

ON THE CONFIGURATION OF HERMAN RINGS OF MEROMORPHIC FUNCTIONS

NÚRIA FAGELLA AND JÖRN PETER

ABSTRACT. We prove some results concerning the possible configurations of Herman rings for transcendental meromorphic functions. We show that one pole is enough to obtain cycles of Herman rings of arbitrary period and give a sufficient condition for a configuration to be realizable.

1. INTRODUCTION

Given a meromorphic map $f : \mathbb{C} \rightarrow \widehat{\mathbb{C}}$, we consider the dynamical system generated by the iterates of f , denoted by $f^n = f \circ \dots \circ f$. If f has a limit at ∞ , then $f : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$ is a rational map. Otherwise, f is a transcendental map, i.e., it has an essential singularity at ∞ . If the essential singularity has no preimages, i.e., if f has no *poles*, we speak about entire transcendental functions. Else f is known as a (transcendental) meromorphic function.

There is a dynamically natural partition of the phase space into the *Fatou set* $\mathcal{F}(f)$, where the iterates of f are well defined and form a normal family, and the *Julia set* $J(f)$, which is the complement.

Background on iteration theory of rational maps can be found for example in [7], [18] or [22]. For transcendental maps, the reader can check the survey in [8] or the book [16].

There are several differences between transcendental and rational maps. One important such difference concerns the *singular values* or singularities of the inverse function. For a rational map f , all branches of the inverse function are locally well defined except on the set of *critical values*, i.e., points $v = f(c)$ where $f'(c) = 0$. If f is transcendental, there is another obstruction: some inverse branches are not well defined either in any neighborhood of the *asymptotic values*. A point $a \in \mathbb{C}$ is called an asymptotic value if there exists a path $\gamma(t) \xrightarrow{t \rightarrow \infty} \infty$ such that $f(\gamma(t)) \xrightarrow{t \rightarrow \infty} a$. Although critical values always form a discrete set (even finite in the case of rational maps), this needs not be the case for asymptotic values, unless f is of finite order.

This fact motivated the definition and study of special classes of transcendental maps like, for example, the class \mathcal{S} of functions of *finite type*, which are those with a finite number of singular values. Entire or meromorphic functions in \mathcal{S} share many properties with rational maps, like for example the fact that all components of the

Fatou set are eventually periodic [12, 13]. There is a classification of the periodic components of the Fatou set for a rational map or a transcendental map in class \mathcal{S} : Such a component can either be a rotation domain (a Siegel disk if it is simply connected or a Herman ring if it is doubly connected) or the basin of attraction of an attracting, superattracting or parabolic periodic point. Each of these components is somehow associated to a singular value. More precisely, all basins must contain one such point, while the orbit of a singular value must always accumulate on each boundary component of a rotation domain.

If we allow f to have infinitely many singular values then there are more possibilities: namely a *Fatou component* (i.e., a connected component of $\mathcal{F}(f)$) can be *wandering*, that is it will never be iterated to a periodic component; or it can belong to a cycle of *Baker domains*, i.e., a domain on which some subsequence of iterates tend to infinity (the essential singularity) at a linear rate.

In this paper we are concerned with one particular type of Fatou components, namely Herman rings. For $0 < r < s < \infty$, let $A_{r,s}$ denote the standard annulus of inner radius r and outer radius s . For $\theta \in [0, 1)$ let $\mathcal{R}_\theta(z) = e^{2\pi i \theta} z$ be the rigid rotation of angle $2\pi\theta$. A p -periodic Fatou component $A \in \widehat{\mathbb{C}}$ is called a *Herman ring* of f , if there exist $r < 1$ and $\theta \in (0, 1) \setminus \mathbb{Q}$ and a conformal change of variables $L : A_{r,1} \rightarrow A$ such that L conjugates f^p to \mathcal{R}_θ , i.e., $f^p \circ L = L \circ \mathcal{R}_\theta$. In this case, we call L the *linearizing function* and θ the *rotation number* of A , which is (mod 1) determined up to sign. It follows that Herman rings are doubly connected, and foliated by invariant simple closed curves, on which all orbits under f^p are dense. An n -cycle of Herman rings consists of n -periodic Herman rings A_1, \dots, A_n which satisfy $f(A_i) \subset A_{i+1}$ for all $i < n$, and $f(A_n) = A_1$.

