This is a preprint of: "Bifurcation of Limit cycles from a 4-dimensional center in R^m in resonance 1:N", Luis Barreira, Jaume Llibre, Clàudia Valls, *J. Math. Anal. Appl.*, vol. 389, 754–768, 2012.
DOI: [10.1016/j.jmaa.2011.12.018]

BIFURCATION OF LIMIT CYCLES FROM A 4–DIMENSIONAL CENTER IN $\mathbb{R}^m$ IN RESONANCE $1:N$

LUIS BARREIRA, JAUME LLIBRE, AND CLAUDIA VALLS

Abstract. For every positive integer $N \geq 2$ we consider the linear differential center $\dot{x} = Ax$ in $\mathbb{R}^m$ with eigenvalues $\pm i$, $\pm Ni$ and 0 with multiplicity $m-4$. We perturb this linear center inside the class of all polynomial differential systems of the form linear plus a homogeneous nonlinearity of degree N, i.e. $\dot{x} = Ax + \varepsilon F(x)$ where every component of $F(x)$ is a linear polynomial plus a homogeneous polynomial of degree N. When the displacement function of order ε of the perturbed system is not identically zero, we study the maximal number of limit cycles that can bifurcate from the periodic orbits of the linear differential center. In particular, we give explicit upper bounds for the number of limit cycles.

1. Introduction

In the qualitative theory of polynomial differential systems the study of their limit cycles is one of the main topics. We recall that for a differential system a *limit cycle* is a periodic orbit isolated in the set of all its periodic orbits. Two main questions arise in this setting in dimension two: the study of the number of limit cycles depending on the degree of the polynomial (see [10, 11] for details in dimension two), and the study of how many limit cycles emerge from the periodic orbits of a center when we perturb it inside a given class of differential equations (see [8] for details). These problems have been studied intensively in dimension two. Our main aim is to bring this study to higher dimension.

In this paper we study how many limit cycles emerge from the periodic orbits of a center when we perturb it inside a given class of differential equations in dimension higher than two. More precisely given $m \geq 5$ we consider the linear differential center

$$\frac{dx}{dt} = \dot{x} = Ax \tag{1}$$

in $\mathbb{R}^m$, where

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -N & 0 & \cdots & 0 \\ 0 & 0 & N & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

for some positive integer N. We perturb system (1) as follows

$$\dot{x} = Ax + \varepsilon F(x), \tag{2}$$

where ε is a small parameter, and $F\colon \mathbb{R}^m \to \mathbb{R}^m$ is a polynomial of the form $F = (F_1^1 + F_N^1, \ldots, F_1^m + F_N^m)$ with F_1^k and F_N^k arbitrary homogeneous polynomials

---

2000 *Mathematics Subject Classification.* 34C29, 34C25, 47H11.

Key words and phrases. periodic orbit, averaging theory, limit cycles, resonance $1:N$.

of degree 1 and N respectively in the variables $x = (x_1, \dots, x_m)$, with the exception that $F_1^k = \lambda_k x_k$ for $k = 5, \dots, m$.

The main reason for considering only perturbations of system (2) of the form linear plus nonlinear homogeneous polynomials of degree N is that the huge computations for studying the number of limit cycles which can bifurcate from the periodic orbits of system (2) become intractable in other cases. These kind of perturbations have already been considered by many authors for differential equations in the plane, see for instance [3, 4, 5, 6, 7, 12, 16, 18].

For $\varepsilon = 0$ the differential system (2) in $\mathbb{R}^m$ has at the origin a singular point with eigenvalues $\pm i$, $\pm Ni$ and 0 with multiplicity $m - 4$. So in particular this singular point has a 4–dimensional center in resonance $1 : N$. We want to study how many limit cycles can bifurcate from the periodic orbits of this center when we perturb it in $\mathbb{R}^m$ with $m > 4$ inside the class of polynomial vector fields of the form linear plus a homogeneous nonlinearity of degree N. This study is interesting for the following two main reasons:

(i) These last years hundreds of papers studied the limit cycles of planar polynomial differential systems, see the book [8] and the references quoted there. The main reason of these studies is the unsolved 16th Hilbert problem, see [9]. In particular many of theses papers studied the limit cycles bifurcating from the periodic orbits of a linear center. On the other hand we note that very few papers have been dedicated to study the perturbation of the periodic orbits of a linear differential systems in $\mathbb{R}^m$ with $m > 2$ inside the class of polynomial differential systems of a given degree in $\mathbb{R}^m$. This is one of mains objectives of this paper.

(ii) If the bifurcated periodic orbits tend to the origin, then these periodic orbits come in fact from a Hopf bifurcation of the origin. In such a situation our study is interesting because we are given information about the periodic orbits that can bifurcate by Hopf from a doubly degenerate singular point. First, it is degenerate because the eigenvalues $\pm i$ and $\pm Ni$, with $N \geq 2$ a positive integer, are in resonant; and second, the remainder eigenvalues are zero.

In order to formulate our main result we need to consider a non-degeneracy condition formulated in terms of the so-called displacement function of order ε (see (5)). This is a somewhat technical assumption and thus we shall leave its description to section 2. Generically the first order part of the displacement function is not zero, but when this occurs we must study the zeros of the n–th order part of the displacement function if $n > 1$ is the smallest n for which the n–th order part of the displacement function is not identically zero, see for more details [1, 13].

Theorem 1. *Assume that $N \geq 2$, $m \geq 5$, and that for $\varepsilon \neq 0$ sufficiently small the displacement function of order ε is not identically zero. Then the maximum number of limit cycles of the differential system* (2) *bifurcating from the periodic orbits of the 4–dimensional linear differential center* (1) *provided by the displacement function of first order in ε is at most*

(a) $2^m + 2^{m-1}3^2 + 2^{m-2}3^{m-4}5$ *if $N = 2$, and*
(b) $2N^{m-2}(N + 1)^2 + 2N(N + 3)(N + 4)^{m-4}$ *if $N > 2$.*

Theorem 1 is proved in section 4 using the averaging theory described in section 2. **It** improves and extends previous results of system (2) restricted to $\mathbb{R}^4$, see [2] and [14].

Strictly speaking the techniques of this paper are essentially not new because they where used previously in the papers [2] and [14], but there were applied to differential systems in $\mathbb{R}^4$ such that when $\varepsilon = 0$ the unperturbed linear differential systems have nonzero eigenvalues. The fact that now we allow the existence of zero eigenvalues forces to adapt and modify some previous technicalities, mainly in the changes of variables for writing the initial differential system in the normal form for applying the averaging method.

