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A NOTE ON THE FIRST INTEGRALS OF THE ABC SYSTEM

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Abstract. Without loss of generality the ABC systems reduce to two cases: either $A = 0$ and $B, C \geq 0$, or $A = 1$ and $0 < B, C \leq 1$. In the first case it is known that the ABC system is completely integrable, here we provide its explicit first integrals. In the second case Ziglin in [14] proved that the ABC system with $0 < B < 1$ and $C > 0$ sufficiently small has no real meromorphic first integrals. We improve Ziglin's result showing that there are no C^1 first integrals under convenient assumptions.

1. Introduction

The ABC system was introduced by Arnold for studying the steady state solutions of Euler's hydrodynamic partial differential equations. Here we study the existence and non–existence of first integrals for this differential system defined in the 3–dimensional torus. As fas as we know we provide by first time sufficient conditions in order that the ABC system has no C^1 first integrals.

The nonlinear ordinary differential equations appear in a natural way in many branches of applied mathematics, physics, chemist, economy, etc. In particular the ABC system was introduced by Arnold in [1] and studied in [5, 6] with regard to the hydrodynamic instability criterion of Friedlander and Vishik [4], putting in particular special interest in the existence of a single hyperbolic periodic solution of an ABC system which implies that the associated steady state solution of Euler's equation is hydrodynamically unstable, see also [3].

Here we are interested in the integrability of the ABC system. More precisely, the goal of this paper is to study the existence of first integrals of the ABC systems

$$
\begin{aligned}
\dot{x} &= A \sin z + C \cos y, \\
\dot{y} &= B \sin x + A \cos z, \\
\dot{z} &= C \sin y + B \cos x,
\end{aligned}
\tag{1}
$$

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in $\mathbb{T}^3$ with the coordinates x, y, z (mod. 2π) and the parameters $A, B, C \in \mathbb{R}$. As usual the dot denotes derivative with respect to time t.

In all the paper we assume that $A \geq 0$, $B \geq 0$ and $C \geq 0$ because doing the change of variables $z \mapsto z + \pi$ we can consider $A \geq 0$, and similarly with the parameters B and C.

If $ABC = 0$, then doing cyclic permutations of x, y, z and of the parameters A, B, C, if necessary, we can consider $A = 0$.

When $ABC \neq 0$ doing a rescaling of time $ds = Mdt$ where $M = \max\{A, B, C\}$ and taking cyclic permutations with respect to the variables x, y, z and the parameters A, B, C, if necessary, we can take $A = 1$ and $0 < C \leq B \leq 1$ or $0 < B < C \leq 1$.

Here *a first integral* is a C^1 non–constant function $H : U \to \mathbb{R}$ whose domain of definition is an open subset U of $\mathbb{T}^3$ such that $\mathbb{T}^3 \setminus U$ has zero Lebesgue measure and H is constant on the solutions of system (1) contained in U, i.e. for any solution $(x(t), y(t), z(t))$ of (1) we have that

$$\frac{dH}{dt}(x(t), y(t), z(t)) = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} + \frac{\partial H}{\partial z}\dot{z} = 0,$$

for $(x(t), y(t), z(t)) \in U$.

The *gradient* of H is defined as

$$\nabla H(x, y, z) = \left(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}, \frac{\partial H}{\partial z}\right).$$

We say that two first integrals $H_1 : U \to \mathbb{R}$ and $H_2 : U \to \mathbb{R}$ are *linearly independent* if their gradients are independent in all the points of U except into a set of Lebesgue measure zero. Note that our first integrals are independent of the time.

The differential system (1) is *completely integrable* in the open subset U of $\mathbb{T}^3$ if there are two linearly independent first integrals H_1 and H_2 defined on U.

Proposition 1. *The ABC systems* (1) *either has no C^1 first integrals, or it is completely integrable.*

The proof of Proposition 1 is given in section 2

It is known that the ABC systems are completely integrable if $A = 0$, see for instance [17], but since we do not find in the literature the explicit expressions of the first integrals in this case we provide them in the next result.

Theorem 2. *The ABC systems* (1) *with $A = 0$ are always completely integrable.*

(a) *If* $B = C = 0$*, then* $H_1 = x$*,* $H_2 = y$ *and* $H_3 = z$ *are first integrals.*
(b) *If* $B = 0$ *and* $C \neq 0$*, then* $H_1 = y$ *and* $H_2 = x \sin y - z \cos y$ *are first integrals.*
(b) *If* $C = 0$ *and* $B \neq 0$*, then* $H_1 = x$ *and* $H_2 = z \sin x - y \cos x$ *are first integrals.*
(d) *If* $BC \neq 0$*, then* $H_1 = C \sin y + B \cos x$ *and* H_2 *given in* (3) *are first integrals.*

The proof of Theorem 2 is given in Section 3.

Ziglin in [14, 15] and Maciejewski and Przybylska in [10] proved the following result.

