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ON THE PERIODIC ORBITS OF THE FOURTH–ORDER DIFFERENTIAL EQUATION $u'''' + qu'' - u = \varepsilon F(u, u', u'', u''')$

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ABSTRACT. We provide sufficient conditions for the existence of periodic solutions of the fourth-order differential equation

$$u'''' + qu'' - u = \varepsilon F(u, u', u'', u'''),$$

where q and ε are real parameters, ε is small and F is a nonlinear function.

1. Introduction and statement of the main results

The objective of this paper is to study the periodic solutions of the fourth-order differential equation

$$u'''' + qu'' - u = \varepsilon F(u, u', u'', u'''), \tag{1}$$

where q and ε are real parameters, ε is small and F is a nonlinear function. The prime denotes derivative with respect to an independent variable l.

Equations of the form (1) appear in many contexts, we only mention some of them in what follows. For instance, Champneys [8] analyzes a class of equations (1) looking mainly for homoclinic orbits.

When $F = \pm u^2$ equation (1) can come from the description of the travelling–wave solutions of the Korteweg–de Vries equation with an additional fifth-order dispersive term. This equation has been used to describe chains of coupled nonlinear oscillators [19] and most notably gravity–capillary shallow water waves [3, 13, 22]. Extended fifth–order Korteweg–de Vries equations have been considered in [6, 9, 10, 15, 16, 18].

Other derivation of (1) with $F = \pm u^2$ appears in describing the displacement u of a compressed strut with bending softness resting on a nonlinear elastic foundation with dimensionless restoring force proportional to $u - u^2$ [11, 12].

Another nonlinearity is $F = \pm u^3$, then equation (1) is called the *Extended Fischer–Kolmogorov* equation or the *Swift-Hohenberg* equation. Equation (1) with this nonlinearity appears when we study travelling–wave solutions of the nonlinear Schrödinger equation with an additional fourth–order dispersion term [5, 14] and in other places, see for instance the book [20] and [2, 7].

We recall that a simple zero r_0^* of a real function $\mathcal{F}(r_0)$ is defined by $\mathcal{F}(r_0^*) = 0$ and $(d\mathcal{F}/dr_0)(r_0^*) \neq 0$.

Our main result on the periodic solutions of the fourth-order differential equation (1) is the following one.

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Theorem 1. *For every positive simple zero r_0^* of the function*

$$\mathcal{F}(r_0)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\sqrt{2}\cos\theta}{\sqrt{q+\sqrt{q^2+4}}}F(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})\,d\theta,$$

where

$$\begin{aligned}
\mathcal{A}= & \ \frac{\left(q-\sqrt{q^2+4}\right)\sqrt{q+\sqrt{q^2+4}}\,r_0\sin\theta}{2\sqrt{2}\sqrt{q^2+4}},\\
\mathcal{B}= & \ -\frac{r_0\cos\theta}{\sqrt{q^2+4}},\\
\mathcal{C}= & \ \frac{\sqrt{q+\sqrt{q^2+4}}\,r_0\sin\theta}{\sqrt{2}\sqrt{q^2+4}},\\
\mathcal{D}= & \ \frac{1}{2}\left(\frac{q}{\sqrt{q^2+4}}+1\right)r_0\cos\theta,
\end{aligned}$$

the differential equation (1) *has a periodic solution $u(l,\varepsilon)$ tending to the periodic solution*

$$u_0(l)=-\frac{\sqrt{2}\ r_0^*}{\sqrt{q^2+4}\sqrt{q+\sqrt{q^2+4}}}\sin\left(\sqrt{\frac{q+\sqrt{q^2+4}}{2}}\ l\right), \tag{2}$$

of $u''''+qu''-u=0$ when $\varepsilon\to 0$.

Theorem 1 is proved in section 3. Its proof is based in the averaging theory for computing periodic orbits, see section 2.

Three easy applications of Theorem 1 are given in the following two corollaries. They are proved in section 4.

Corollary 2. *If $F(u,u',u'',u''')=u'-(u')^3$, then the differential equation* (1) *has a periodic solution $u(l,\varepsilon)$ tending to the periodic solution*

$$u_0(l)=-\frac{2\sqrt{2}}{\sqrt{3(q+\sqrt{q^2+4})}}\sin\left(\sqrt{\frac{q+\sqrt{q^2+4}}{2}}\ l\right),$$

of $u''''+qu''-u=0$ when $\varepsilon\to 0$.