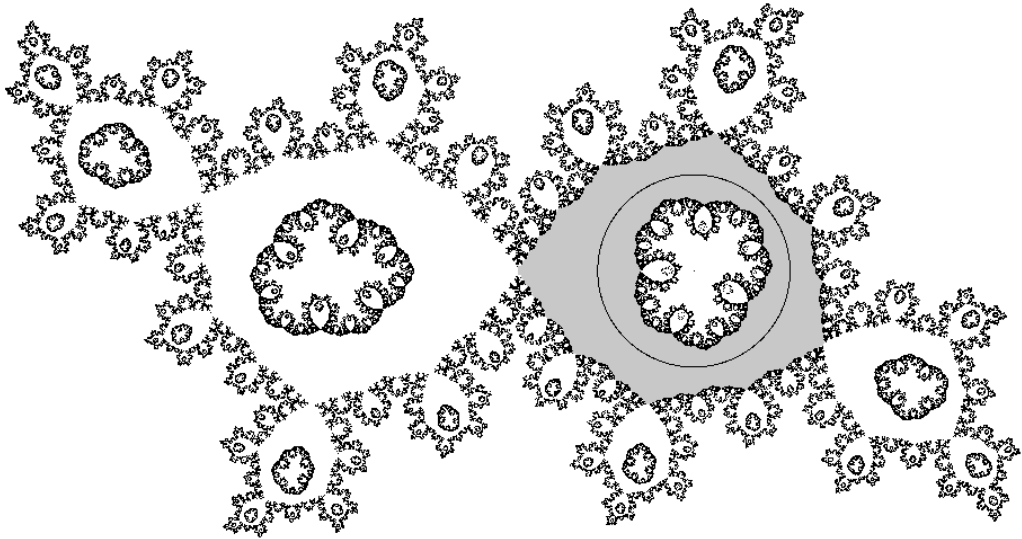


FIGURE 1. Herman ring of a rational map

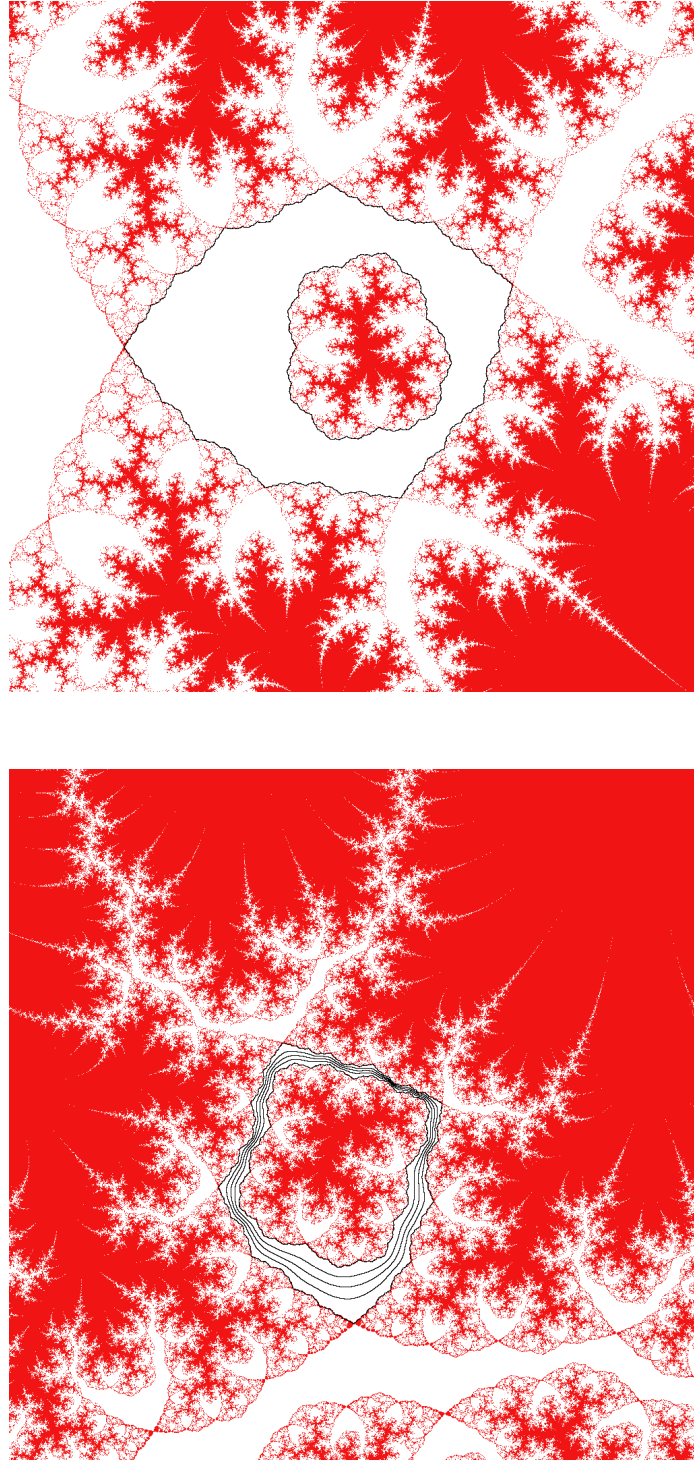


FIGURE 2. Herman rings of transcendental meromorphic functions

For our constructions, we often also work with Siegel disks. These are Fatou domains on which f^p is conjugated to an irrational rotation of the unit disk. We use the terms *linearizing function* and *rotation number* in the same way as for Herman rings.

It is well known that entire functions may not have Herman rings, which is a consequence of the maximum principle. Although Fatou and Julia originally conjectured that Herman rings did not exist for any rational map, this was proven to be false by Herman in [14] who gave an example by extending earlier work of Arnold in [2]. Later on, Shishikura [19] used quasiconformal surgery to construct rational functions with Herman rings. We shall make extensive use of his construction all throughout the paper. More precisely, Shishikura [19] showed that for any $p > 0$ and any $\theta \in (0, 1) \setminus \mathcal{B}$, there exists a rational functions with a p -cycle of Herman rings of rotation number θ , where \mathcal{B} stands for the set of irrational numbers called of *Brjuno type*, a superset of the Diophantine numbers. The periodic cycles of Herman rings in Shishikura's construction are all non-nested, that is, assuming that ∞ does not belong to any of the rings, every ring lies in the unbounded component of the complement of all other rings in the cycle. In the same paper, Shishikura also proved that nested configurations are possible, by explicitly constructing a rational map with a nested 2-cycle of Herman rings.

Shishikura's surgery construction was later generalized to transcendental meromorphic functions in [10], showing the existence of transcendental meromorphic functions with *invariant* Herman rings of arbitrary Brjuno rotation number, and arbitrary configuration. An example with infinitely many invariant Herman rings was constructed in [17] for a transcendental meromorphic function with infinitely many singular values.

In this paper we are concerned with invariant cycles of Herman rings for transcendental meromorphic maps. Our first result concerns the relation of such cycles with the number of poles of f . This result appeared already in [10], but unfortunately, the proof was incorrect.

Theorem A. *Let A_1, \dots, A_n be invariant Herman rings of a transcendental meromorphic function f . Then there exists a pole in every bounded connected component of $\mathbb{C} \setminus \bigcup_{i=1}^n A_i$.*

An obvious corollary of this theorem is the following:

Corollary. *If a transcendental meromorphic function f has n poles, then f has at most n invariant Herman rings.*

On the other hand, we can show that for given n , there is a transcendental meromorphic function that has an n -cycle of Herman rings and just one pole.

Theorem B. *Let $n \in \mathbb{N}, k_1, \dots, k_n \in \mathbb{N}_0$ such that $\sum_{i=1}^n k_i \geq 1$ and $\theta \in \mathcal{B}$. Then there exists a transcendental meromorphic function F which has an n -cycle of Herman rings A_1, \dots, A_n of rotation number θ such that there are exactly k_i poles in the bounded component of $\mathbb{C} \setminus A_i$, and no other poles. Further, F can be chosen to be of finite type (i.e. f has at most finitely many critical and asymptotic values).*

Once the existence of cycles of Herman rings is established, it arises the question of possible configurations (see Section 3 for a precise definition). There is extensive

work by Shishikura [20, 21] (some of it unpublished) concerning this issue for rational maps. For a given configuration of rings he defined an associated abstract *tree* and showed that, for any tree satisfying certain conditions, one can construct a rational map with a cycle of Herman rings realizing such tree. We will not generalize his results in this paper. Instead, we present a surgery construction which “transcendentalizes” rational maps g with Herman rings. Intuitively, we add an essential singularity at a certain point in the Riemann sphere, but still preserve the dynamics in a region which contains all the Herman rings of g .

We show that the procedure of ‘transcendentalizing’ a rational function g without affecting the configuration of Herman rings is always possible. First, we deal with the nicer case (see Theorem C below) where there exists a curve γ which is mapped ‘outside of itself’ under g . In this case, we can say a lot about the dynamics of the resulting transcendental meromorphic function F .

To make clear what we mean by ‘outside of itself’, we use the following definition. For a simple, closed, oriented curve γ in $\hat{\mathbb{C}}$, we denote by $\text{int}(\gamma)$ (resp. $\text{ext}(\gamma)$) the component of $\hat{\mathbb{C}} \setminus \gamma$ which lies to the left (resp. to the right) of γ (see Figure 3 below). Throughout this article, if the orientation of a simple closed curve in \mathbb{C} is not defined explicitly, we always assume that it is oriented such that the interior of γ is the bounded component of $\mathbb{C} \setminus \gamma$.

