\cos^2\theta + f^k_{22j}\sin^2\theta\big)x_j + \sum_{i=3}^m\sum_{j=i}^m\sum_{l=j}^m \delta_{ijl}f^k_{ijl}x_ix_jx_l.
$$

Moreover, $H_1 = H_1(\theta, r, x_3, \dots, x_m)$ and $H_k = H_k(\theta, r, x_3, \dots, x_m)$ for $k = 3, \dots, m$ are sums of terms of order 3 in $r, x_3, \dots, x_m$, multiplied by a function of θ among the ones in

$$\begin{aligned}&\cos\theta,\ \sin\theta,\ \cos\theta\sin\theta,\ \cos^3\theta,\ \sin^3\theta,\\&\cos^2\theta\sin\theta,\ \cos\theta\sin^2\theta,\ \cos^3\theta\sin\theta,\ \cos\theta\sin^3\theta.\end{aligned} \tag{14}$$

This readily implies that

$$\int_0^{2\pi} H_1(\theta, r, x_3, \dots, x_m)\,d\theta = \int_0^{2\pi} H_k(\theta, r, x_3, \dots, x_m)\,d\theta = 0$$

for $k = 3, \dots, m$. Finally, $H_2 = H_2(\theta, r, x_3, \dots, x_m)$ is a function at least of order 3 in the variables $r, x_3, \dots, x_m$.

Reduction of the system to form (5). In the region $\dot\theta \neq 0$ system (11) yields the equations

$$\begin{aligned}\frac{dr}{d\theta} &= \frac{T_{11} + T_{12} + H_1 + O_4(r, \theta, x_3, \dots, x_m)}{\omega + T_2 + H_2/r},\\ \frac{dx_k}{d\theta} &= \frac{\varepsilon^3\mu_k + T_{k1} + T_{k2} + H_k + O_4(r, \theta, x_3, \dots, x_m)}{\omega + T_2 + H_2/r}\end{aligned} \tag{15}$$

for $k = 3, \dots, m$, where for simplicity we have omitted the dependence on the variables θ, r, $x_3, \dots, x_m$. Here, each $O_4(r, \theta, x_3, \dots, x_m)$ denotes a term of order 4 in $r, x_3, \dots, x_m$. We note that this system is 2π-periodic in the independent variable θ. Moreover, performing the rescaling $(r, x_3, \dots, x_m) = \varepsilon(\rho, y_3, \dots, y_m)$, system (15) has the appropriate form to apply averaging theory. Namely, setting

$$\begin{aligned}\tilde T_{k1}(\theta, \rho, y_3, \dots, y_m) &= \frac{1}{\varepsilon^2} T_{k1}(\theta, r, x_3, \dots, x_m),\\ \tilde T_{k2}(\theta, \rho, y_3, \dots, y_m) &= \frac{1}{\varepsilon^3} T_{k2}(\theta, r, x_3, \dots, x_m),\\ \tilde T_{2}(\theta, \rho, y_3, \dots, y_m) &= \frac{1}{\varepsilon} T_{2}(\theta, r, x_3, \dots, x_m),\end{aligned}$$

in the variables $\rho, y_3, \dots, y_m$ the system has the form

$$\begin{aligned}\frac{d\rho}{d\theta} &= \varepsilon f_{11}(\theta, \rho, y_3, \dots, y_m) + \varepsilon^2 f_{12}(\theta, \rho, y_3, \dots, y_m)\\ &\quad + \varepsilon^3 g_1(\theta, \rho, y_3, \dots, y_m, \varepsilon),\\ \frac{dy_k}{d\theta} &= \varepsilon f_{k1}(\theta, \rho, y_3, \dots, y_m) + \varepsilon^2 f_{k2}(\theta, \rho, y_3, \dots, y_m)\\ &\quad + \varepsilon^3 g_k(\theta, \rho, y_3, \dots, y_m, \varepsilon),\end{aligned} \tag{16}$$