Corollary 3. *If $F(u,u',u'',u''')=\sin u'$, then for every positive integer m there exists an $\varepsilon_0>0$ such that for all $\varepsilon\in(0,\varepsilon_0)$ the differential equation* (1) *has at least m periodic solutions.*

Corollary 4. *If $F(u,u',u'',u''')=u'''-(u''')^3$, then the differential equation* (1) *has a periodic solution $u(l,\varepsilon)$ tending to the periodic solution*

$$u_0(l)=-\frac{4}{\sqrt{3}\sqrt{q+\sqrt{q^2+4}}\sqrt{q\left(q+\sqrt{q^2+4}\right)+2}}\sin\left(\sqrt{\frac{q+\sqrt{q^2+4}}{2}}\ l\right),$$

of $u''''+qu''-u=0$ when $\varepsilon\to 0$.

We denote by $g_u(u)$ the derivative of the function $g(u)$ with respect to the variable u. Then we get the following result also proved in section 4.

Corollary 5. *Let $g(u)$ be a C^2 function. If $F(u, u', u'', u''') = g_u(u)$, then the function $\mathcal{F}(r_0)$ of the statement of Theorem 1 is identically zero.*

2. Basic results on averaging theory

In this section we present the basic result from the averaging theory that we shall need for proving the main results of this paper.

We consider the problem of the bifurcation of T–periodic solutions from differential systems of the form

$$\dot{\mathbf{x}}(t) = F_0(t, \mathbf{x}) + \varepsilon F_1(t, \mathbf{x}) + \varepsilon^2 F_2(t, \mathbf{x}, \varepsilon), \tag{3}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ sufficiently small. Here the functions $F_0, F_1 : \mathbb{R} \times \Omega \to \mathbb{R}^n$ and $F_2 : \mathbb{R} \times \Omega \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are $\mathcal{C}^2$ functions, T–periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system

$$\dot{\mathbf{x}}(t) = F_0(t, \mathbf{x}), \tag{4}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory.

Let $\mathbf{x}(t, \mathbf{z}, \varepsilon)$ be the solution of the system (4) such that $\mathbf{x}(0, \mathbf{z}, \varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathbf{x}(t, \mathbf{z}, 0)$ as

$$\dot{\mathbf{y}} = D_{\mathbf{x}} F_0(t, \mathbf{x}(t, \mathbf{z}, 0))\mathbf{y}. \tag{5}$$

In what follows we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (5), and by $\xi : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates; i.e. $\xi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

We assume that there exists a k–dimensional submanifold $\mathcal{Z}$ of Ω filled with T–periodic solutions of (4). Then an answer to the problem of bifurcation of T–periodic solutions from the periodic solutions contained in $\mathcal{Z}$ for system (3) is given in the following result.

Theorem 6. *Let V be an open and bounded subset of $\mathbb{R}^k$, and let $\beta : \mathrm{Cl}(V) \to \mathbb{R}^{n-k}$ be a $\mathcal{C}^2$ function. We assume that*

(i) *$\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha, \beta(\alpha)),\ \alpha \in \mathrm{Cl}(V)\} \subset \Omega$ and that for each $\mathbf{z}_\alpha \in \mathcal{Z}$ the solution $\mathbf{x}(t, \mathbf{z}_\alpha)$ of (4) is T–periodic;*

(ii) *for each $\mathbf{z}_\alpha \in \mathcal{Z}$ there is a fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of (5) such that the matrix $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$ has in the upper right corner the $k \times (n-k)$ zero matrix, and in the lower right corner a $(n-k) \times (n-k)$ matrix Δ_α with $\det(\Delta_\alpha) \neq 0$.*

We consider the function $\mathcal{F} : \mathrm{Cl}(V) \to \mathbb{R}^k$

$$\mathcal{F}(\alpha) = \xi\left(\frac{1}{T}\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha))dt\right). \tag{6}$$

If there exists $a \in V$ with $\mathcal{F}(a) = 0$ and $\det\left((d\mathcal{F}/d\alpha)(a)\right) \neq 0$, then there is a T–periodic solution $\varphi(t, \varepsilon)$ of system (3) such that $\varphi(0, \varepsilon) \to \mathbf{z}_a$ as $\varepsilon \to 0$.