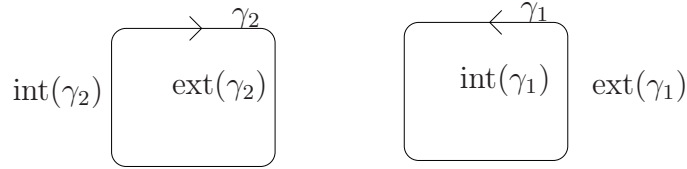


FIGURE 3. The interior and exterior depends on the orientation

To state the theorem we use the notation $\mathcal{C}_{\mathbb{C}}$ or $\mathcal{C}_{\hat{\mathbb{C}}}$ to denote a configuration in \mathbb{C} or $\hat{\mathbb{C}}$. The precise definitions can be found in Section 3 but, intuitively, a configuration is an equivalence class which provides the relative position, orientation and dynamics of the cycles of Herman rings.

Theorem C. *Let g be a rational function with Herman rings and let $\mathcal{C}_{\hat{\mathbb{C}}}(g)$ be the configuration associated with the Herman rings of g . Suppose that there exists a simple closed real-analytic curve γ contained in a non-periodic preimage of some periodic component of the Fatou set such that g is injective on γ and $\text{int}(\gamma) \subset \text{int}(g(\gamma))$, where we choose some orientation of γ and assign $g(\gamma)$ the orientation respected by g . Further suppose that $g^n(\gamma) \cap \text{int}(\gamma) = \emptyset$ for all $n \in \mathbb{N}$ and that there are no periodic Herman rings in $\text{int}(\gamma)$. Then there exists a transcendental meromorphic function F and a configuration $\mathcal{C}_{\mathbb{C}}$ with $\mathcal{C}_{\hat{\mathbb{C}}} \sim \mathcal{C}_{\hat{\mathbb{C}}}(g)$ and $\mathcal{C}_{\mathbb{C}} \sim \mathcal{C}_{\mathbb{C}}(F)$, i.e. F realizes $\mathcal{C}_{\mathbb{C}}$. Further, F is of finite type, has finitely many poles, and is quasiconformally conjugate to a hyperbolic exponential map near ∞ .*

The Herman rings arising from Theorem B are 'non-nested', i.e. if we consider the configuration associated with the cycle we obtain from the theorem, we have either $A_i \subset \text{ext}(A_j)$ or $A_i \subset \text{int}(A_j)$ for all $i \in \{1, \dots, n\}$ and $j \neq i$.

As a corollary to Theorem C, we can prove that there exists a transcendental meromorphic function f with a 'nested' 2-cycle A_1, A_2 of Herman rings, i.e. $A_1 \subset \text{int}(A_2)$ and $A_2 \subset \text{ext}(A_1)$. The existence of a rational function satisfying this configuration has been proven first by Shishikura, and it is easy to see that the function he constructs satisfies the hypothesis of Theorem C.

Corollary. *There exists a transcendental meromorphic function F which has a 2-cycle of Herman rings A_1, A_2 such that $A_1 \subset \text{int}(A_2)$ and $A_2 \subset \text{ext}(A_1)$. Further, F can be chosen to be of finite type and has finitely many poles.*

It turns out we can generalize the method of the proof of Theorem C for any other rational map with Herman rings, hence proving that any rational configuration is also possible for transcendental meromorphic maps. More precisely we prove the following (see Section 3 for precise definitions).

Theorem D. *Let g be a rational function with Herman rings. Let $\mathcal{C}'_{\mathbb{C}}$ be a configuration with $\mathcal{C}'_{\mathbb{C}} \sim \mathcal{C}_{\mathbb{C}}(g)$. Then there exists a transcendental meromorphic function F with $\mathcal{C}_{\mathbb{C}}(F) \sim \mathcal{C}'_{\mathbb{C}}$, i.e. F realizes $\mathcal{C}'_{\mathbb{C}}$.*

The paper is organized as follows. In section 2 we review some facts about quasiconformal surgery that we will need throughout the paper. In section 3 we provide the terminology and definitions we will use, in particular we will define what we mean by a configuration. Section 4 deals with the proofs of Theorems A, B and C. In section 5 we give the proof of the main theorem (Theorem D).

2. TOOLS FROM QUASICONFORMAL SURGERY

We call a function quasiregular (resp. quasimeromorphic) if g is locally the composition $f \circ \phi$ of a holomorphic (resp. meromorphic) map f and a quasiconformal map ϕ . The measurable Riemann mapping theorem (see e.g. [1]) ensures that the Beltrami equation

$$\frac{f_{\bar{z}}}{f_z} = \mu$$

can be solved almost everywhere by a K -quasiconformal map ϕ under the very weak condition that the Beltrami coefficient (or complex dilatation) μ is measurable and satisfies $\|\mu\|_{\infty} < 1$ (we say that μ is bounded), and the constant K can be chosen as $K = \frac{1+\|\mu\|_{\infty}}{1-\|\mu\|_{\infty}}$. The importance of this theorem for quasiconformal surgery is the following: Suppose that f is a quasiregular or quasimeromorphic function and μ is a bounded Beltrami coefficient which is f -invariant, i.e. the pullback

$$f^*\mu(u) := \frac{f_{\bar{z}}(u) + \mu(f(u))\overline{f_z(u)}}{f_z(u) + \mu(f(u))\overline{f_{\bar{z}}(u)}}$$

satisfies $f^*\mu = \mu$ almost everywhere. Let ϕ be the solution of the Beltrami equation, i.e. $\phi^*\mu_0 = \mu$ (where μ_0 is the function that is 0 everywhere). If we define $g := \phi \circ f \circ \phi^{-1}$, it follows that μ_0 is g -invariant, and a corollary of Weyl's lemma yields that g is holomorphic (resp. meromorphic) since g is also quasiregular (resp. quasimeromorphic). Hence, the central question that arises is the following: Given a quasiregular (or quasimeromorphic) map g , under which conditions does a g -invariant bounded Beltrami coefficient exist? The most general answer to this question is the following result (see e.g. [15]).

Theorem. *Let g be a quasiregular map. Then there exists a bounded, g -invariant Beltrami coefficient μ if and only if the iterates g^n are uniformly K -quasiregular for some $K < \infty$.*

Since the set of poles of a quasimeromorphic map together with all their preimages has zero measure, the proof of this theorem can be carried over to quasimeromorphic maps as well, we only have to replace 'quasiregular' by 'quasimeromorphic' in the statement (see [9]).

Although this theorem is very useful in many situations, it is not always easy to see that the family of iterates is uniformly K -quasiregular (or K -quasimeromorphic).

In [19], Shishikura proved another sufficient condition for the existence of a bounded invariant Beltrami coefficient.