for some functions $g_1, g_3, \dots, g_m$, where

$$\begin{aligned}f_{11}(\theta, \rho, y_3, \dots, y_m) &= \frac{\tilde T_{11}(\theta, \rho, y_3, \dots, y_m)}{\omega},\\ f_{12}(\theta, \rho, y_3, \dots, y_m) &= \frac{\tilde T_{12}(\theta, \rho, y_3, \dots, y_m) + \tilde H_1(\theta, \rho, y_3, \dots, y_m)}{\omega}\\ &\quad - \frac{\tilde T_2(\theta, \rho, y_3, \dots, y_m)\tilde T_{11}(\theta, \rho, y_3, \dots, y_m)}{\omega^2},\end{aligned}$$

and

$$
\begin{aligned}
f_{k1}(\theta,\rho,y_3,\dots,y_m) &= \frac{\tilde T_{k1}(\theta,\rho,y_3,\dots,y_m)}{\omega},\\
f_{k2}(\theta,\rho,y_3,\dots,y_m) &= \frac{\mu_k+\tilde T_{k2}(\theta,\rho,y_3,\dots,y_m)+\tilde H_k(\theta,\rho,y_3,\dots,y_m)}{\omega}\\
&\quad -\frac{\tilde T_2(\theta,\rho,y_3,\dots,y_m)\tilde T_{k1}(\theta,\rho,y_3,\dots,y_m)}{\omega^2},
\end{aligned}
$$

for $k=3,\dots,m$. Moreover, each $\tilde H_k=\tilde H_k(\theta,\rho,y_3,\dots,y_m)$ is a sum of terms of order 3 in $\rho,y_3,\dots,y_m$, multiplied by a function of θ among the ones in (14). We note that

$$\int_0^{2\pi} f_{j1}(\theta,\rho,y_3,\dots,y_m)\,d\theta=\int_0^{2\pi}\tilde H_j(\theta,\rho,y_3,\dots,y_m)\,d\theta=0,$$

for $j=1,3,4,\dots,m$ (the first integral vanishes in view of the conditions (H1)–(H4)). Thus, system (16) can be written as system (5) taking $x=(\rho,y_3,\dots,y_k)$, $t=\theta$, $T=2\pi$,

$$f=(f_{11},f_{31},\dots,f_{m1}),\quad g=(f_{12},f_{32},\dots,f_{m2}),$$

and

$$R=(g_1,g_3,\dots,g_m).$$

It is easy to verify that system (16) satisfies the assumptions of the averaging theory of second order described in Theorem 4, taking $\Omega=U\cap(\mathbb{R}^+\times\mathbb{R}^{m-2})$ for some open disc U centered at the origin in $\mathbb{R}^{m-1}$, and taking $\varepsilon_0>0$ sufficiently small.

Functions in Theorem 4. To apply Theorem 4 (see (7)) we need to compute the functions g and f^1 (see (4)), which we write in the form

$$g=\begin{pmatrix} f_{12}(\rho,y_3,\dots,y_m)\\ f_{32}(\rho,y_3,\dots,y_m)\\ \vdots\\ f_{m2}(\rho,y_3,\dots,y_m)\end{pmatrix}\quad\text{and}\quad f^1=\begin{pmatrix} F_1(\rho,y_3,\dots,y_m)\\ F_3(\rho,y_3,\dots,y_m)\\ \vdots\\ F_m(\rho,y_3,\dots,y_m)\end{pmatrix},$$

for some functions $F_1,F_3,\dots,F_m$. We also let

$$G_i(\rho,y_3,\dots,y_m)=\frac{1}{2\pi}\int_0^{2\pi}\big[f_{i2}(\theta,\rho,y_3,\dots,y_m)+F_i(\theta,\rho,y_3,\dots,y_m)\big]\,d\theta. \tag{17}$$

Now we start computing the functions G_i, which are the components of the sum g^0+f^{10} (see (6)). We first observe that