More important than the result of Theorem 1 is the computation of the averaged system associated to the differential system (2), because its singular points with Jacobian nonzero provide the limit cycles of the differential system (2) when the displacement function of order ε is not identically zero. When N and m are relatively small all the computations for arriving to the averaged system can be made explicitly, and consequently the upper bound for the number of limit cycles given in Theorem 1 can be improved. Thus we have the following result.

Theorem 2. *Assume that for $\varepsilon \neq 0$ sufficiently small the displacement function of order ε is not identically zero. Then the maximum number of limit cycles of the differential system* (2) *bifurcating from the periodic orbits of the* 4*–dimensional linear differential center* (1) *provided by the displacement function of first order in ε is at most*

(a) 20 *if $N = 2$ and $m = 5$, and*
(b) 46 *if $N = 3$ and $m = 5$.*

Theorem 2 is proved in section 5.

We note that in order to obtain the general (non-sharp) bounds in Theorem 1 we use the Bézout Theorem, while for $N = 2, 3$ and $m = 5$ one can make explicit calculations, thus allowing the improvement of the general bounds in these particular cases. Indeed, in Theorem 1 the upper bounds are 296 and 1116, for $N = 2$ and $m = 5$, and for $N = 3$ and $m = 5$, respectively.

2. First–order averaging theory

The aim of this section is to present the first-order averaging method obtained in [1]. We first briefly recall the basic elements of averaging theory. Roughly speaking, the method gives a quantitative relation between the solutions of a nonautonomous periodic system and the solutions of its averaged system, which is autonomous. The following theorem provides a first order approximation for periodic solutions of the original system.

We consider the differential system

$$\dot{x}(t) = \varepsilon H(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{3}$$

where $H\colon \mathbb{R} \times D \to \mathbb{R}^n$ and $R\colon \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. We define $h\colon D \to \mathbb{R}^n$ by

$$h(z) = \int_0^T H(s, z)\, ds, \tag{4}$$

and denote by $d_B(h, V, a)$ the Brouwer degree of h at some neighborhood V of a (see [15] for the definition).

Theorem 3. *We assume that*

(i) *H and R are locally Lipschitz with respect to x;*
(ii) *for $a \in D$ with $h(a) = 0$, there exists a neighborhood V of a such that $h(z) \neq 0$ for all $z \in V \setminus \{a\}$ and $d_B(h, V, a) \neq 0$.*

Then for $\varepsilon \neq 0$ sufficiently small there exists an isolated T-periodic solution $\phi(\cdot, \varepsilon)$ of system (3) *such that $\phi(a, 0) = a$.*

The system

$$\dot{x} = \varepsilon h(x),$$

is called the *averaged system* associated to system (3).

Hypothesis (i) ensures the existence and uniqueness of the solution of each initial value problem on the interval $[0, T]$. Hence, for each $z \in D$, it is possible to denote by $x(\cdot, z, \varepsilon)$ the solution of system (3) with the initial value $x(0, z, \varepsilon) = z$. We also consider the function $\zeta\colon D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ defined by

$$\zeta(z, \varepsilon) = \int_0^T \big(\varepsilon H(t, x(t, z, \varepsilon)) + \varepsilon^2 R(t, x(t, z, \varepsilon), \varepsilon)\big)\, dt. \tag{5}$$

This is called the *displacement function of order* ε. It follows from the proof of Theorem 3 that for every $z \in D$ the following relations hold:

$$x(T, z, \varepsilon) - x(0, z, \varepsilon) = \zeta(z, \varepsilon), \quad \text{and} \quad \zeta(z, \varepsilon) = \varepsilon h(z) + O(\varepsilon^2),$$

where h is given by (4) and where the symbol $O(\varepsilon^2)$ denotes a function bounded on every compact subset of $D \times (-\varepsilon_0, \varepsilon_0)$ multiplied by ε^2.

We note that in order to see that $d_B(h, V, a) \neq 0$ it is sufficient to check that the Jacobian of $D_z h(z)$ at $z = a$ is not zero, see for more details [15].

3. Averaged system

Writing

$$F_1 = (F_1^1, F_1^2, F_1^3, F_1^4, 0, \ldots, 0), \quad F_N = (F_N^1, F_N^2, F_N^3, F_N^4, F_N^5, \ldots, F_N^m),$$

system (2) becomes

$$\begin{aligned}
\dot{x}_1 &= -x_2 + \varepsilon(F_1^1(x) + F_N^1(x)),\\
\dot{x}_2 &= x_1 + \varepsilon(F_1^2(x) + F_N^2(x)),\\
\dot{x}_3 &= -N x_4 + \varepsilon(F_1^3(x) + F_N^3(x)),\\
\dot{x}_4 &= N x_3 + \varepsilon(F_1^4(x) + F_N^4(x)),\\
\dot{x}_k &= \varepsilon(\lambda_k x_k + F_N^k(x)), \quad k = 5, \ldots, m.
\end{aligned} \tag{6}$$

Lemma 4. *Doing the change of variables from $(x_1, x_2, x_3, x_4, x_5, \ldots, x_m)$ to the new variables $(\theta, r, \rho, s, y_5, \ldots, y_m)$ given by*

$$x_1 = r\cos\theta,\ x_2 = r\sin\theta,\ x_3 = \rho\cos(N(\theta + s)),\ x_4 = \rho\sin(N(\theta + s)),\ x_k = y_k,$$

for $k = 5, \ldots, m$, and taking θ as the new independent variable, system (6) is transformed into the system

$$\begin{aligned}
\frac{dr}{d\theta} &= \varepsilon H_1(\theta, r, \rho, s, y_5, \ldots, y_m) + O(\varepsilon^2),\\
\frac{d\rho}{d\theta} &= \varepsilon H_2(\theta, r, \rho, s, y_5, \ldots, y_m) + O(\varepsilon^2),\\
\frac{ds}{d\theta} &= \varepsilon H_3(\theta, r, \rho, s, y_5, \ldots, y_m) + O(\varepsilon^2),\\
\frac{dy_k}{d\theta} &= \varepsilon H_k(\theta, r, \rho, s, y_5, \ldots, y_m) + O(\varepsilon^2), \quad k = 5, \ldots, m,
\end{aligned} \tag{7}$$

where

$$\begin{aligned}
H_1 &= (F_1^1 + F_N^1)\cos\theta + (F_1^2 + F_N^2)\sin\theta,\\
H_2 &= (F_1^3 + F_N^3)\cos(N(\theta+s)) + (F_1^4 + F_N^4)\sin(N(\theta+s)),\\
H_3 &= \frac{1}{N\rho}\big((F_1^4 + F_N^4)\cos(N(\theta+s)) - (F_1^3 + F_N^3)\sin(N(\theta+s))\big)\\
&\quad - \frac{1}{r}\big((F_1^2 + F_N^2)\cos\theta - (F_1^1 + F_N^1)\sin\theta\big),\\
H_k &= \lambda_k y_k + F_N^k.
\end{aligned}$$