Lemma. *Let $m \in \mathbb{N}$ and $g : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ be a quasiregular (resp. quasimeromorphic) map. Suppose that there exists an integer $N \geq 0$, disjoint open sets E_i and quasiconformal mappings $\phi_i : E_i \rightarrow E'_i$ (where $i = 1, \dots, m$) satisfying the following conditions:*

- (a) $g(E) \subset E$, where $E = \bigcup_{i=1}^m E_i$
- (b) $\phi \circ g \circ \phi_i^{-1}$ is analytic in $E'_i = \phi_i(E_i)$, where $\phi : E \rightarrow \hat{\mathbb{C}}$ is defined by $\phi|_{E_i} = \phi_i$
- (c) $g_{\bar{z}} = 0$ a.e. on $\hat{\mathbb{C}} \setminus g^{-N}(E)$

Then there exists a quasiconformal mapping φ of $\hat{\mathbb{C}}$ such that $\varphi \circ g \circ \varphi^{-1}$ is a rational (resp. transcendental meromorphic) function. Moreover, $\varphi \circ \phi_i^{-1}$ is conformal in E'_i and $\varphi_{\bar{z}} = 0$ a.e. on $\hat{\mathbb{C}} \setminus \bigcup_{n \geq 0} g^{-n}(E)$.

The quasiconformal maps we construct in this paper arise from pasting together several quasiconformal maps on analytic curves. Most of the times, we prescribe the boundary values of the function on such a curve γ , choose some conformal map f defined in the interior (or exterior) of γ and modify f near γ to fit the boundary conditions. This modification is done by quasiconformal interpolation. We will use the following fundamental result implicitly several times throughout this paper - see e.g. [9] or [10] for a proof.

Lemma (qc interpolation). *Let A and \tilde{A} be annuli bounded by real analytic Jordan curves $\gamma^{(i)}, \gamma^{(o)}, \tilde{\gamma}^{(i)}, \tilde{\gamma}^{(o)}$ respectively, where (i) (resp. (o)) stands for inner (resp. outer) boundary. Let $f^{(i)} : \gamma^{(i)} \rightarrow \tilde{\gamma}^{(i)}$ (resp. $f^{(i)} : \gamma^{(i)} \rightarrow \tilde{\gamma}^{(o)}$) and $f^{(o)} : \gamma^{(o)} \rightarrow \tilde{\gamma}^{(o)}$*

(resp. $f^{(o)} : \gamma^{(o)} \rightarrow \tilde{\gamma}^{(i)}$) be orientation-preserving (resp. orientation-reversing) diffeomorphisms. Then, there exists a quasiconformal map $f : A \rightarrow \tilde{A}$ such that $f|_{\gamma^{(i)}} = f^{(i)}$ and $f|_{\gamma^{(o)}} = f^{(o)}$.

3. HERMAN RINGS AND CONFIGURATIONS

Let γ and δ be simple, oriented, closed curves in $\hat{\mathbb{C}}$. Note that $\text{int}(\gamma) = \text{int}(\delta)$ means that $\gamma = \delta$ (as sets) and that γ and δ have in addition the same orientation.

By a configuration $\mathcal{C}_{\hat{\mathbb{C}}} = (\Gamma, \pi)_{\hat{\mathbb{C}}}$, we mean a finite collection $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ of disjoint, oriented, simple closed curves in $\hat{\mathbb{C}}$, together with a permutation π of $\{1, \dots, n\}$.

Two configurations $\mathcal{C}_{\hat{\mathbb{C}}} = (\Gamma, \pi)_{\hat{\mathbb{C}}}$ and $\mathcal{C}'_{\hat{\mathbb{C}}} = (\Gamma', \pi')_{\hat{\mathbb{C}}}$ (where $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ and $\Gamma' = \{\gamma'_1, \dots, \gamma'_n\}$) can be considered equivalent (we write $\mathcal{C}_{\hat{\mathbb{C}}} \sim \mathcal{C}'_{\hat{\mathbb{C}}}$) if one of the following conditions is satisfied:

- $\pi = \pi'$ and there exists a homeomorphism ψ of $\hat{\mathbb{C}}$ such that $\text{int}(\psi(\gamma_i)) = \text{int}(\gamma'_i)$
- $\pi = \pi'$ and the orientations of γ_i and γ'_i are reversed for all i that form one or more cycles of π
- there exists a permutation τ of $\{1, \dots, n\}$ such that

$$\tau^{-1} \circ \pi \circ \tau = \pi' \text{ and } \text{int}(\gamma_{\tau(i)}) = \text{int}(\gamma'_i)$$

Naturally, they are also equivalent if $\mathcal{C}'_{\hat{\mathbb{C}}}$ arises from $\mathcal{C}_{\hat{\mathbb{C}}}$ by combining more than one of the above properties.

Let f be a rational function which has exactly n periodic Herman rings A_1, \dots, A_n .

For $i = 1, \dots, n$ let $\pi(i) := j$ if $f(A_i) \subset A_j$. Let δ_i be an oriented curve in A_i , where the orientation is chosen such that $\text{int}(f(\delta_i)) \setminus A_{\pi(i)} = \text{int}(\delta_{\pi(i)}) \setminus A_{\pi(i)}$ (i.e. $f(\delta_i)$ and $\delta_{\pi(i)}$ have the same orientation in $A_{\pi(i)}$), and define $\Delta := \{\delta_1, \dots, \delta_n\}$.

We define the configuration of the Herman rings of f by $\mathcal{C}_{\hat{\mathbb{C}}}(f) := (\Delta, \pi)_{\hat{\mathbb{C}}}$ (it is immediate that the equivalence class does not change if we choose different curves δ'_i or renumber them). See Figure 4.

We say that the configuration $\mathcal{C}_{\hat{\mathbb{C}}} = (\Gamma, \pi)_{\hat{\mathbb{C}}}$ is realized by f if $\mathcal{C}_{\hat{\mathbb{C}}}(f) \sim \mathcal{C}_{\hat{\mathbb{C}}}$.

It is not hard to see that that a configuration $(\Gamma, \pi)_{\hat{\mathbb{C}}}$ (where $\Gamma = \{\gamma_1, \dots, \gamma_n\}$) is realized by f if and only if $f(A_i) = A_{\pi(i)}$ for all $i \in \{1, \dots, n\}$ and there exist oriented simple closed curves $\delta_i \in A_i$ and a homeomorphism $\phi : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ with the following properties:

- If $f(\delta_i)$ is oriented such that f respects the orientation of δ_i and $f(\delta_i)$, then $\text{int}(f(\delta_i)) \setminus A_{\pi(i)} = \text{int}(\delta_{\pi(i)}) \setminus A_{\pi(i)}$ for all i
- $\text{int}(\phi(\delta_i)) = \text{int}(\gamma_i)$ for all i

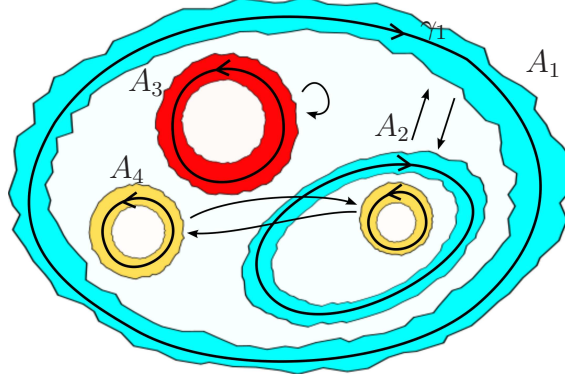


FIGURE 4. $\mathcal{C}_{\hat{\mathbb{C}}}(f) = \{\{\gamma_1, \dots, \gamma_5\}, \{2, 1, 3, 5, 4\}\}$ is a representative of the equivalence class.