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} f_{12}(\theta,\rho,y_3,\dots,y_m)\,d\theta\\
&=\frac{1}{2\pi\omega}\int_0^{2\pi}\tilde T_{12}(\theta,\rho,y_3,\dots,y_m)\,d\theta\\
&\quad-\frac{1}{2\pi\omega^2}\int_0^{2\pi}\tilde T_{11}(\theta,\rho,y_3,\dots,y_m)\tilde T_2(\theta,\rho,y_3,\dots,y_m)\,d\theta,
\end{aligned}
\tag{18}
$$

and

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} f_{k2}(\theta,\rho,y_3,\dots,y_m)\,d\theta\\
&=\mu_k+\frac{1}{2\pi\omega}\int_0^{2\pi}\tilde T_{k2}(\theta,\rho,y_3,\dots,y_m)\,d\theta\\
&\quad-\frac{1}{2\pi\omega^2}\int_0^{2\pi}\tilde T_{k1}(\theta,\rho,y_3,\dots,y_m)\tilde T_2(\theta,\rho,y_3,\dots,y_m)\,d\theta
\end{aligned}$$

for $k=3,\dots,m$. Furthermore, following (4) we let

$$Y_j=Y_j(\theta,\rho,y_3,\dots,y_m)=\int_0^{\theta}\tilde T_{j1}(\psi,\rho,y_3,\dots,y_m)\,d\psi \tag{19}$$

for $j=1,3,\dots,m$. Then the function y^1 in (4) is given by

$$y^1=\frac{1}{\omega}(Y_1,Y_3,\dots,Y_m).$$

Moreover,

$$F_i(\theta,\rho,y_3,\dots,y_m)=\frac{1}{\omega}\bigg(\frac{\partial f_{i1}}{\partial\rho}Y_1+\sum_{j=3}^{m}\frac{\partial f_{i1}}{\partial y_j}Y_j\bigg). \tag{20}$$

Computation of the functions G_i. Now we proceed with the explicit computation of the integrals giving the functions G_i. We first observe that

$$\begin{aligned}
&\frac{1}{2\pi\omega}\int_0^{2\pi}\tilde T_{12}(\theta,\rho,y_3,\dots,y_m)\,d\theta\\
&=\frac{1}{16\omega}\bigg[\rho^3\big(f^1_{111}+f^1_{122}+f^2_{112}+f^2_{222}\big)\\
&\quad+8\rho\sum_{i=3}^{m}\sum_{j=i}^{m}\big(\delta_{1ij}f^1_{1ij}+\delta_{2ij}f^2_{2ij}\big)y_iy_j\bigg].
\end{aligned} \tag{21}$$

Moreover, with the help of an algebraic manipulator such as Mathematica we obtain

$$\begin{aligned}
&-\frac{1}{2\pi\omega^2}\int_0^{2\pi}\tilde T_{11}(\theta,\rho,y_3,\dots,y_m)\tilde T_2(\theta,\rho,y_3,\dots,y_m)\,d\theta\\
&=\frac{\rho^3}{8}\Big(\frac12 f^1_{11}f^1_{12}+\frac12 f^1_{12}f^1_{22}-\frac12 f^1_{11}f^2_{11}-\frac12 f^2_{11}f^2_{12}+\frac12 f^1_{22}f^2_{22}-\frac12 f^2_{12}f^2_{22}\Big)\\
&\quad+\frac{\rho}{4}\sum_{i=3}^{m}\sum_{j=i}^{m}\delta_{ij}\Big(f^1_{ij}f^1_{12}-\frac12 f^1_{ij}f^2_{11}+\frac12 f^1_{ij}f^2_{22}\\
&\quad-\frac12 f^2_{ij}f^1_{11}+\frac12 f^2_{ij}f^1_{22}-f^2_{ij}f^2_{12}\Big)y_iy_j
\end{aligned} \tag{22}$$

Adding (21) and (22) we obtain the first component of g^0. Now we consider the remaining components. For $k = 3, \dots, m$ we have

$$\begin{aligned}
&\frac{1}{2\pi\omega}\int_0^{2\pi} \tilde{T}_{k2}(\theta,\rho,y_3,\dots,y_m)\,d\theta \\
&= \frac{1}{\omega}\biggl[\frac{\rho^2}{4}\sum_{j=3}^m \bigl(f_{11j}^k + f_{22j}^k\bigr)y_j + \sum_{i=3}^m\sum_{j=i}^m\sum_{l=j}^m \delta_{ijl}f_{ijl}^k y_i y_j y_l\biggr],
\end{aligned}$$