Proof. In the variables $(\theta, r, \rho, s, y_5, \ldots, y_m)$ system (6) becomes

$$\begin{aligned}
\dot\theta &= 1 + \frac{\varepsilon}{r}\big(\cos\theta(F_1^2 + F_N^2) - \sin\theta(F_1^1 + F_N^1)\big),\\
\dot r &= \varepsilon H_1(\theta, r, \rho, s, y_5, \ldots, y_m),\\
\dot\rho &= \varepsilon H_2(\theta, r, \rho, s, y_5, \ldots, y_m),\\
\dot s &= \varepsilon H_3(\theta, r, \rho, s, y_5, \ldots, y_m),\\
\dot y_k &= \varepsilon H_k(\theta, r, \rho, s, y_5, \ldots, y_m), \qquad k = 5, \ldots, m.
\end{aligned} \tag{8}$$

For ε sufficiently small, $\dot\theta(t) > 0$ for each $(t, (\theta, r, \rho, s, y_5, \ldots, y_m)) \in \mathbb{R}\times D$. Now we eliminate the variable t in the above system by considering θ as the new independent variable. It is clear that the right-hand side of the new system is well defined and continuous in $\mathbb{R}\times D\times(-\varepsilon_0, \varepsilon_0)$, 2π-periodic with respect to the independent variable θ, and locally Lipschitz with respect to $(r, \rho, s, y_5, \ldots, y_m)$. From (8) equation (7) is obtained after an expansion with respect to the small parameter ε. □

We recall a technical result from [2] that we shall use later on.

Lemma 5. *Let N be a nonnegative integer, and let α and β be real numbers. Given nonnegative integers i, j, k, l, there exist constants c_{uv} and d_{uv} such that*

$$\cos^i\alpha\sin^j\alpha\cos^k\beta\sin^l\beta$$

is equal to

$$\sum_{u=0}^{[(i+j)/2]}\sum_{v=0}^{[(k+l)/2]} c_{uv}\cos\big((i+j-2u)\alpha \pm (k+l-2v)\beta\big)$$

if $j + l$ is even, and is equal to

$$\sum_{u=0}^{[(i+j)/2]}\sum_{v=0}^{[(k+l)/2]} d_{uv}\sin\big((i+j-2u)\alpha \pm (k+l-2v)\beta\big)$$

if $j + l$ is odd. Here $[x]$ denotes the integer part function of $x \in \mathbb{R}$.

Now we compute the functions $h_j(r, \rho, s, y_5, \ldots, y_m)$ for $j = 1, \ldots, m$ of system (7) given in (4). We write

$$F_1^g = \sum_{j=1}^m a_j^g x_j \quad\text{and}\quad F_N^g = \sum_{i_1+i_2+\cdots+i_m=N} a^g_{i_1\cdots i_m} x_1^{i_1}x_2^{i_2}\cdots x_m^{i_m},$$

for $g = 1, \ldots, m$. We also write

$$h_j(r, \rho, s, y_5, \ldots, y_m) = \int_0^{2\pi} H_j(\theta, r, \rho, s, y_5, \ldots, y_m)\,d\theta$$

for $j = 1, 2, 3, 5, \ldots, m$.

Proposition 6. *We have*

$$
\begin{aligned}
h_1(r,\rho,s,y_5,\dots,y_m) &= a_1 r + r^{N-1}\rho(b_1\sin(Ns)+c_1\cos(Ns))\\
&\quad + \sum_{2l+i_5+\cdots+i_m=0}^{N} d^1_{li_5\cdots i_m} r^{N-2l-i_5-\cdots-i_m}\rho^{2l} y_5^{i_5}\cdots y_m^{i_m},
\end{aligned}
$$

for some constants a_1, b_1, c_1 *and* $d^1_{li_5\cdots i_m}$ *depending on the coefficients of the perturbation.*

Proof. We write the function H_1 as

$$H_1 = H_1^1 + H_1^N = (F_1^1\cos\theta + F_1^2\sin\theta) + (F_N^1\cos\theta + F_N^2\sin\theta).$$

Then

$$
\begin{aligned}
h_1^1(r,s,\rho,y_5,\dots,y_m) &= \int_0^{2\pi} H_1^1(\theta,r,s,\rho,y_5,\dots,y_m)\,d\theta\\
&= \sum_{j=1}^m \int_0^{2\pi} \big(a_j^1\cos\theta + a_j^2\sin\theta\big)x_j\,d\theta = \pi(a_1^1+a_2^2)r,
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
h_1^N(r,s,\rho,y_5,\dots,y_m) &= \int_0^{2\pi} H_1^N(\theta,r,s,\rho,y_5,\dots,y_m)\,d\theta\\
&= \sum_{i_1+\cdots+i_m=N}\int_0^{2\pi}\big(a^1_{i_1\cdots i_m}x_1^{i_1}\cdots x_m^{i_m}\cos\theta + a^2_{i_1\cdots i_m}x_1^{i_1}\cdots x_m^{i_m}\sin\theta\big)\,d\theta\\
&= \sum_{i_1+\cdots+i_m=N}\int_0^{2\pi} a^1_{i_1\cdots i_m} r^{i_1+i_2}\rho^{i_3+i_4}\cos^{i_1+1}\theta\sin^{i_2}\theta\\
&\quad\cdot\cos^{i_3}(N(\theta+s))\sin^{i_4}(N(\theta+s))y_5^{i_5}\cdots y_m^{i_m}\,d\theta\\
&\quad+\sum_{i_1+\cdots+i_m=N}\int_0^{2\pi} a^2_{i_1\cdots i_m} r^{i_1+i_2}\rho^{i_3+i_4}\cos^{i_1}\theta\sin^{i_2+1}\theta\\
&\quad\cdot\cos^{i_3}(N(\theta+s))\sin^{i_4}(N(\theta+s))y_5^{i_5}\cdots y_m^{i_m}\,d\theta.
\end{aligned}
$$