Theorem 6 goes back to Malkin [17] and Roseau [21], for a shorter proof see [4].

Theorem 6 will be used in the next section for proving our Theorem 1.

3. Proof of Theorem 1

Introducing the variables $(x,y,z,v)=(u,u',u'',u''')$ we write the fourth–order differential equation (1) as a first–order differential system defined in an open subset Ω of $\mathbb{R}^4$. Thus we have the differential system

$$
\begin{aligned}
x' &= y,\\
y' &= z,\\
z' &= v,\\
v' &= x - qz + \varepsilon F(x,y,z,v).
\end{aligned}
\tag{7}
$$

Of course as before the dot denotes derivative with respect to the independent variable l. System (7) with $\varepsilon = 0$ will be called the *unperturbed system*, otherwise we have the *perturbed system*. The unperturbed system has a unique singular point, the origin with eigenvalues

$$
\pm\sqrt{\frac{q+\sqrt{q^2+4}}{2}}\,i, \quad \pm\sqrt{\frac{\sqrt{q^2+4}-q}{2}}.
$$

We shall write system (7) in such a way that the linear part at the origin will be in its real Jordan normal form. Then, doing the change of variables $(x,y,z,v)\to(X,Y,Z,V)$ given by

$$
\begin{pmatrix} X\\ Y\\ Z\\ V \end{pmatrix} =
\begin{pmatrix}
0 & \dfrac{q-\sqrt{q^2+4}}{2} & 0 & 1\\
-\sqrt{\dfrac{2}{q+\sqrt{q^2+4}}} & 0 & \sqrt{\dfrac{q+\sqrt{q^2+4}}{2}} & 0\\
\sqrt{\dfrac{2}{\sqrt{q^2+4}-q}} & \dfrac{q-\sqrt{q^2+4}}{2} & \sqrt{\dfrac{\sqrt{q^2+4}-q}{2}} & 1\\
-\sqrt{\dfrac{2}{\sqrt{q^2+4}-q}} & \dfrac{q-\sqrt{q^2+4}}{2} & -\sqrt{\dfrac{\sqrt{q^2+4}-q}{2}} & 1
\end{pmatrix}
\begin{pmatrix} x\\ y\\ z\\ v \end{pmatrix},
$$

the differential system (7) becomes

$$
\begin{aligned}
X' &= -\sqrt{\frac{q+\sqrt{q^2+4}}{2}}\,Y + \varepsilon G(X,Y,Z,V),\\
Y' &= \sqrt{\frac{q+\sqrt{q^2+4}}{2}}\,X,\\
Z' &= \sqrt{\frac{\sqrt{q^2+4}-q}{2}}\,Z + \varepsilon G(X,Y,Z,V),\\
V' &= -\sqrt{\frac{\sqrt{q^2+4}-q}{2}}\,V + \varepsilon G(X,Y,Z,V),
\end{aligned}
\tag{8}
$$

where $G(X,Y,Z,V) = F(A,B,C,D)$ with

$$
\begin{aligned}
A = \;& \frac{1}{4\sqrt{2(q^2+4)}}\left[\sqrt{q^2+4}\left(\sqrt{\sqrt{q^2+4}-q}\,(Z-V) - 2\sqrt{q+\sqrt{q^2+4}}\,Y\right)\right.\\
& \left. +q\left(2\sqrt{q+\sqrt{q^2+4}}\,Y + \sqrt{\sqrt{q^2+4}-q}\,(Z-V)\right)\right],\\
B = \;& \frac{1}{2\sqrt{q^2+4}}\,(-2X+Z+V),\\
C = \;& \frac{1}{2\sqrt{2(q^2+4)}}\left(2\sqrt{q+\sqrt{q^2+4}}\,Y + \sqrt{\sqrt{q^2+4}-q}\,(Z-V)\right),\\
D = \;& \frac{1}{4\sqrt{q^2+4}}\left(\sqrt{q^2+4}\,(V+2X+Z) - q(V-2X+Z)\right),
\end{aligned}
$$

Note that the linear part of the differential system (8) at the origin is in its real normal form of Jordan.