and again using Mathematica we obtain

$$\begin{aligned}
&-\frac{1}{2\pi\omega^2}\int_0^{2\pi} \tilde{T}_{k1}(\theta,\rho,y_3,\dots,y_m)\tilde{T}_2(\theta,\rho,y_3,\dots,y_m)\,d\theta \\
&= -\frac{1}{2\omega^2}\sum_{i=3}^m\sum_{j=i}^m\sum_{l=3}^m \delta_{i,j}\bigl(f_{ij}^2 f_{1l}^k - f_{ij}^1 f_{2l}^k\bigr)y_i y_j y_l \\
&\quad + \frac{\rho^2}{8\omega^2}\sum_{j=3}^m \Bigl(f_{2j}^1 f_{22}^k + f_{1j}^2 f_{22}^k - 2f_{2j}^2 f_{12}^k + f_{12}^1 f_{1j}^k \\
&\quad - \frac{3}{2}f_{11}^2 f_{1j}^k - \frac{1}{2}f_{22}^2 f_{1j}^k + \frac{1}{2}f_{22}^k f_{2j}^k + \frac{3}{2}f_{22}^1 f_{2j}^k - f_{12}^2 f_{2j}^k\Bigr).
\end{aligned}$$

These formulas conclude the computation of the integral in (18).

Now we compute the integrals $\int_0^{2\pi} F_i(\theta,\rho,y_3,\dots,y_m)\,d\theta$. Setting $j = 1$ in (19) we obtain

$$Y_1 = Y_1(\theta,\rho,y_3,\dots,y_m) = \int_0^\theta \tilde{T}_{11}(\psi,\rho,y_3,\dots,y_m)\,d\psi,$$

and it follows from (12) that

$$\begin{aligned}
Y_1 = \frac{\rho^2}{12}\biggl[&-3\Bigl(f_{12}^1 + \frac{1}{2}f_{11}^2 + \frac{3}{2}f_{22}^2\Bigr)\cos\theta - \Bigl(f_{12}^1 + \frac{1}{2}f_{11}^2 - \frac{1}{2}f_{22}^2\Bigr)\cos(3\theta) \\
&+ 3\Bigl(\frac{3}{2}f_{11}^1 + \frac{1}{2}f_{22}^1 + f_{12}^2\Bigr)\sin\theta + \Bigl(\frac{1}{2}f_{11}^1 - \frac{1}{2}f_{22}^1 - f_{12}^2\Bigr)\sin(3\theta)\biggr] \\
&- \frac{\rho}{4}\sum_{j=3}^m \bigl[(f_{2j}^1 + f_{1j}^2)y_j\cos(2\theta) + 2f_{2j}^2 y_j\sin(2\theta)\bigr] \\
&+ \sum_{i=3}^m\sum_{j=i}^m \delta_{ij}\bigl(f_{ij}^1\sin\theta - f_{ij}^2\cos\theta\bigr)y_i y_j.
\end{aligned} \tag{23}$$

Moreover, for $j = 3, \dots, m$ it follows from (19) and (13) that

$$Y_j = -\frac{\rho^2}{4}\bigl[f_{12}^k\cos(2\theta) + f_{22}^k\sin(2\theta)\bigr] - \rho\sum_{j=3}^m \bigl(f_{2j}^k\cos\theta - f_{1j}^k\sin\theta\bigr)y_j. \tag{24}$$

Now we observe that by (12) and (13),

$$
\begin{aligned}
\frac{\partial \tilde{T}_{11}}{\partial \rho} &= 2\rho\Big[\frac{1}{2}f_{11}^1\cos^3\theta + \Big(f_{12}^1+\frac{1}{2}f_{11}^2\Big)\cos^2\theta\sin\theta \\
&\quad + \Big(\frac{1}{2}f_{22}^1 + f_{12}^2\Big)\cos\theta\sin^2\theta + \frac{1}{2}f_{22}^2\sin^3\theta\Big] \\
&\quad + \sum_{j=3}^{m}\big[f_{2j}^2(\sin^2\theta-\cos^2\theta) + \big(f_{2j}^1+f_{1j}^2\big)\cos\theta\sin\theta\big]y_j,\\
\frac{\partial \tilde{T}_{11}}{\partial y_j} &= \rho\big[f_{2j}^2(\sin^2\theta-\cos^2\theta) + \big(f_{2j}^1+f_{1j}^2\big)\cos\theta\sin\theta\big] \\
&\quad + 2\sum_{i=j}^{m}\delta_{ji}\big(f_{ji}^1\cos\theta + f_{ji}^2\sin\theta\big)y_i,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial \tilde{T}_{k1}}{\partial \rho} &= 2\rho\Big[-\frac{1}{2}f_{22}^k(\cos^2\theta-\sin^2\theta) + f_{12}^k\cos\theta\sin\theta\Big] \\
&\quad + \sum_{j=3}^{m}\big(f_{1j}^k\cos\theta + f_{2j}^k\sin\theta\big)y_j,\\
\frac{\partial \tilde{T}_{k1}}{\partial y_j} &= \rho\big(f_{1j}^k\cos\theta + f_{2j}^k\sin\theta\big).
\end{aligned}
$$