By Lemma 5 we obtain

$$
\begin{aligned}
h_1^N(r,s,\rho,y_5,\dots,y_m) &= \sum_{i_1+\cdots+i_m=N} r^{i_1+i_2}\rho^{i_3+i_4}y_5^{i_5}\cdots y_m^{i_m}\\
&\quad\cdot\int_0^{2\pi}\sum_{u=0}^{[(i_1+i_2+1)/2]}\sum_{v=0}^{[(i_3+i_4)/2]} C_{uv}^{i_1\cdots i_m}(\theta)\,d\theta,
\end{aligned}
$$

where

$$
\begin{aligned}
C_{uv}^{i_1\cdots i_m} &= c_{uv}^{i_1\cdots i_m}\cos\big((i_1+i_2+1-2u)\theta \pm (i_3+i_4-2v)N(\theta+s)\big)\\
&\quad + d_{uv}^{i_1\cdots i_m}\sin\big((i_1+i_2+1-2u)\theta \pm (i_3+i_4-2v)N(\theta+s)\big),
\end{aligned}
$$

for some constants $c_{uv}^{i_1\cdots i_m}$ and $d_{uv}^{i_1\cdots i_m}$. Therefore all the integrals with respect to θ are zero except possibly when

$$i_1+i_2+1-2u = N(i_3+i_4-2v). \tag{10}$$

We observe that $0 \le i_1+i_2+1-2u \le N+1$. So there are only two possibilities: either $i_3+i_4-2v=1$ or $i_3+i_4-2v=0$.

If $i_3+i_4-2v=1$, then it follows from (10) that

$$i_5+\cdots+i_m = N-(i_1+i_2+i_3+i_4) = -2(u+v).$$

Therefore $u = v = 0 = i_5 = \cdots = i_m = 0$, and hence, $i_1+i_2 = N-1$ and $i_3+i_4 = 1$. This yields the term

$$r^{N-1}\rho\big(b_1 \sin(Ns) + c_1(\cos Ns)\big). \tag{11}$$

If $i_3 + i_4 - 2v = 0$, then $2v + i_5 + \cdots + i_m = N - i_1 - i_2$, and $2v + i_5 + \cdots + i_m$ runs from 0 to N. This yields the terms

$$\sum_{2v+i_5+\cdots+i_m=0}^{N} d^1_{vi_5\cdots i_m} r^{N-2v-i_5-\cdots-i_m} \rho^{2v} y_5^{i_5} \cdots y_m^{i_m}. \tag{12}$$

The proposition follows adding the terms from (9), (11) and (12). □

Proposition 7. *We have*

$$h_2(r, \rho, s, y_5, \ldots, y_m) = a_2\rho + r^N (b_2 \sin(Ns) + c_2 \cos(Ns)) + \sum_{2v+i_5+\cdots+i_m=1}^{N+1} d^2_{vi_5\cdots i_m} r^{N+1-2v-i_5-\cdots-i_m} \rho^{2v-1} y_5^{i_5} \cdots y_m^{i_m},$$

for some constants a_2, b_2, c_2 and $d^2_{vi_5\cdots i_m}$ depending on the coefficients of the perturbation.

Proof. As in Proposition 6 we write the function H_2 as

$$\begin{aligned} H_2 = H_2^1 + H_2^N = &\big(F_1^3 \cos(N(\theta+s)) + F_1^4 \sin(N(\theta+s)\big) \\ &+ \big(F_N^3 \cos(N(\theta+s)) + F_N^4 \sin(N(\theta+s)\big) \end{aligned}$$

Then

$$\begin{aligned} h_2^1(r, s, \rho, y_5, \ldots, y_m) &= \int_0^{2\pi} H_2^1(\theta, r, s, \rho, y_5, \ldots, y_m)\, d\theta \\ &= \sum_{j=1}^{m} \int_0^{2\pi} \big(a_j^3 \cos(N(\theta+s)) + a_j^4 \sin(N(\theta+s))\big) x_j \, d\theta \\ &= \pi(a_3^3 + a_4^4)\rho, \end{aligned} \tag{13}$$

and using Lemma 5 we obtain

$$\begin{aligned} h_2^N(r, s, \rho, y_5, \ldots, y_m) &= \int_0^{2\pi} H_2^N(\theta, r, s, \rho, y_5, \ldots, y_m)\, d\theta \\ &= \sum_{i_1+\cdots+i_m=N} \int_0^{2\pi} a^3_{i_1\cdots i_m} r^{i_1+i_2} \rho^{i_3+i_4} \cos^{i_1}\theta \sin^{i_2}\theta \\ &\quad \cdot \cos^{i_3+1}(N(\theta+s)) \sin^{i_4}(N(\theta+s)) y_5^{i_5} \cdots y_m^{i_m}\, d\theta \\ &\quad + \sum_{i_1+\cdots+i_m=N} \int_0^{2\pi} a^4_{i_1\cdots i_m} r^{i_1+i_2} \rho^{i_3+i_4} \cos^{i_1}\theta \sin^{i_2}\theta \\ &\quad \cdot \cos^{i_3}(N(\theta+s)) \sin^{i_4+1}(N(\theta+s)) y_5^{i_5} \cdots y_m^{i_m}\, d\theta \\ &= \sum_{i_1+\cdots+i_m=N} r^{i_1+i_2} \rho^{i_3+i_4} y_5^{i_5} \cdots y_m^{i_m} \\ &\quad \cdot \int_0^{2\pi} \sum_{u=0}^{[(i_1+i_2)/2]} \sum_{v=0}^{[(i_3+i_4+1)/2]} D_{uv}^{i_1\cdots i_m}(\theta)\, d\theta, \end{aligned}$$

where

$$\begin{aligned} D_{uv}^{i_1\cdots i_m} = &c_{uv}^{i_1\cdots i_m} \cos\big((i_1+i_2-2u)\theta \pm (i_3+i_4+1-2v)N(\theta+s)\big) \\ &+ d_{uv}^{i_1\cdots i_m} \sin\big((i_1+i_2-2u)\theta \pm (i_3+i_4+1-2v)N(\theta+s)\big), \end{aligned}$$

for some constants $c_{uv}^{i_1\cdots i_m}$ and $d_{uv}^{i_1\cdots i_m}$. All the integrals with respect to θ are zero except possibly when

$$i_1 + i_2 - 2u = N(i_3 + i_4 + 1 - 2v). \tag{14}$$

We observe that $0 \le i_1 + i_2 - 2u \le N$. So there are only two possibilities: either $i_3 + i_4 + 1 - 2v = 1$ or $i_3 + i_4 + 1 - 2v = 0$.