The main objective of this paper is to show that every configuration that is realizable by a rational function is also realizable by a transcendental meromorphic function. Since ∞ is a special point for transcendental meromorphic maps, we define a configuration $\mathcal{C}_{\mathbb{C}} := (\Gamma, \pi)_{\mathbb{C}}$ of curves in \mathbb{C} by replacing every instance of $\hat{\mathbb{C}}$ in the previous definition by \mathbb{C} . We then call $(\mathcal{C}, \pi)_{\mathbb{C}}$ realizable if there exists a transcendental meromorphic function f which satisfies the above conditions (again with $\hat{\mathbb{C}}$ replaced by \mathbb{C}).

Notice that the variety of configurations in \mathbb{C} is larger than in $\hat{\mathbb{C}}$ - two configurations that are equivalent in $\hat{\mathbb{C}}$ need not be equivalent as configurations in \mathbb{C} . An example is the following: Let $r_1 < r_2$ and $\gamma_i := \{|z| = r_i\}$, oriented such that $\gamma_1 \subset \text{int}(\gamma_2)$ and $\gamma_2 \subset \text{ext}(\gamma_1)$. Let $\gamma'_1 := S^1$ and $\gamma'_2 := \{|z - 3| = 1\}$, again oriented such that $\gamma'_1 \subset \text{int}(\gamma'_2)$ and $\gamma'_2 \subset \text{ext}(\gamma'_1)$. Let \mathcal{C} consist of γ_1 and γ_2 , and \mathcal{C}' of γ'_1 and γ'_2 , both with $\pi = \pi' := \text{id}$. Then $(\mathcal{C}, \pi)_{\hat{\mathbb{C}}}$ and $(\mathcal{C}', \pi')_{\hat{\mathbb{C}}}$ are equivalent, but $(\mathcal{C}, \pi)_{\mathbb{C}}$ and $(\mathcal{C}', \pi')_{\mathbb{C}}$ are not, since a homeomorphism f of \mathbb{C} cannot send the bounded component of $\mathbb{C} \setminus \gamma'_2$ to the unbounded component of $\mathbb{C} \setminus \gamma_2$, as it would be required.

Nevertheless, we will show at the end of this paper that if $\mathcal{C}_{\hat{\mathbb{C}}}$ is a realizable configuration, then all configurations $\mathcal{C}'_{\hat{\mathbb{C}}}$ with $\mathcal{C}_{\hat{\mathbb{C}}} \sim \mathcal{C}'_{\hat{\mathbb{C}}}$ are also realizable (by a transcendental meromorphic function).

4. PROOF OF THEOREMS A, B AND C

In this section, we prove Theorems A, B and C. We first recall the statements.

Theorem. *Let A_1, \dots, A_n be invariant Herman rings of a transcendental meromorphic function f . Then there exists a pole in every bounded connected component of $\mathbb{C} \setminus \bigcup_{i=1}^n A_i$.*

Proof. Suppose that the above statement does not hold, i.e. there exists a bounded component U of $\mathbb{C} \setminus \bigcup A_i$ which does not contain any pole of f . There exists an index i

such that A_i 'surrounds' U , i.e. A_i is contained in the unbounded component of $\mathbb{C} \setminus U$ and $\partial A_i \cap \partial U \neq \emptyset$. For every k , let γ_k be an invariant curve in A_k , oriented such that $\text{int}(\gamma_k)$ is the bounded component of $\mathbb{C} \setminus \gamma_k$. Let $\gamma_{k_1}, \dots, \gamma_{k_m}$ be the curves which are contained in $\text{int}(\gamma_i)$, but not in the interior of any other curve $\gamma_j \subset \text{int}(\gamma_i)$. If there are no such curves, then we obtain a contradiction using the maximum principle. Otherwise we glue a rotation inside of each γ_{k_l} , transforming $A_{k_l} \cup \text{int}(\gamma_{k_l})$ into a Siegel disc following fundamental ideas of Shishikura [19]. To make this construction precise we fix l and start by setting $\gamma := \gamma_{k_l}$ and $A := A_{k_l}$. Further, let θ be the rotation number of A .

Let $L : A_{R,1} \rightarrow A$ be the linearizing map. Then there is an $r \in (R, 1)$ such that $L(\{|z| = r\}) = \gamma$.

Let \tilde{L} be a quasiconformal extension of L (for example, the Douady-Earle extension, see [11]) from $\overline{D(0, r)}$ to $\overline{\text{int}(\gamma)}$ and let $\mathcal{R}_\theta(z) := e^{2\pi i \theta} z$ be the rigid rotation by $2\pi\theta$.

Define

$$h(z) := \begin{cases} \tilde{L} \circ \mathcal{R}_\theta \circ \tilde{L}^{-1}(z) & \text{if } z \in \text{int}(\gamma) \\ f(z) & \text{otherwise.} \end{cases}$$

Then it is easy to see that h is a quasimeromorphic map that coincides with f on γ (since \tilde{L} is a quasiconformal extension of the linearizer). Further, the set $E := \text{int}(\gamma)$ satisfies the conditions of Shishikura's lemma (with $N = 0$), so there exists a quasiconformal map $\varphi : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ fixing ∞ such that $F = \varphi^{-1} \circ h \circ \varphi$ is a transcendental meromorphic function. It follows easily that $\varphi(\gamma)$ is an invariant curve in a Siegel disk of F .

Repeating this construction with each of the A_{k_l} , we end up with a transcendental meromorphic function which has a Herman ring B that is some quasiconformal image of A_i , but all Herman rings inside have been transformed into Siegel disks. In particular there is no pole in $\text{int}(B)$ which is a contradiction to the maximum principle.