Using Mathematica it follows from (23) and (24) that

$$
\begin{aligned}
\frac{1}{2\pi\omega^2}\int_0^{2\pi}\frac{\partial \tilde{T}_{11}}{\partial \rho}Y_1\,d\theta &= \frac{\rho}{4\omega^2}\sum_{i=3}\sum_{j=i}^{m}\delta_{ij}\Big(f_{ij}^1f_{12}^1 + \frac{1}{2}f_{ij}^1f_{11}^2 + \frac{3}{2}f_{ij}^1f_{22}^2 \\
&\quad - \frac{3}{2}f_{ij}^2f_{11}^1 - \frac{1}{2}f_{ij}^2f_{22}^1 - f_{12}^2f_{ij}^2\Big)y_iy_j,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{1}{2\pi\omega^2}\sum_{j=3}^{m}\int_0^{2\pi}\frac{\partial \tilde{T}_{11}}{\partial y_j}Y_j\,d\theta &= \frac{\rho^3}{8\omega^2}\sum_{j=3}^{m}\Big(-\frac{1}{2}f_{2j}^1f_{22}^j - \frac{1}{2}f_{1j}^2f_{22}^j + f_{2j}^2f_{12}^j\Big)\\
&\quad + \frac{\rho}{\omega^2}\sum_{j=3}^{m}\sum_{i=j}^{m}\sum_{l=3}^{m}\delta_{ij}\big(f_{ij}^2f_{1l}^j - f_{ij}^1f_{21}^j\big)y_iy_l.
\end{aligned}
$$

Summarizing, by (20) we conclude that

$$
\begin{aligned}
&\frac{\omega^2}{2\pi}\int_0^{2\pi} F_1(\rho, y_3, \ldots, y_m)\,d\theta \\
&= \frac{\rho^3}{8}\bigg(\frac{1}{2}f_{11}^1 f_{12}^1 + \frac{1}{2}f_{12}^1 f_{22}^1 - \frac{1}{2}f_{11}^1 f_{11}^2 - \frac{1}{2}f_{11}^2 f_{12}^2 + \frac{1}{2}f_{22}^1 f_{22}^2 \\
&\quad - \frac{1}{2}f_{12}^2 f_{22}^2 - \frac{1}{2}\sum_{j=3}^m f_{2j}^1 f_{22}^j - \frac{1}{2}\sum_{j=3}^m f_{1j}^2 f_{22}^j + \sum_{j=3}^m f_{2j}^2 f_{12}^j\bigg) \\
&\quad + \frac{\rho}{2}\sum_{i=3}\sum_{j=i}^m \delta_{ij}\big(f_{ij}^1 f_{12}^1 + f_{ij}^1 f_{22}^2 - f_{ij}^2 f_{11}^1 - f_{ij}^2 f_{12}^2\big) y_i y_j \\
&\quad + \rho \sum_{j=3}^m \sum_{i=j}^m \sum_{l=3}^m \delta_{ij}\big(f_{ij}^2 f_{1l}^j - f_{ij}^1 f_{2l}^j\big) y_i y_l \\
&\quad + \frac{\omega}{16}\bigg[\rho^3\big(f_{111}^1 + f_{122}^1 + f_{112}^2 + f_{222}^2\big) + 8\rho\sum_{i=3}^m \sum_{j=i}^m \big(\delta_{1ij} f_{1ij}^1 + \delta_{2ij} f_{2ij}^2\big) y_i y_j\bigg].
\end{aligned}
$$