Theorem 3. *System* (1) *with either* $A = 1$*,* $0 < B < 1$ *and* $C = \varepsilon > 0$ *sufficiently small, or* $A = B$ *has no real meromorphic first integrals such that in a neighborhood* U *of a convenient solution* γ *the fundamental group of* γ *(at some point) can be represented by loops in* U*.*

In the next theorem we improve Theorem 3 by proving the absence of C^1 first integrals instead of real meromorphic first integrals under convenient assumptions.

Theorem 4. *System* (1) *with* $A = 1$*,* $0 < B \leq 1$ *and* $C = \varepsilon > 0$ *sufficiently small has no a* C^1 *first integral* $H(x, y, z)$ *if in a neighborhood of a convenient periodic orbit* γ_ε *the vectors* $\nabla H(x, y, z)$ *and* $(\sin z + C \cos y, B \sin x + \cos z, C \sin y + B \cos x)$ *are linearly independent at the points* $(x, y, z) \in \gamma_\varepsilon$*. Of course if such a local* C^1 *first integrals do not exist, then it cannot* C^1 *first integrals.*

The proof of Theorem 4 is given in Section 4. For this prove we use in an essential way a non-trivial result of Chicone [3] on the periodic orbits of the ABC system.

These last years the Ziglin's and the Morales-Ramis' theories study the real non-meromorphic integrability of autonomous real differential systems using the monodromy group of the complex variational equations associated to an explicit non-equilibrium orbit of the system, see for details [11]. Both theories are in some sense inspired in Kovalevskaya's ideas used in the study of the first integrals of the rigid body with one fixed point (see Arnold [2]). But Kovalevskaya's ideas go back to Poincaré (see [12]), who used the eigenvalues of the monodromy matrix associated to a periodic orbit for studying the C^1 non-integrability of the differential system, see for more details section 4. It seems that this result of Poincaré was forgot by the mathematical community until that modern Russian mathematicians (especially Kozlov) have recently publish on it, see [2, 8].

When $A = 1$ and $B = C < 1$, or $A = B = 1$ and $C < 1$ except for at most a countable values of C, it was proved in [16] the absence of real meromorphic first integrals for system (1). In [17] it was proved also the absence of real meromorphic first integrals for system (1) when $A = B = C = 1$.

2. Proof of Proposition 1

To prove Proposition 1 we will use some definitions and results. Let $\mathbf{x} = (x, y, z)$ and $J = J(\mathbf{x})$ be a non-negative C^1 function non-identically zero on any open subset of $\mathbb{R}^3$, then J is a *Jacobi multiplier* of the differential system (1) if

$$\int_{\Omega} J(\mathbf{x})\, d\mathbf{x} = \int_{\phi_t(\Omega)} J(\mathbf{x})\, d\mathbf{x}, \tag{2}$$

where Ω is any open subset of $\mathbb{R}^3$ and ϕ_t is the flow defined by the differential system (1). The following result is due to Jacobi, for a proof in the general case of dimension n see [7, Theorem 2.7].

Theorem 5. *Consider the differential system* (1) *in* $\mathbb{R}^3$*, and assume that it admits a Jacobi multiplier* $J = J(\mathbf{x})$ *and one first integral. Then the system admits an additional first integral functionally independent with the previous one, and consequently the differential system* (1) *is completely integrable.*

In general given a function J is not easy to verify (2) for knowing if it is a Jacobi multiplier. However we have the following result of Whittaker (see [13] for the general case of dimension n), which plays a main role for detecting Jacobi multipliers.

Proposition 6. *Let* $J = J(\mathbf{x})$ *be a non-negative* C^1 *function non-identically zero on any open subset of* $\mathbb{R}^3$*. Then* J *is a Jacobi multiplier of the differential system* (1) *if and only if the divergence of the differential system*

$$\begin{aligned} \dot{x} &= J(\mathbf{x})(A \sin z + C \cos y), \\ \dot{y} &= J(\mathbf{x})(B \sin x + A \cos z), \\ \dot{z} &= J(\mathbf{x})(C \sin y + B \cos x), \end{aligned}$$

is zero.

Proof of Proposition 1. Since the divergence of system (1) is zero it follows from Proposition 6 that $J = 1$ is a Jacobi multiplier of system (1). Therefore Theorem 5 implies that either system (1) has no C^1 first integrals, or if it has one C^1 first integral then it is completely integrable. This completes the proof. □

3. Proof of Theorem 2

To prove Theorem 2 we need to study systems (1) with $A = 0$.

Case (a): $B = C = 0$. Then system (1) becomes $\dot{x} = 0$, $\dot{y} = 0$, $\dot{z} = 0$, so statement (a) holds.

Case (b): $B = 0$, $C \neq 0$. Now system (1) becomes $\dot{x} = C\cos y$, $\dot{y} = 0$, $\dot{z} = C\sin y$ and statement (b) follows easily.

Case (c): $B \neq 0$, $C = 0$. This case is equivalent to (b) doing the change of variables $(x, y, z) \to (z, x, y)$ and $(A, B, C) \to (C, A, B)$.