□ See Figure 5

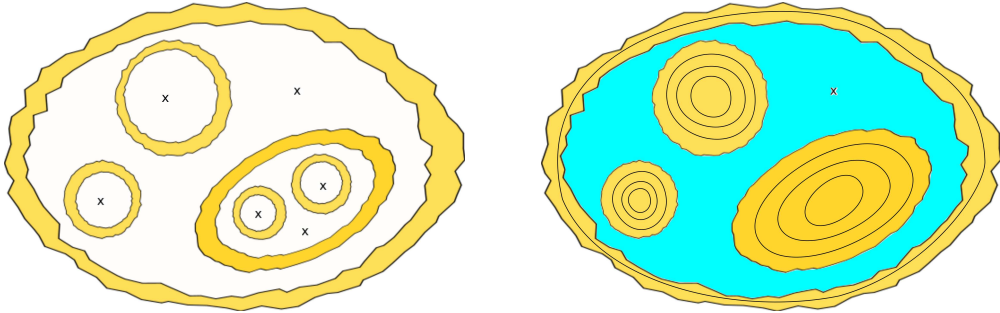


FIGURE 5. Sketch of the proof of Theorem A.

We now prove Theorem B which shows that for given n , a transcendental meromorphic function that has an n -cycle of Herman rings and just one pole can be constructed.

Theorem. *Let $n \in \mathbb{N}$, $k_1, \dots, k_n \in \mathbb{N}_0$ such that $\sum_{i=1}^n k_i \geq 1$ and $\theta \in \mathcal{B}$. Then there exists a transcendental meromorphic function F which has an n -cycle of Herman rings A_1, \dots, A_n of rotation number θ such that there are exactly k_i poles in the bounded component of $\mathbb{C} \setminus A_i$, and no other poles. Further, F can be chosen to be of finite type (i.e. f has at most finitely many critical and asymptotic values).*

Proof. Choose a transcendental entire function f which has an n -cycle of Siegel disks $\Delta_1, \dots, \Delta_n$, and choose an f^n -invariant curve γ_1 in Δ_1 . Then $\gamma_1 = L(\{|z| = r\})$, where $L : \mathbb{D} \rightarrow \Delta_1$ is the linearizing map. We define

$$\gamma_i := f^{i-1}(\gamma_1) \text{ for } i = 2, \dots, n.$$

Now let p_i be polynomials as follows:

If $k_i = 0$, let $p_i := \text{id}$. Otherwise let p_i be a polynomial of degree $k_i + 1$ which has a Siegel disk around 0 of rotation number $-\theta/n_0$, where n_0 is the number of non-zero entries in (k_1, \dots, k_n) (note that $-\theta/n_0$ is also a Brjuno number). Then it follows that $p_n \circ p_{n-1} \circ \dots \circ p_1$ has a Siegel disk D_1 around 0 of rotation number $-\theta$. Let

$$D_i := p_{i-1}(D_{i-1}) \text{ for all } i = 2, \dots, n.$$

Let $\delta_1 := \tilde{L}(\{|z| = r\})$, where \tilde{L} is the linearizing map of D_1 . Then δ_1 is a curve in D_1 which is invariant under $p_n \circ \dots \circ p_1$. Define

$$\delta_i := p_{i-1}(\delta_{i-1}) \text{ for all } i = 2, \dots, n.$$

Now construct an orientation-reversing real-analytic homeomorphism $\psi_1 : \gamma_1 \rightarrow \delta_1$ as follows. Since we have

$$\tilde{L}^{-1} \circ p_n \circ \dots \circ p_1 \circ \tilde{L}(z) = e^{-2\pi i \theta} z,$$

conjugating the left side by $c(z) := \bar{z}$ yields a rigid rotation by θ .

Further it is clear that $L^{-1} \circ f^n \circ L$ is also a rigid rotation by θ . Hence, by defining

$$\psi_1 := \tilde{L} \circ c \circ L^{-1},$$

ψ_1 is an orientation-reversing real-analytic homeomorphism mapping γ_1 to δ_1 and satisfying

$$\psi_1^{-1} \circ p_n \circ \dots \circ p_1 \circ \psi_1 = f^n.$$

For $i = 2, \dots, n$, define orientation-reversing maps $\psi_i : \gamma_i \rightarrow \delta_i$ by

$$\psi_i \circ f = p_{i-1} \circ \psi_{i-1}.$$

Let $E'_1 := \tilde{L}(A_{R_1, R_2})$ (where $0 < R_1 < r < R_2 < 1$) and $E'_i := p_{i-1}(E'_{i-1})$ for $i = 2, \dots, n$. Similarly, let $E_1 := L(A_{R_1, R_2})$ and $E_i := f^{i-1}(E_{i-1})$ for $i = 2, \dots, n$.

Now construct quasiconformal maps $\phi_i : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ as follows: Let ϕ_i be conformal outside E_i , mapping the center of Δ_i to ∞ and ∞ to 0. Extend ϕ_i by quasiconformal

interpolation to $\overline{E_i \cap \text{ext}(\gamma_i)}$ such that ϕ_i equals ψ_i on γ_i . In the same way extend ϕ_i quasiconformally to $\overline{E_i \cap \text{int}(\gamma_i)}$.

Define

$$g(z) := \begin{cases} f(z) & \text{if } z \notin \bigcup_{i=1}^n \text{int}(\gamma_i) \\ \phi_{i+1}^{-1} \circ p_i \circ \phi_i(z) & \text{if } z \in \text{int}(\gamma_i) \end{cases}.$$

Then g is quasiregular. Observe that the poles of g in $\text{int}(\gamma_i)$ are exactly the non-zero preimages under ϕ_i of the zeros of p_i . Hence there are as many poles in $\text{int}(\gamma_i)$ as $\deg(p_i) - 1$.

By construction, we have $\gamma_i \subset E_i \subset \Delta_i$ and $E := \bigcup_{i=1}^n E_i$ is invariant under g . Further, by defining $\phi := \phi_i|_{E_i}$ on E_i , the function $\phi \circ g \circ \phi^{-1}$ is holomorphic on $\phi(E_i)$ for all i , and g is holomorphic on $\hat{\mathbb{C}} \setminus g^{-1}(E)$. Hence the hypothesis of Shishikura's lemma is satisfied, so the existence of F follows. Further, if f was chosen to be of finite type, then F is also. \square

The Herman rings constructed here are all non-nested, i.e. if we choose a periodic curve δ_1 in one of the periodic Herman rings of F and assign $f^m(\delta_1)$ the orientation such that f^m respects the orientations of δ_1 and $f^m(\delta_1)$, then we have either $A_j \subset \text{int}(A_i)$ or $A_j \subset \text{ext}(A_i)$ for $j \neq i$. The existence of a transcendental meromorphic function with a nested cycle of Herman rings follows from a construction by Shishikura together with Theorem C, which we will prove now.