Moreover, for $k = 3, \ldots, m$ we have

$$
\frac{1}{2\pi\omega^2}\int_0^{2\pi} \frac{\partial \tilde{T}_{k1}}{\partial y_j} Y_j\, d\theta = 0,
$$

and

$$
\begin{aligned}
&\frac{1}{2\pi\omega^2}\int_0^{2\pi} \frac{\partial \tilde{T}_{k1}}{\partial \rho} Y_1\, d\theta \\
&= -\frac{1}{2\omega^2}\sum_{i=3}^m \sum_{j=i}^m \sum_{l=3}^m \delta_{ij}\big(f_{ij}^2 f_{1l}^k - f_{ij}^1 f_{2l}^k\big) y_i y_j y_l \\
&\quad + \frac{\rho^2}{8\omega^2}\sum_{j=3}^m \bigg(f_{2j}^1 f_{22}^k + f_{1j}^2 f_{22}^k - 2 f_{2j}^2 f_{12}^k - f_{12}^1 f_{1j}^k - \frac{1}{2} f_{11}^2 f_{1j}^k \\
&\quad - \frac{3}{2} f_{22}^2 f_{1j}^k + \frac{3}{2} f_{22}^k f_{2j}^k + \frac{1}{2} f_{22}^1 f_{2j}^k + f_{12}^2 f_{2j}^k\bigg) y_j
\end{aligned}
$$

Therefore, for $j = 3, \ldots, k$ we have

$$
\begin{aligned}
&\frac{\omega^2}{2\pi}\int_0^{2\pi} F_j(\rho, y_3, \ldots, y_m)\,d\theta \\
&= -\sum_{i=3}^m \sum_{j=i}^m \sum_{l=3}^m \delta_{ij}\big(f_{ij}^2 f_{1l}^k - f_{ij}^1 f_{2l}^k\big) y_i y_j y_l \\
&\quad + \frac{\rho^2}{4}\sum_{j=3}^m \big(f_{2j}^1 f_{22}^k + f_{1j}^2 f_{22}^k - 2 f_{2j}^2 f_{12}^k \\
&\quad - f_{11}^2 f_{1j}^k - f_{22}^2 f_{1j}^k + f_{22}^k f_{2j}^k + f_{22}^1 f_{2j}^k\big) y_j \\
&\quad + \omega\bigg[\frac{\rho^2}{4}\sum_{j=3}^m \big(f_{11j}^k + f_{22j}^k\big) y_j + \sum_{i=3}^m \sum_{j=i}^m \sum_{l=j}^m \delta_{ijl} f_{ijl}^k y_i y_j y_l\bigg].
\end{aligned}
$$

This concludes the computation of the functions G_i, and the proof of Theorem 5 is complete.

5. Proof of Theorem 7

When $m = 3$ the averaged system (9) becomes

$$\begin{aligned} \frac{d\rho}{d\theta} &= \Delta_1 \rho^3 + \Delta_2 \rho y_3^2, \\ \frac{dy_3}{d\theta} &= \Delta_3 \rho^2 y_3 + \Delta_4 y_3^3. \end{aligned}$$

We can easily verify that it has as a single equilibrium point with $\rho > 0$, given by

$$p = \left(\frac{|2\omega\mu_3|^{1/3}\sqrt{|\Delta_2|}}{|\Delta_1|^{1/6}|\Delta_3\Delta_2 - \Delta_1\Delta_4|^{1/3}}, \frac{(2\omega\mu_3\Delta_1)^{1/3}}{(\Delta_3\Delta_2 - \Delta_1\Delta_4)^{1/3}}. \right).$$

A simple computation shows that the Jacobian matrix at the point p has determinant $12|\omega\mu_3|\sqrt{|\Delta_1\Delta_2|} \neq 0$, and thus we can apply Theorem 4.

6. Details of Example 1

The purpose of this section is to provide the details of Example 1, showing that the averaging theory of second order can be applied to system (10).