If $i_3 + i_4 + 1 - 2v = 1$, then by (14) we obtain that

$$N - i_3 - i_4 - i_5 - \cdots - i_m - 2u = N,$$

and hence $i_3 + i_4 + i_5 + \cdots + i_m + 2u = 0$. This implies that $i_3 = i_4 = \cdots = i_m = 0$ and $u = 0$. Then $i_1 + i_2 = N$, which yields the term

$$r^N \big(b_2 \sin(Ns) + c_2 \cos(Ns)\big). \tag{15}$$

If $i_3 + i_4 + 1 - 2v = 0$, then

$$2v + i_5 + \cdots + i_m - 1 = N - i_1 - i_2.$$

Thus $2v + i_5 + \cdots + i_m$ runs from 1 to $N + 1$, yielding the terms

$$\sum_{2v+i_5+\cdots+i_m=1}^{N+1} d^2_{vi_5\cdots i_m} r^{N+1-2v-i_5-\cdots-i_m} \rho^{2v-1} y_5^{i_5} \cdots y_m^{i_m}. \tag{16}$$

The proposition follows adding the terms of (13), (15) and (16). □

Proposition 8. *We have*

$$\begin{aligned} h_3(r, \rho, s, y_5, \ldots, y_m) = {} & a_3 + r^{N-2}\rho(b_3 \sin(Ns) + c_3 \cos(Ns)) \\ & + r^N \rho^{-1} \big(d_3 \sin(Ns) + e_3 \cos(Ns)\big) \\ & + \sum_{2v+i_5+\cdots+i_m=0}^{N} d^3_{vi_5\cdots i_m} r^{N-1-2v-i_5-\cdots-i_m} \rho^{2v} y_5^{i_5} \cdots y_m^{i_m} \\ & + \sum_{2v+i_5+\cdots+i_m=1}^{N+1} d^4_{vi_5\cdots i_m} r^{N+1-2v-i_5-\cdots-i_m} \rho^{2v-2} y_5^{i_5} \cdots y_m^{i_m}, \end{aligned}$$

for some constants a_3, b_3, c_3, d_3, e_3, $d^3_{vi_5\cdots i_m}$ and $d^4_{vi_5\cdots i_m}$ depending on the coefficients of the perturbation.

Proof. We have $H_3 = H_3^1 + H_3^N$ where

$$\begin{aligned} H_3^1 &= \frac{1}{N\rho}\Big(F_1^4 \cos\big(N(\theta+s)\big) - F_1^3 \sin\big(N(\theta+s)\big)\Big) - \frac{1}{r}\Big(F_1^2 \cos\theta - F_1^1 \sin\theta\Big), \\ H_3^N &= \frac{1}{N\rho}\Big(F_N^4 \cos\big(N(\theta+s)\big) - F_N^3 \sin\big(N(\theta+s)\big)\Big) - \frac{1}{r}\Big(F_N^2 \cos\theta - F_N^1 \sin\theta\Big). \end{aligned}$$

Proceeding in a similar manner to the proofs of Propositions 6 and 7 we get

$$\begin{aligned} h_3^1(r, \rho, s, y_5, \cdots, y_m) &= \int_0^{2\pi} H_3^1(\theta, r, \rho, s, y_5, \ldots, y_m)\, d\theta \\ &= \frac{\pi(a_3^4 - a_4^3)}{N} - \pi(a_1^2 - a_1^1). \end{aligned} \tag{17}$$

Now we calculate

$$h_3^N(r, \rho, s, y_5, \cdots, y_m) = \int_0^{2\pi} H_3^N(\theta, r, \rho, s, y_5, \ldots, y_m)\, d\theta.$$

In a similar manner to the proofs of Propositions 6 and 7 we get

$$
\begin{aligned}
h_3^N(r,\rho,s,y_5,\cdots,y_m) = \frac{1}{N} & \sum_{i_1+\cdots+i_m=N} r^{i_1+i_2}\rho^{i_3+i_4-1}y_5^{i_5}\cdots y_m^{i_m} \\
& \cdot \int_0^{2\pi} \sum_{u=0}^{[(i_1+i_2)/2]} \sum_{v=0}^{[(i_3+i_4+1)/2]} E_{uv}^{i_1\cdots i_m}(\theta)\,d\theta \\
& - \sum_{i_1+\cdots+i_m=N} r^{i_1+i_2-1}\rho^{i_3+i_4}y_5^{i_5}\cdots y_m^{i_m} \\
& \cdot \int_0^{2\pi} \sum_{u=0}^{[(i_1+i_2+1)/2]} \sum_{v=0}^{[(i_3+i_4)/2]} F_{uv}^{i_1\cdots i_m}(\theta)\,d\theta
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
E_{uv}^{i_1\cdots i_m} &= c_{uv}^{i_1\cdots i_m}\cos\big((i_1+i_2-2u)\theta \pm (i_3+i_4+1-2v)N(\theta+s)\big) \\
&\quad + d_{uv}^{i_1\cdots i_m}\sin\big((i_1+i_2-2u)\theta \pm (i_3+i_4+1-2v)N(\theta+s)\big),
\end{aligned} \tag{19}
$$

and

$$
\begin{aligned}
F_{uv}^{i_1\cdots i_m} &= f_{uv}^{i_1\cdots i_m}\cos\big((i_1+i_2+1-2u)\theta \pm (i_3+i_4-2v)N(\theta+s)\big) \\
&\quad + g_{uv}^{i_1\cdots i_m}\sin\big((i_1+i_2+1-2u)\theta \pm (i_3+i_4-2v)N(\theta+s)\big).
\end{aligned} \tag{20}
$$

The terms whose integrals need not be zero satisfy

$$i_1+i_2-2u = N(i_3+i_4+1-2v)$$

in equation (19), and

$$i_1+i_2+1-2u = N(i_3+i_4-2v)$$

in equation (20).