Now we pass from the cartesian variables (X,Y,Z,V) to the cylindrical ones (r,θ,Z,Y) of $\mathbb{R}^4$, where $X = r\cos\theta$ and $Y = r\sin\theta$. In these new variables the differential system (8) can be written as

$$
\begin{aligned}
r' &= \varepsilon\cos\theta\, H(r,\theta,Z,V),\\
\theta' &= \sqrt{\frac{q+\sqrt{q^2+4}}{2}} - \varepsilon\frac{\sin\theta}{r}H(r,\theta,Z,V),\\
Z' &= \sqrt{\frac{\sqrt{q^2+4}-q}{2}}\,Z + \varepsilon H(r,\theta,Z,V),\\
V' &= -\sqrt{\frac{\sqrt{q^2+4}-q}{2}}\,V + \varepsilon H(r,\theta,Z,V),
\end{aligned}
\tag{9}
$$

where $H(r,\theta,Z,V) = F(a,b,c,d)$ with

$$
\begin{aligned}
a = \;& \frac{1}{4\sqrt{2(q^2+4)}}\left[2\left(q-\sqrt{q^2+4}\right)\sqrt{q+\sqrt{q^2+4}}\,r\sin\theta\right.\\
& \left. -\sqrt{\sqrt{q^2+4}-q}\left(q+\sqrt{q^2+4}\right)(V-Z)\right],\\
b = \;& \frac{1}{2\sqrt{q^2+4}}\,(Z+V-2\,r\cos\theta)\,,\\
c = \;& \frac{1}{2\sqrt{2(q^2+4)}}\left(\sqrt{\sqrt{q^2+4}-q}\,(Z-V) + 2\sqrt{q+\sqrt{q^2+4}}\,r\sin\theta\right),\\
d = \;& \frac{1}{4\sqrt{q^2+4}}\left(\sqrt{q^2+4}(Z+V+2\,r\cos\theta) - q(Z+V-2\,r\cos\theta)\right).
\end{aligned}
$$

Now we change the independent variable from l to θ, and denoting the derivative with respect to θ by a dot the differential system (9) becomes

$$
\begin{aligned}
\dot r =\ & \varepsilon\sqrt{\frac{2}{q+\sqrt{q^2+4}}}\cos\theta\, H + O(\varepsilon^2),\\
\dot Z =\ & \sqrt{\frac{\sqrt{q^2+4}-q}{q+\sqrt{q^2+4}}}\,Z\\
& +\varepsilon\frac{\sqrt{2}\left((q+\sqrt{q^2+4})r+\sqrt{\sqrt{q^2+4}-q}\sqrt{q+\sqrt{q^2+4}}Z\sin\theta\right)}{\left(q+\sqrt{q^2+4}\right)^{3/2} r}H + O(\varepsilon^2),\\
\dot V =\ & -\sqrt{\frac{\sqrt{q^2+4}-q}{q+\sqrt{q^2+4}}}\,V\\
& +\varepsilon\frac{\sqrt{2}\left((q+\sqrt{q^2+4})r-\sqrt{\sqrt{q^2+4}-q}\sqrt{q+\sqrt{q^2+4}}V\sin\theta\right)}{\left(q+\sqrt{q^2+4}\right)^{3/2} r}H + O(\varepsilon^2),
\end{aligned} \tag{10}
$$

where $H = H(r,\theta,Z,V)$.

We shall apply Theorem 6 to the differential system (10). We note that system (10) can be written as system (3) taking

$$
\mathbf{x} = \begin{pmatrix} r\\ Z\\ V\end{pmatrix},\quad t=\theta,\quad F_0(t,\mathbf{x}) = \begin{pmatrix} 0\\ \sqrt{\dfrac{\sqrt{q^2+4}-q}{q+\sqrt{q^2+4}}}\,Z\\ -\sqrt{\dfrac{\sqrt{q^2+4}-q}{q+\sqrt{q^2+4}}}\,V\end{pmatrix},
$$

and

$$
F_1(t,\mathbf{x}) = \begin{pmatrix} \sqrt{\dfrac{2}{q+\sqrt{q^2+4}}}\cos\theta\, H\\ \dfrac{\sqrt{2}\left((q+\sqrt{q^2+4})r+\sqrt{\sqrt{q^2+4}-q}\sqrt{q+\sqrt{q^2+4}}Z\sin\theta\right)}{\left(q+\sqrt{q^2+4}\right)^{3/2} r}H\\ \dfrac{\sqrt{2}\left((q+\sqrt{q^2+4})r-\sqrt{\sqrt{q^2+4}-q}\sqrt{q+\sqrt{q^2+4}}V\sin\theta\right)}{\left(q+\sqrt{q^2+4}\right)^{3/2} r}H\end{pmatrix}.
$$