Case (d): $BC \neq 0$. In this case system (1) becomes

$$\dot{x} = C\cos y, \quad \dot{y} = B\sin x, \quad \dot{z} = C\sin y + B\cos x.$$

It is easy to check that this system has the analytic first integral $H_1 = C\sin y + B\cos x$. It is not difficult to check that another first integral is

$$H_2 = \frac{w_1 F(u|v) + z w_2}{w_2} \tag{3}$$

where F is the elliptic integral of first kind, and

$$\begin{aligned}
u &= i\sinh^{-1}\left(\sqrt{\frac{B\cos x + B + C + C\sin y}{B\cos x - B + C + C\sin y}}\tan\frac{x}{2}\right),\\
v &= \frac{(B\cos x + B - C + C\sin y)(B\cos x - B + C + C\sin y)}{(B\cos x - B - C + C\sin y)(B\cos x + B + C + C\sin y)},\\
w_1 &= \frac{2i\sqrt{2}(B\cos x + C\sin y)\cos y}{\sqrt{(-B\cos x + B + C - C\sin y)(B\cos x - B + C + C\sin y)}},\\
w_2 &= \sqrt{\cos(2y) + 1}\sqrt{\frac{B\cos x + B + C + C\sin y}{B\cos x - B + C + C\sin y}}.
\end{aligned}$$

Note that H_2 is an analytic first integral defined in a convenient open subset U of $\mathbb{T}^3$ such that $\mathbb{T}^3 \setminus U$ has zero Lebesgue measure.

4. Proof of Theorem 4

Now we will use the multipliers of a periodic orbit for studying the C^1 non-integrability of the ABC system. To state the results we shall introduce some notation.

We consider the autonomous differential system

$$\dot{\mathbf{x}} = f(\mathbf{x}), \tag{4}$$

where $f\colon U \to \mathbb{R}^3$ is C^2, U is an open subset of $\mathbb{R}^3$. We write its general solution as $\phi(t, \mathbf{x}_0)$ with $\phi(0, \mathbf{x}_0) = \mathbf{x}_0 \in U$ and t belonging to its maximal interval of definition.

We say that the solution $\phi(t, \mathbf{x}_0)$ is T*-periodic* with $T > 0$ if and only if $\phi(T, \mathbf{x}_0) = \mathbf{x}_0$ and $\phi(t, \mathbf{x}_0) \neq \mathbf{x}_0$ for $t \in (0, T)$. The *periodic orbit* associated to the periodic solution $\phi(t, \mathbf{x}_0)$ is $\gamma = \{\phi(t, \mathbf{x}_0), t \in [0, T]\}$. The *variational equation* associated to the T-periodic solution $\phi(t, \mathbf{x}_0)$ is

$$\dot{M} = \left(\frac{\partial f(\mathbf{x})}{\partial \mathbf{x}} \Big|_{\mathbf{x}=\phi(t,\mathbf{x}_0)} \right) M, \tag{5}$$

where M is a 3×3 matrix. Of course $\partial f(\mathbf{x})/\partial \mathbf{x}$ denotes the Jacobian matrix of f with respect to $\mathbf{x}$. The *monodromy matrix* associated to the T-periodic solution $\phi(t, \mathbf{x}_0)$ is the solution $M(T, \mathbf{x}_0)$ of (5) satisfying that $M(0, \mathbf{x}_0)$ is the identity matrix. The eigenvalues of the monodromy matrix associated to the periodic solution $\phi(t, \mathbf{x}_0)$ are called the *multipliers* of the periodic orbit.

In [9] the authors proved the following result which goes back to Poincaré (see [12]).

Theorem 7. *Consider the C^2 differential system* (4)*. If there is a periodic orbit γ having only one multiplier equal to* 1*, then system* (4) *has no C^1 first integrals $H(\mathbf{x})$ defined in a neighborhood of γ if the vectors $\nabla H(\mathbf{x})$ and $f(\mathbf{x})$ are linearly independent at the points $\mathbf{x} \in \gamma$.*

Proof of Theorem 4. It was proved in [3] (see Theorems 5.1 and 5.2) that under the assumptions of Theorem 4 system (1) has a periodic orbit γ_ε such that the Poincaré map at this periodic orbit has the following two eigenvalues

$$\lambda_{1,2} = 1 \pm \sqrt{\varepsilon(2\pi N p(x_0, z_0) \sec^2 z)} + O(\varepsilon),$$

where $O(\varepsilon)$ denotes the terms of order ε, $N \geq 1$ is an integer number and $p(x_0, z_0) \neq 0$ for some chosen values of (x_0, z_0). Therefore for ε sufficiently small we have that $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$. Hence using Theorem 7 system (1) has no C^1 first integrals $H(x, y, z)$ defined in a neighborhood of γ_ε if the vectors $\nabla H(x, y, z)$ and $(\sin z +$

$C\cos y, B\sin x + \cos z, C\sin y + B\cos x)$ are linearly independent on the points $(x, y, z) \in \gamma_\varepsilon$. The proof of the theorem is completed. □

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