The arguments in the proof of Proposition 7 show that in (18) the terms that may remain in the first sum are

$$
\begin{aligned}
& r^N\rho^{-1}\big(d_3\sin(Ns)+e_3\cos(Ns)\big) \\
& + \sum_{2v+i_5+\cdots+i_m=1}^{N+1} d_{vi_5\cdots i_m}^4 r^{N+1-2v-i_5-\cdots-i_m}\rho^{2v-2}y_5^{i_5}\cdots y_m^{i_m},
\end{aligned} \tag{21}
$$

and the arguments in the proof of Proposition 6 show that the terms that may remain in the second sum are

$$
\begin{aligned}
& r^{N-2}\rho\big(b_3\sin(Ns)+c_3\cos(Ns)\big) \\
& + \sum_{2v+i_5+\cdots+i_m=0}^{N} d_{vi_5\cdots i_m}^3 r^{N-1-2v-i_5-\cdots-i_m}\rho^{2v}y_5^{i_5}\cdots y_m^{i_m}.
\end{aligned} \tag{22}
$$

The proposition follows adding the terms in (17), (21) and (22). □

Proposition 9. *For $k=5,\ldots,m$, we have*

$$h_k(r,\rho,s,y_5,\ldots,y_m) = \lambda_k y_k + \sum_{2v+i_5+\cdots+i_m=0}^{N} d_{vi_5\cdots i_m}^5 r^{N-2v-i_5-\cdots-i_m}\rho^{2v}y_5^{i_5}\cdots y_m^{i_m},$$

for some constants $d_{vi_5\cdots i_m}^5$ depending on the coefficients of the perturbation.

Proof. As in the former proofs, we write $H_k = H_k^1 + H_k^N$ where $H_k^1 = \lambda_k y_k$ and $H_k^N = F_N^k$, and we compute the function

$$h_k^N(r,s,\rho,y_5,\ldots,y_m) = \int_0^{2\pi} H_k^N(\theta,r,s,\rho,y_5,\ldots,y_m)\,d\theta.$$

Proceeding as in the proofs of Propositions 6 or 7 we obtain

$$\begin{aligned} h_k^N(r,\rho,s,y_5,\cdots,y_m) &= \sum_{i_1+\cdots+i_m=N} \int_0^{2\pi} a^k_{i_1\cdots i_m} r^{i_1+i_2}\rho^{i_3+i_4}\cos^{i_1}\theta\sin^{i_2}\theta \\ &\qquad \cdot\cos^{i_3}(N(\theta+s))\sin^{i_4}(N(\theta+s))y_5^{i_5}\cdots y_m^{i_m}\,d\theta \\ &= \sum_{i_1+\cdots+i_m=N} r^{i_1+i_2}\rho^{i_3+i_4}y_5^{i_5}\cdots y_m^{i_m} \\ &\qquad \cdot\int_0^{2\pi}\sum_{u=0}^{[(i_1+i_2)/2]}\sum_{v=0}^{[(i_3+i_4)/2]} G_{uv}^{i_1\cdots i_m}(\theta)\,d\theta, \end{aligned}$$

where

$$\begin{aligned} G_{uv}^{i_1\cdots i_m} &= g_{uv}^{i_1\cdots i_m}\cos\big((i_1+i_2-2u)\theta\pm(i_3+i_4-2v)N(\theta+s)\big) \\ &\quad + h_{uv}^{i_1\cdots i_m}\sin\big((i_1+i_2-2u)\theta\pm(i_3+i_4-2v)N(\theta+s)\big). \end{aligned}$$

All the integrals with respect to θ are zero except possibly when

$$i_1+i_2-2u = N(i_3+i_4-2v). \tag{23}$$

Again we observe that $0\le i_1+i_2-2u\le N$. So there are only two possibilities: either $i_3+i_4-2v=1$ or $i_3+i_4-2v=0$.

If $i_3+i_4-2v=1$, then by (23) we obtain

$$N-i_3-i_4-i_5-\cdots-i_m-2u=N,$$

and thus $i_3=i_4=\cdots=i_m=0$, which contradicts to the fact that $i_3+i_4-2v=1$. Therefore, this case does not occur.

If $i_3+i_4-2v=0$, then

$$2v+i_5+\cdots+i_m=N-i_1-i_2.$$

Hence $2v+i_5+\cdots+i_m$ runs from 0 to N, and we obtain the terms

$$\sum_{2v+i_5+\cdots+i_m=0}^{N} d^5_{vi_5\cdots i_m} r^{N-2v-i_5-\cdots-i_m}\rho^{2v}y_5^{i_5}\cdots y_m^{i_m}.$$

This yields the desired statement. □

4. Proof of Theorem 1

We recall a technical result proved in [2].

Lemma 10. *If N, α and β are nonnegative integers with $\alpha+\beta=N$, then*

$$\int_0^{2\pi}\cos^\alpha\theta\sin^\beta\theta\cos(N(\theta+s))\,d\theta = \begin{cases} \dfrac{(-1)^{\beta/2}\pi}{2^{N-1}}\cos(Ns) & \text{if } \beta \text{ is even,} \\ \dfrac{(-1)^{(\beta+1)/2}\pi}{2^{N-1}}\sin(Ns) & \text{if } \beta \text{ is odd,} \end{cases}$$

and

$$\int_0^{2\pi}\cos^\alpha\theta\sin^\beta\theta\sin(N(\theta+s))\,d\theta = \begin{cases} \dfrac{(-1)^{\beta/2}\pi}{2^{N-1}}\sin(Ns) & \text{if } \beta \text{ is even,} \\ -\dfrac{(-1)^{(\beta+1)/2}\pi}{2^{N-1}}\cos(Ns) & \text{if } \beta \text{ is odd.} \end{cases}$$

We will use the following proposition.