We shall study the periodic solutions of system (4) in our case, i.e. the periodic solutions of the unperturbed system (10). Clearly these periodic solutions are

$$
(r(\theta), Z(\theta), V(\theta)) = (r_0, 0, 0),
$$

for any $r_0 > 0$; i.e. are all the circles of the plane $Z = V = 0$ of system (9). Of course all these periodic solutions in the coordinates (r, Z, V) have period 2π in the variable θ.

We shall describe the different elements which appear in the statement of Theorem 6 in the particular case of the differential system (10). Thus we have that $k = 1$ and $n = 3$. Let $r_1 > 0$ be arbitrarily small and let $r_2 > 0$ be arbitrarily large. Then we take the open bounded subset V of $\mathbb{R}$ as $V = (r_1, r_2)$, $\alpha = r_0$ and $\beta : [r_1, r_2] \to \mathbb{R}^2$ defined as $\beta(r_0) = (0, 0)$. The set $\mathcal{Z}$ is

$$\mathcal{Z} = \{\mathbf{z}_\alpha = (r_0, 0, 0),\ r_0 \in [r_1, r_2]\}.$$

Clearly for each $\mathbf{z}_\alpha \in \mathcal{Z}$ we can consider that the solution $\mathbf{x}(t) = \mathbf{z}_\alpha = (r_0, 0, 0)$ is 2π–periodic.

Computing the fundamental matrix $M_{\mathbf{z}_\alpha}(t)$ of the linear differential system (5) associated to the 2π–periodic solution $\mathbf{z}_\alpha = (r_0, 0, 0)$ such that $M_{\mathbf{z}_\alpha}(0)$ be the identity of $\mathbb{R}^3$, we get

$$M(t) = M_{\mathbf{z}_\alpha}(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\sqrt{\frac{\sqrt{q^2+4}-q}{q+\sqrt{q^2+4}}}\,\theta} & 0 \\ 0 & 0 & e^{-\sqrt{\frac{\sqrt{q^2+4}-q}{q+\sqrt{q^2+4}}}\,\theta} \end{pmatrix}.$$

Note that the matrix $M_{\mathbf{z}_\alpha}(t)$ does not depend of the particular periodic orbit $\mathbf{z}_\alpha$. Since the matrix

$$M^{-1}(0) - M^{-1}(2\pi) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 - e^{-\sqrt{\frac{\sqrt{q^2+4}-q}{q+\sqrt{q^2+4}}}\,2\pi} & 0 \\ 0 & 0 & 1 - e^{\sqrt{\frac{\sqrt{q^2+4}-q}{q+\sqrt{q^2+4}}}\,2\pi} \end{pmatrix},$$

satisfies the assumptions of statement (ii) of Theorem 6 we can apply it to system (10).

Now $\xi : \mathbb{R}^3 \to \mathbb{R}$ is $\xi(r, Z, V) = r$. We calculate the function

$$\begin{aligned} \mathcal{F}(r_0) = \quad & \mathcal{F}(\alpha) = \xi\left(\frac{1}{T}\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t) F_1(t, \mathbf{x}(t, \mathbf{z}_\alpha))dt\right), \\ = \quad & \frac{1}{2\pi}\int_0^{2\pi} \frac{\sqrt{2}\cos\theta}{\sqrt{q+\sqrt{q^2+4}}} F(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})\, d\theta, \end{aligned}$$

where the expression of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are the ones given in the statement of Theorem 1. Then, by Theorem 6 we have that for every simple zero $r_0^* \in [r_1, r_2]$ of the function $\mathcal{F}(r_0)$ we have a periodic solution $(r, Z, V)(\theta, \varepsilon)$ of system (10) such that

$$(r, Z, V)(\theta, \varepsilon) \to (r_0^*, 0, 0) \quad \text{as } \varepsilon \to 0.$$

Going back through the changes of coordinates we get a periodic solution $(r,\theta,Z,V)(l,\varepsilon)$ of system (9) such that

$$(r,\theta,Z,V)(l,\varepsilon)\to\left(r_0^*,\sqrt{\frac{q+\sqrt{q^2+4}}{2}}\;l,0,0\right)\quad \text{as } \varepsilon\to 0.$$