Theorem. *Let g be a rational function with Herman rings and let $\mathcal{C}_{\hat{\mathbb{C}}}(g)$ be the configuration associated with the Herman rings of g . Suppose that there exists a simple closed real-analytic curve γ contained in a non-periodic preimage of some periodic component of the Fatou set such that g is injective on γ and $\text{int}(\gamma) \subset \text{int}(g(\gamma))$, where we choose some orientation of γ and assign $g(\gamma)$ the orientation respected by g . Further suppose that $g^n(\gamma) \cap \text{int}(\gamma) = \emptyset$ for all $n \in \mathbb{N}$ and that there are no periodic Herman rings in $\text{int}(\gamma)$. Then there exists a transcendental meromorphic function F and a configuration $\mathcal{C}_{\mathbb{C}}$ with $\mathcal{C}_{\hat{\mathbb{C}}} \sim \mathcal{C}_{\hat{\mathbb{C}}}(g)$ and $\mathcal{C}_{\mathbb{C}} \sim \mathcal{C}_{\mathbb{C}}(F)$, i.e. F realizes $\mathcal{C}_{\mathbb{C}}$. Further, F is of finite type, has finitely many poles, and is quasiconformally conjugate to a hyperbolic exponential map near ∞ .*

Proof. Let $E(z) = \lambda(e^z - 1)$ be the exponential map with an attracting fixed point at 0 of multiplier λ such that the modulus of the annulus $A_{|\lambda|,1}$ equals the modulus of the annulus bounded by γ and $g(\gamma)$.

Let δ be a real-analytic curve around 0 which is contained in the domain where the linearizing map is injective. By definition of E , the annulus bounded by δ and $E(\delta)$ (which is contained in the interior of δ) is also equal to the modulus of $A_{|\lambda|,1}$. Let $\phi : \text{int}(\gamma) \rightarrow \text{ext}(\delta)$ be a conformal map. Then $\phi|_{\gamma}$ is an orientation-reversing real-analytic homeomorphism mapping γ to δ . If we define

$$\phi(z) := E \circ \phi \circ g^{-1}(z)$$

on $g(\gamma)$, then $\phi|_{g(\gamma)}$ is also an orientation-reversing map sending $g(\gamma)$ to $E(\delta)$. Let $p := \phi^{-1}(\infty)$. Our goal is to construct an essential singularity at p by gluing the dynamics of E in the exterior of δ inside of γ (as indicated in Figure 6).

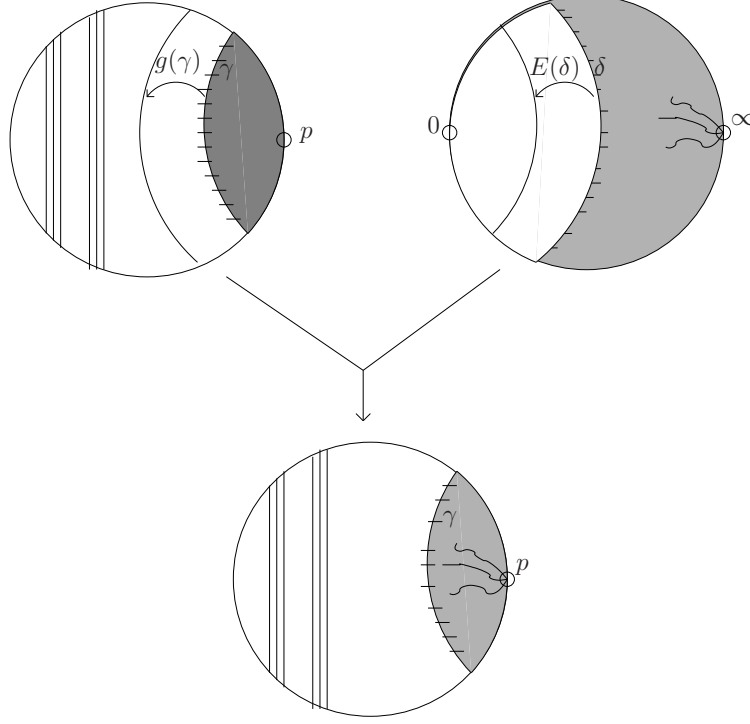


FIGURE 6. Sketch of the construction of an essential singularity. The vertical lines indicate the periodic Herman rings of g

Since γ is contained in the preimage B of some periodic Fatou component, there exists a neighborhood U of γ which is contained in B . Hence $g(U)$ is a neighborhood of $g(\gamma)$.

We want to extend ϕ quasiconformally to all of $\hat{\mathbb{C}}$. Choose a conformal map ψ_1 that maps the annulus A bounded by γ and $g(\gamma)$ to the annulus bounded by δ and $E(\delta)$, sending γ to δ and $g(\gamma)$ to $E(\delta)$. Then modify ψ_1 on $A \cap U$ and on $A \cap g(U)$ quasiconformally such that ψ_1 equals ϕ on γ and $g(\gamma)$. Similarly, let $\psi_2 : \text{ext}(g(\gamma)) \rightarrow \text{int}(E(\delta))$ be a conformal map and modify it quasiconformally on $g(U)$ such that ψ_2 equals ϕ on $g(\gamma)$. Note that the modification of ψ_1 and ψ_2 is possible because both $\phi|_\gamma$ and $\phi|_{g(\gamma)}$ are orientation-reversing. Extend ϕ by setting $\phi := \psi_1$ in A and $\phi := \psi_2$ in $\text{ext}(g(\gamma))$. The resulting map is quasiconformal and holomorphic outside of $U \cup g(U)$.

Now let h be the quasimeromorphic map defined by

$$h(z) = \begin{cases} g(z) & \text{if } z \notin \text{int}(\gamma) \\ \phi^{-1} \circ E \circ \phi(z) & \text{if } z \in \text{int}(\gamma) \text{ and } z \neq p. \end{cases}$$

We want to see that h is quasiconformally conjugate to a transcendental meromorphic function. So we have to show that the iterates h^n are uniformly K -quasimeromorphic for some K . But this is easy to see since the orbit of every point in \mathbb{C} passes at most once through the region where h is not holomorphic, i.e. through $U \cup g(U)$ (by the hypothesis that $\text{int}(\gamma) \cap g^n(\gamma) = \emptyset$ for all n). Hence there is a quasiconformal map $\varphi : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$ mapping p to ∞ such that $F = \varphi \circ h \circ \varphi^{-1}$ is a transcendental meromorphic function.

Near ∞ , the dynamics of F are the same as the dynamics of E .

We now list some properties of the function F . It has one finite asymptotic value, namely $\varphi(\phi^{-1}(-\lambda))$. The other asymptotic value is the essential singularity ∞ . Because E has no critical points, F has at most as many critical values as g . The above properties yield that F is of finite type.

Further, since ∞ is also an omitted value of E , F has at most as many poles as g . The φ -images of the curves that form the configuration $\mathcal{C}_{\hat{\mathbb{C}}}(g)$ define a new configuration $\mathcal{C}'_{\hat{\mathbb{C}}} = (\Gamma', \pi)_{\hat{\mathbb{C}}}$ which is equivalent to $\mathcal{C}_{\hat{\mathbb{C}}}(g)$, and by construction, F realizes $\mathcal{C}'_{\hat{\mathbb{C}}}$. Hence the theorem is proved. \square

Remark. *The condition that g is injective on γ can also be dropped. Indeed, if g is $n : 1$ on γ , we choose $E_n(z) := z \mapsto \lambda z^{n-1}(e^z - 1)$ instead of E and the same argument can be used with obvious modifications.*

5. THE GENERAL CASE - PROOF OF THEOREM D

In this section we show the main theorem of this paper - namely, every configuration of Herman rings that can be realized by a rational function can also be realized by a transcendental meromorphic function (in the sense of section 3). We first recall the statement.