Proposition 11. *The function* $h_3(r,\rho,s,y_5,\dots,y_m)$ *is given by*

$$
\begin{aligned}
h_3(r,\rho,s,y_5,\dots,y_m) &= a_3 + r^{N-2}\rho\big(-c_1\sin(Ns)+b_1\cos(Ns)\big)\\
&\quad + \frac{1}{N} r^N\rho^{-1}\big(-c_2\sin(Ns)+b_2\cos(Ns)\big)\\
&\quad + \sum_{2v+i_5+\cdots+i_m=0}^{N} d^3_{vi_5\cdots i_m} r^{N-1-2v-i_5-\cdots-i_m}\rho^{2v}y_5^{i_5}\cdots y_m^{i_m}\\
&\quad + \sum_{2v+i_5+\cdots+i_m=1}^{N+1} d^4_{vi_5\cdots i_m} r^{N+1-2v-i_5-\cdots-i_m}\rho^{2v-2}y_5^{i_5}\cdots y_m^{i_m},
\end{aligned}
$$

where b_1, c_1 *are the constants in Proposition 6, and* b_2, c_2 *are the constants in Proposition 7.*

Proof. Using the notation of Proposition 8 we shall prove that $b_3=-c_1$, $c_3=b_1$, $d_3=-c_2/N$ and $e_3=b_2/N$. In order to simplify the proof, let $a^1_{i_1 i_2\cdots i_m}x_1^{i_1}x_2^{i_2}\cdots x_m^{i_m}$ be a monomial in F^1_N such that $i_1+i_2=N-1$, $i_3=0$, $i_4=1$ and $i_5=\cdots=i_m=0$. When we compute h_1 and h_3, this monomial appears in h_1 as

$$
\int_0^{2\pi} a^1_{i_1\cdots i_m}\cos^{i_1+1}\theta\sin^{i_2}\theta\sin(N(\theta+s))\,d\theta, \tag{24}
$$

and in h_3 as

$$
\int_0^{2\pi} a^1_{i_1\cdots i_m}\cos^{i_1}\theta\sin^{i_2+1}\theta\sin(N(\theta+s))\,d\theta. \tag{25}
$$

By Lemma 10, the term in (24) is equal to

$$
\begin{cases}
\dfrac{(-1)^{i_2/2}}{2^{N-1}}a^1_{i_1\cdots i_m}\sin(Ns), & \text{if } i_2 \text{ is even},\\[2mm]
-\dfrac{(-1)^{(i_2+1)/2}}{2^{N-1}}a^1_{i_1\cdots i_m}\cos(Ns), & \text{if } i_2 \text{ is odd},
\end{cases}
$$

and the term in (25) is equal to

$$
\begin{cases}
\dfrac{(-1)^{(i_2+1)/2}}{2^{N-1}}a^1_{i_1\cdots i_m}\sin(Ns), & \text{if } i_2+1 \text{ is even},\\[2mm]
\dfrac{(-1)^{i_2/2}}{2^{N-1}}a^1_{i_1\cdots i_m}\cos(Ns), & \text{if } i_2+1 \text{ is odd}.
\end{cases}
$$

For i_2 odd the coefficient of the monomial appears in a sum determining the coefficient of $r^{N-1}\rho\cos(Ns)$ in h_1, and also appears in a sum determining the coefficient of $r^{N-2}\rho\sin(Ns)$ in h_3 with the opposite sign. In a similar way for i_2 even the coefficient of the monomial appears in a sum determining the coefficient of $r^{N-1}\rho\sin(Ns)$ in h_1, and appears in a sum determining the coefficient of $r^{N-2}\rho\cos(Ns)$ in h_3 with the same sign.

We can do the same for all monomials in F^2_N, F^3_N and F^4_N, and thus we conclude that $b_3=-c_1$, $c_3=b_1$, $d_3=-c_2/N$ and $e_3=b_2/N$. □

Now we have all the ingredients to prove Theorem 1.

Proof of Theorem 1. It follows from Propositions 6, 7, 8, 9 and 11 that

$$
\begin{aligned}
h_1 = &\quad a_1 r + r^{N-1}\rho\big(b_1\sin(Ns) + c_1\cos(Ns)\big) \\
&+ \sum_{2v+i_5+\cdots+i_m=0}^{N} d^1_{vi_5\cdots i_m} r^{N-2v-i_5-\cdots-i_m}\rho^{2v} y_5^{i_5}\cdots y_m^{i_m}, \\
h_2 = &\quad a_2\rho + r^N\big(b_2\sin(Ns) + c_2\cos(Ns)\big) \\
&+ \sum_{2v+i_5+\cdots+i_m=1}^{N+1} d^2_{vi_5\cdots i_m} r^{N+1-2v-i_5-\cdots-i_m}\rho^{2v-1} y_5^{i_5}\cdots y_m^{i_m}, \\
h_3 = &\quad a_3 + r^{N-2}\rho\big(-c_1\sin(Ns) + b_1\cos(Ns)\big) \\
&+ \frac{1}{N} r^N\rho^{-1}\big(-c_2\sin(Ns) + b_2\cos(Ns)\big) \\
&+ \sum_{2v+i_5+\cdots+i_m=0}^{N} d^3_{vi_5\cdots i_m} r^{N-1-2v-i_5-\cdots-i_m}\rho^{2v} y_5^{i_5}\cdots y_m^{i_m} \\
&+ \sum_{2v+i_5+\cdots+i_m=1}^{N+1} d^4_{vi_5\cdots i_m} r^{N+1-2v-i_5-\cdots-i_m}\rho^{2v-2} y_5^{i_5}\cdots y_m^{i_m}, \\
h_k = &\quad \lambda_k y_k + \sum_{2v+i_5+\cdots+i_m=0}^{N} d^5_{vi_5\cdots i_m} r^{N-2v-i_5-\cdots-i_m}\rho^{2v} y_5^{i_5}\cdots y_m^{i_m},
\end{aligned}
\tag{26}
$$

where $h_j = h_j(r, \rho, s, y_5, \ldots, y_m)$.

According to the results of Section 2 we must study the real solutions of the system

$$h_k(r, \rho, s, y_5, \ldots, y_m) = 0 \quad \text{for } k = 1, 2, 3, 5, \ldots, m \tag{27}$$

that have nonzero Jacobian. In order that these solutions can provide limit cycles of system (2) we must look for those such that $r^2 + \rho^2 \neq 0$ (we recall that this kind of polar coordinates are introduced in Lemma 4). We distinguish three cases.

Case 1: $r = 0$ and $\rho \neq 0$. If $N > 2$ then in the system (27) the variable s does not appear. So the Jacobian of the system is always zero, and consequently the number of limit cycles of system (2) provided by the averaging theory is zero in this case.

In this case, if $N = 2$ then it is easy to check that all the equations of system (27) (except the first one which is identically zero) are polynomial equations of degree two in the variables r, ρ, $y_5, \ldots, y_m$, $\cos(2s)$ and $\sin(2s)$. Therefore, adding to system (27) the equation $\cos^2(2s) + \sin^2(2s) = 1$ by the Bézout Theorem (see [17]) the maximum number of limit cycles that can appear in this subcase is 2^{m-1}. Since for each solution $w_0 = \cos(2s)$ and $z_0 = \sin(2s)$ of $\cos^2(2s) + \sin^2(2s) = 1$ we can find $s_1, s_2 \in [0, 2\pi)$ such that $\sin(2s_i) = z_0$ and $\cos(2s_i) = w_0$ for $i = 1, 2$, we get that the total number of solutions of system (27) is at most 2^m.