In coordinates r, θ, x_3, x_4 system (10) becomes

$$\begin{aligned} \dot{r} &= x_4^2 \cos\theta + \frac{r x_3 (x_4 + x_3)\, \cos^2\theta}{\omega} - \frac{4r^3 \cos^4\theta}{3\omega} + r^2 \cos^2\theta \sin\theta, \\ \dot{\theta} &= -\frac{x_4^2 \sin\theta}{r} + \frac{4r^2 \cos^3\theta \sin\theta}{3\omega} - r\cos\theta \sin^2\theta \\ &\quad + \frac{\omega^2 \cos^2\theta - x_3 x_4 \cos\theta \sin\theta - x_3^2 \cos\theta \sin\theta + \omega^2 \sin^2\theta}{\omega}, \\ \dot{x}_3 &= (\varepsilon^3 + x_4 x_3^2)\mu_3 - 2r^2(x_4 + x_3)\mu_3 \cos^2\theta, \\ \dot{x}_4 &= \frac{-x_4 x_3^2 + \varepsilon^3 \omega\mu_4 + x_4 x_3^2 \omega\mu_4}{\omega} - 2r^2(x_4 + x_3)\mu_4 \cos^2\theta + r x_3 \sin\theta. \end{aligned}$$

Performing the rescaling $\rho = r\varepsilon$, $y_3 = x_3\varepsilon$ and $y_4 = x_4\varepsilon$, in the region $\dot{\theta} \neq 0$ we obtain the reduced system

$$\begin{aligned} \frac{d\rho}{d\theta} &= \frac{\varepsilon \cos\theta (y_4^2 + \rho^2 \cos\theta \sin\theta)}{\omega} \\ &\quad - \frac{\varepsilon^2 \cos\theta}{3\rho\omega^2} \Big(-3\rho^2 y_3 y_4 \cos^3\theta - 3\rho^2 y_3^2 \cos^3\theta + 4\rho^4 \cos^5\theta \\ &\quad - 3y_4^4 \sin\theta - 6\rho^2 y_4^2 \cos\theta \sin^2\theta - 3\rho^2 y_3 y_4 \cos\theta \sin^2\theta \\ &\quad - 3\rho^2 y_3^2 \cos\theta \sin^2\theta + 4\rho^4 \cos^3\theta \sin^2\theta - 3\rho^4 \cos^2\theta \sin^3\theta \Big), \\ \frac{dy_3}{d\theta} &= \frac{\varepsilon^2 \mu_3 (1 + y_3^2 y_4 - 2\rho^2 y_4 \cos^2\theta - 2\rho^2 y_3 \cos^2\theta)}{\omega}, \end{aligned}$$

$$\begin{aligned}\frac{dy_4}{d\theta} &= \frac{\varepsilon\rho y_3\sin\theta}{\omega} + \frac{\varepsilon^2}{\omega^2}\Big(- y_4y_3^2\cos^2\theta + \omega\mu_4\cos^2\theta + y_4y_3^2\omega\mu_4\cos^2\theta \\ &\quad - 2\rho^2y_4\omega\mu_4\cos^4\theta - 2\rho^2y_3\omega\mu_4\cos^4\theta + y_4^2y_3\sin^2\theta - y_4y_3^2\sin^2\theta \\ &\quad + \omega\mu_4\sin^2\theta + y_4y_3^2\omega\mu_4\sin^2\theta - 2\rho^2y_4\omega\mu_4\cos^2\theta\sin^2\theta \\ &\quad - 2\rho^2y_3\omega\mu_4\cos^2\theta\sin^2\theta + \rho^2y_3\cos\theta\sin^3\theta\Big).\end{aligned}$$

We can write this system in the form (16) with

$$f_{11} = \frac{\cos\theta(y_4^2+\rho^2\cos\theta\sin\theta)}{\omega}, \ f_{31}=0, \ f_{41} = \frac{\rho y_3\sin\theta}{\omega},$$