Theorem. *Let g be a rational function with Herman rings. Let $\mathcal{C}'_{\hat{\mathbb{C}}}$ be a configuration with $\mathcal{C}'_{\hat{\mathbb{C}}} \sim \mathcal{C}_{\hat{\mathbb{C}}}(g)$. Then there exists a transcendental meromorphic function F with $\mathcal{C}_{\hat{\mathbb{C}}}(F) \sim \mathcal{C}'_{\hat{\mathbb{C}}}$, i.e. F realizes $\mathcal{C}'_{\hat{\mathbb{C}}}$.*

Proof. Denote the periodic Herman rings of g by A_1, \dots, A_n . Let $\mathcal{C}'_{\hat{\mathbb{C}}} = (\Gamma', \pi')_{\hat{\mathbb{C}}}$ and $\mathcal{C}_{\hat{\mathbb{C}}}(g) = (\Gamma, \pi)_{\hat{\mathbb{C}}}$. By changing simultaneously the orientations of the curves corresponding to one or more cycles that form π' and renumbering the curves in Γ' if necessary, we can assume that $\pi = \pi'$ and that there exists an orientation-preserving homeomorphism α of $\hat{\mathbb{C}}$ such that $\text{int}(\alpha(\gamma_i)) = \text{int}(\gamma'_i)$ for all $i = 1, \dots, n$ (where we assume that $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ and $\Gamma' = \{\gamma'_1, \dots, \gamma'_n\}$).

We consider the point $w := \alpha^{-1}(\infty)$ and choose a non-periodic preimage B of some periodic Herman ring which is contained in the same connected component of $\hat{\mathbb{C}} \setminus \bigcup_{i=1}^n A_i$ as w . The fact that such a component exists is clear because the preimages of every Herman ring accumulate everywhere on the Julia set of g . Now choose an oriented, real-analytic, closed curve $\gamma \subset B$ such that there are no periodic Herman rings of

g in $\text{int}(\gamma)$ and g is injective on γ (for example, let γ be the boundary of a small disc around some non-critical point in B). We have $g^n(\gamma) \cap \text{int}(\gamma) = \emptyset$ for all $n \in \mathbb{N}$ because B is non-periodic.

Let $p \in \text{int}(\gamma)$. We modify α on the component of $\hat{\mathbb{C}} \setminus \bigcup_{i=1}^n A_i$ that contains w in such a way that the resulting map (we call it α as well) satisfies $p = \alpha^{-1}(\infty)$.

Assign $g(\gamma)$ the orientation such that g respects the orientations of γ and $g(\gamma)$. If $\text{int}(\gamma) \subset \text{int}(g(\gamma))$, we can proceed as in the proof of Theorem C. Therefore we now have to deal with the case that $\text{int}(\gamma) \subset \text{ext}(g(\gamma))$ (the other cases cannot occur since $g^n(\gamma) \cap \text{int}(\gamma) = \emptyset$ for all $n \in \mathbb{N}$).

The construction resembles the one done in the previous section, but notice that now we cannot find only one quasiconformal map with which to conjugate, because if we tried to do the construction as in the proof of Theorem C, the homeo that maps $g(\gamma)$ to $E(\delta)$ would now be orientation-preserving, but the homeo that maps γ to δ would be orientation-reversing, so we would not be able to extend these maps to a single quasiconformal homeo of the Riemann sphere. It follows that we have to work with two quasiconformal maps instead.

As before, let $E(z) := \lambda(e^z - 1)$ be an exponential map with $\lambda < 1$, and let δ be a real-analytic curve around 0 which is mapped 1-1 inside of itself under E .

We choose a conformal map ψ_1 mapping the interior of γ to the exterior of δ that sends p to ∞ . Now choose a conformal map ψ_2 mapping the exterior of γ to the interior of δ , sending ∞ to 0, and modify it quasiconformally in a small (one-sided) neighborhood U of γ which is contained in B such that the resulting function (call it ψ_2 again) coincides with ψ_1 on γ . By pasting ψ_1 and ψ_2 , we obtain a quasiconformal map Φ_1 of $\hat{\mathbb{C}}$ sending γ to δ which is conformal outside U .

For $z \in g(\gamma)$, we define

$$\hat{\psi}(z) := E \circ \Phi_1 \circ g^{-1}(z)$$

and extend it in the same way to a quasiconformal self-map Φ_2 of the sphere which maps the interior of $g(\gamma)$ to the exterior of $E(\delta)$ and vice versa, is conformal outside $g(U)$ and sends $g(p)$ to ∞ and ∞ to $-\lambda$.

Now we define

$$h(z) = \begin{cases} g(z) & \text{if } z \notin \text{int}(\gamma) \\ \Phi_2^{-1} \circ E \circ \Phi_1(z) & \text{if } z \in \text{int}(\gamma) \text{ and } z \neq p. \end{cases}$$

By the construction of $\hat{\psi}$ and therefore Φ_2 , h is quasimeromorphic with an essential singularity at p . As before, it is easy to see that the orbit of every point in \mathbb{C} passes through the region of non-holomorphicity of h at most once, hence the iterates h^n are uniformly K -quasimeromorphic for some K . This ensures the existence of a transcendental meromorphic function $F = \varphi \circ h \circ \varphi^{-1}$, where φ is a quasiconformal self-map of $\hat{\mathbb{C}}$ sending p to ∞ .

Note that F does not have more critical points than g , so F has only a finite number of critical values. Unlike in section 4, F now has 2 finite asymptotic values, namely $\varphi(g(p))$ and $\varphi(\Phi_2^{-1}(-\lambda))$.

In fact, the essential singularity ∞ is not an asymptotic value of F - if it were, then there would exist an asymptotic value of E in the interior of $E(\delta)$, which is clearly impossible by the definition and properties of δ .

Further, F has infinitely many poles - the set of poles equals $\varphi(E^{-1}(\Phi_2(p)))$.

Finally, the map $\alpha \circ \varphi^{-1}$ is a homeomorphism of \mathbb{C} satisfying the condition for F to realize the configuration $(\Gamma', \pi)_{\mathbb{C}}$, which finishes the proof of Theorem D. \square

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(N. Fagella) DEPARTAMENT DE MATEMÀTICA APLICADA I ANÀLISI, UNIVERSITAT DE BARCELONA,
08007 BARCELONA, SPAIN

E-mail address: `fagella@maia.ub.es`

(J. Peter) MATHEMATISCHES SEMINAR, CHRISTIAN-ALBRECHTS-UNIVERSITÄT ZU KIEL, 24098
KIEL, GERMANY

E-mail address: `peter@math.uni-kiel.de`