Consequently we obtain a periodic solution $(X,Y,Z,V)(l,\varepsilon)$ of system (8) such that

$$(X,Y,Z,V)(l,\varepsilon)\to\left(r_0^*\cos\left(\sqrt{\frac{q+\sqrt{q^2+4}}{2}}\;l\right),r_0^*\sin\left(\sqrt{\frac{q+\sqrt{q^2+4}}{2}}\;l\right),0,0\right)$$

as $\varepsilon\to 0$. Therefore, since

$$\begin{aligned} x= &\ \frac{1}{4\sqrt{2(q^2+4)}}\left(\sqrt{q^2+4}\left(\sqrt{\sqrt{q^2+4}-q}(Z-V)-2\sqrt{q+\sqrt{q^2+4}}\,Y\right)\right.\\ &\left.+q\left(2\sqrt{q+\sqrt{q^2+4}}\,Y+\sqrt{\sqrt{q^2+4}-q}(Z-V)\right)\right), \end{aligned}$$

we have a periodic solution $(x,y,z,v)(l,\varepsilon)$ of system (7) such that

$$x(l,\varepsilon)\to-\frac{\sqrt{2}\,r_0^*}{\sqrt{q^2+4}\sqrt{q+\sqrt{q^2+4}}}\sin\left(\sqrt{\frac{q+\sqrt{q^2+4}}{2}}\;l\right)$$

as $\varepsilon\to 0$. Of course, it is easy to check that the previous expression provides a periodic solution of the linear differential equation $u''''+qu''-u=0$. Hence Theorem 1 is proved.

4. Proof of the three corollaries

Proof of Corollary 2. If $F(u,u',u'',u''')=u'-(u')^3$, then the function $\mathcal{F}(r_0)$ of the statement of Theorem 1 is

$$\mathcal{F}(r_0)=-\frac{r_0\left(q^2+4-\dfrac{3r_0^2}{4}\right)}{\sqrt{2}\,(q^2+4)^{3/2}\sqrt{q+\sqrt{q^2+4}}}.$$

From this expression and (2) the corollary follows easily. □

Proof of Corollary 3. If $F(u,u',u'',u''')=\sin u'$, it is not difficult to show that

$$\mathcal{F}(r_0)=-\frac{\sqrt{2}\;J_1\left(\dfrac{r_0}{\sqrt{q^2+4}}\right)}{\sqrt{q+\sqrt{q^2+4}}},$$

where J_1 is the Bessel function of first kind. This function has infinitely many simple zeros which tends to be equidistant when $r_0\to\infty$, see for more details [1]. Hence the corollary is proved. □

Proof of Corollary 4. If $F(u,u',u'',u''')=u'''-(u''')^3$, then the function $\mathcal{F}(r_0)$ of the statement of Theorem 1 is

$$\mathcal{F}(r_0)=-\frac{\sqrt{q+\sqrt{q^2+4}}\,r_0\left(\left(3r_0^2-8\right)q^2+3\sqrt{q^2+4}r_0^2q+6r_0^2-32\right)}{16\sqrt{2}\left(q^2+4\right)^{3/2}}.$$

From this expression and (2) the corollary follows easily. □

Proof of Corollary 5. If $F(u,u',u'',u''')=g_u(u)$, then we have that

$$\frac{\sqrt{2}\cos\theta}{\sqrt{q+\sqrt{q^2+4}}}F(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D})=$$
$$\frac{\sqrt{2}\cos\theta}{\sqrt{q+\sqrt{q^2+4}}}g'\left(\frac{\left(q-\sqrt{q^2+4}\right)\sqrt{q+\sqrt{q^2+4}}r_0\sin\theta}{2\sqrt{2}\sqrt{q^2+4}}\right).$$

Since the primitive of this function is

$$-\frac{2\sqrt{q^2+4}}{\pi\left(\sqrt{q^2+4}-q\right)\left(q+\sqrt{q^2+4}\right)r_0}g\left(\frac{\left(q-\sqrt{q^2+4}\right)\sqrt{q+\sqrt{q^2+4}}r_0\sin\theta}{2\sqrt{2}\sqrt{q^2+4}}\right).$$

We get that $\mathcal{F}(r_0)=0$. □

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