and

$$\begin{aligned} f_{12} &= -\frac{\cos\theta}{3\rho\omega^2}\Big(- 3\rho^2y_3y_4\cos^3\theta - 3\rho^2y_3^2\cos^3\theta + 4\rho^4\cos^5\theta \\ &\quad - 3y_4^4\sin\theta - 6\rho^2y_4^2\cos\theta\sin^2\theta - 3\rho^2y_4y_3\cos\theta\sin^2\theta \\ &\quad - 3\rho^2y_3^2\cos\theta\sin^2\theta + 4\rho^4\cos^3\theta\sin^2\theta - 3\rho^4\cos^2\theta\sin^3\theta\Big), \\ f_{32} &= \frac{\mu_3(1+y_4y_3^2-2\rho^2y_4\cos^2\theta-2\rho^2y_3\cos^2\theta)}{\omega}, \\ f_{42} &= \frac{1}{\omega^2}\Big(- y_4y_3^2\cos^2\theta + \omega\mu_4\cos^2\theta + y_4y_3^2\omega\mu_4\cos^2\theta \\ &\quad - 2\rho^2y_4\omega\mu_4\cos^4\theta - 2\rho^2y_3\omega\mu_4\cos^4\theta + y_4^2y_3\sin^2\theta - y_4y_3^2\sin^2\theta \\ &\quad + \omega\mu_4\sin^2\theta + y_4y_3^2\omega\mu_4\sin^2\theta - 2\rho^2y_4\omega\mu_4\cos^2\theta\sin^2\theta \\ &\quad - 2\rho^2y_3\omega\mu_4\cos^2\theta\sin^2\theta + \rho^2y_3\cos\theta\sin^3\theta\Big).\end{aligned}$$

Note that

$$\int_0^{2\pi} f_{11}(\theta,\rho,y_3,y_4)\,d\theta = \int_0^{2\pi} f_{41}(\theta,\rho,y_3,y_4)\,d\theta = 0.$$

Now we study the averages of second order. Setting

$$Y_1(\theta,\rho,y_3,y_4) = \int_0^\theta f_{11}(\psi,\rho,y_3,y_4)\,d\theta, \quad Y_3=0,$$

and

$$Y_4(\theta,\rho,y_3,y_4) = \int_0^\theta f_{41}(\psi,\rho,y_3,y_4)\,d\theta,$$

with the notation of (17) we obtain

$$\begin{aligned} G_1 &= \frac{1}{2\pi}\int_0^{2\pi} f_{12}(\theta,\rho,y_3,y_4)\,d\theta \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial f_{11}(\theta,\rho,y_3,y_4)}{\partial\rho}Y_1(\theta,\rho,y_3,y_4)\,d\theta \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial f_{11}(\theta,\rho,y_3,y_4)}{\partial y_4}Y_4(\theta,\rho,y_3,y_4)\,d\theta \\ &= \frac{\rho}{2\omega^2}(-\rho^2+y_4^2-y_3y_4+y_3^2),\end{aligned}$$

$$
\begin{aligned}
G_3 &= \frac{1}{2\pi}\int_0^{2\pi} f_{32}(\theta,\rho,y_3,y_4)\,d\theta \\
&= \frac{\mu_3}{\omega}(1 + y_4 y_3^2 - \rho^2(y_3+y_4)), \\
G_4 &= \frac{1}{2\pi}\int_0^{2\pi} f_{42}(\theta,\rho,y_3,y_4)\,d\theta \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial f_{41}(\theta,\rho,y_3,y_4)}{\partial\rho} Y_1(\theta,\rho,y_3,y_4)\,d\theta \\
&\quad + \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial f_{41}(\theta,\rho,y_3,y_4)}{\partial y_4} Y_4(\theta,\rho,y_3,y_4)\,d\theta \\
&= \frac{1}{\omega^2}\big[y_4(y_4-y_3)y_3 + (1 + y_4 y_3^2 - \rho^2(y_3+y_4))\omega\mu_4\big].
\end{aligned}
$$

Computing the zeros of (G_1, G_3, G_4) yields

$$p_1 = (1,1,0), \quad p_2 = (1,0,1), \quad p_3 = (1,1,1),$$

in coordinates ρ, y_3, y_4. For the function (G_1, G_3, G_4), the Jacobians at the points p_1 and p_2 have determinant $-3\mu_3/\omega^5 \neq 0$, and the Jacobian at the point p_3 has determinant $3\mu_3/\omega^5 \neq 0$. Thus, we can apply Theorem 4. This concludes the details of the example.

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