Case 2: $b_2 = c_2 = 0$, $\rho = 0$ and $r \neq 0$. Then the degree of the polynomial equations of system (27) in the variables r, ρ, $y_5, \ldots, y_m$, $\cos(Ns)$ and $\sin(Ns)$ are N, $N+1$, $N+1$,$N, \ldots, N$ respectively. Therefore, adding to system (27) the equation $\cos^2(Ns) + \sin^2(Ns) = 1$ by the Bézout Theorem the maximum number of limit cycles that can appear in this case is $2N^{m-3}(N+1)^2$. Since for each solution $w_0 = \cos(Ns)$ and $z_0 = \sin(Ns)$ of $\cos^2(Ns) + \sin^2(Ns) = 1$ we can find $s_1, \ldots, s_N \in [0, 2\pi)$ such that $\sin(Ns_i) = z_0$ and $\cos(Ns_i) = w_0$ for $i = 1, \ldots, N$, we obtain that the total number of solutions of system (27) is at most $2N^{m-2}(N+1)^2$.

Case 3: $r\rho \neq 0$. Now we perform the change of variables

$$r^{N-1} = B, \quad \rho/r = A, \quad \sin(Ns) = z, \quad \cos(Ns) = w, \quad y_k/r = C_k$$

for $k = 5, \ldots, m$. In the new variables, the functions

$$\tilde{h}_1 = h_1/r, \quad \tilde{h}_2 = h_2/r, \quad \tilde{h}_3 = \rho h_3/r, \quad \tilde{h}_4 = z^2 + w^2 - 1, \quad \tilde{h}_k = h_k/r$$

for $k = 5, \ldots, m$ are given by

$$\begin{aligned}
\tilde{h}_1 &= a_1 + AB(b_1 z + c_1 w) + BP_1(A^2, C_5, \ldots, C_m), \\
\tilde{h}_2 &= a_2 A + B(b_2 z + c_2 w) + ABP_2(A^2, C_5, \ldots, C_m), \\
\tilde{h}_3 &= a_3 A + BA^2(-c_1 z + b_1 w) + \frac{1}{N} B(-c_2 z + b_2 w) \\
&\quad + ABP_3(A^2, C_5, \ldots, C_m) + BA^{-1} P_4(A^2, C_5, \ldots, C_m), \\
\tilde{h}_4 &= z^2 + w^2 - 1, \\
\tilde{h}_k &= \lambda_k C_k + BP_k(A^2, C_5, \ldots, C_m),
\end{aligned} \tag{28}$$

for $k = 5, \ldots, m$, where

$$P_i(A^2, C_5, \cdots, C_m) = \sum_{2l+i_5+\cdots+i_m=0}^{N} d^i_{li_5\cdots i_m} A^{2l} C_5^{i_5} \cdots C_m^{i_m}$$

for $i = 1, 3, k$ and

$$P_i(A^2, C_5, \cdots, C_m) = \sum_{2l+i_5+\cdots+i_m=1}^{N+1} d^i_{li_5\cdots i_m} A^{2l} C_5^{i_5} \cdots C_m^{i_m}$$

for $i = 2, 4$.

Solving $(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3) = (0, 0, 0)$, we find the solution

$$z = \frac{1}{A} Z(A^2, C_5, \cdots, C_m), \; w = \frac{1}{A} W(A^2, C_5, \cdots, C_m), \; B = B(A^2, C_5, \cdots, C_m),$$

where

$$Z = \frac{Z_1}{Z_2}, \quad W = \frac{W_1}{Z_2}, \quad \text{and} \quad B = \frac{B_1}{B_2}$$

with

$$\begin{aligned}
Z_1 = \;& -N(a_2 b_1 P_1 - a_1 b_1 P_2 + a_3 c_1 P_2 - a_2 c_1 P_3) A^4 + \\
& (-a_2 b_2 P_1 + a_3 c_2 N P_1 + a_1 b_2 P_2 - a_1 c_2 N P_3 + a_2 c_1 N P_4) A^2 - a_1 c_2 N P_4, \\
Z_2 = \;& a_2 \left(b_1^2 + c_1^2\right) N A^4 - a_1 \left(b_2^2 + c_2^2\right) + \\
& (a_2 b_1 b_2 - a_1 b_1 N b_2 + a_3 c_1 N b_2 + a_2 c_1 c_2 - a_3 b_1 c_2 N - a_1 c_1 c_2 N) A^2, \\
W_1 = \;& -N(a_2 c_1 P_1 - a_3 b_1 P_2 - a_1 c_1 P_2 + a_2 b_1 P_3) A^4 + \\
& (-a_2 c_2 P_1 - a_3 b_2 N P_1 + a_1 c_2 P_2 + a_1 b_2 N P_3 - a_2 b_1 N P_4) A^2 + a_1 b_2 N P_4, \\
B_1 = \;& -a_2 \left(b_1^2 + c_1^2\right) N A^4 + a_1 \left(b_2^2 + c_2^2\right) + \\
& (-a_2 b_1 b_2 + a_1 b_1 N b_2 - a_3 c_1 N b_2 - a_2 c_1 c_2 + a_3 b_1 c_2 N + a_1 c_1 c_2 N) A^2, \\
B_2 = \;& \left(b_1^2 + c_1^2\right) N P_2 A^4 - b_2^2 P_1 - c_2^2 P_1 + b_2 c_1 N P_4 - b_1 c_2 N P_4 + \\
& (-b_1 b_2 N P_1 - c_1 c_2 N P_1 + b_1 b_2 P_2 + c_1 c_2 P_2 + b_2 c_1 N P_3 - b_1 c_2 N P_3) A^2.
\end{aligned}$$

Therefore in the variables $(A^2, C_5, \ldots, C_m)$, B is a quotient of a polynomial of degree 2 by a polynomial of degree $N+3$, Z is a quotient of a polynomial of degree $N+3$ by a polynomial of degree 2, and W is a quotient of a polynomial of degree $N+3$ by a polynomial of degree 2.

Substituting z and w in the equation $\tilde{h}_4 = 0$, we obtain a quotient of a polynomial of degree $2(N+3)$ by a polynomial of degree 5 in the variables $(A^2, C_5, \ldots, C_m)$.

Substituting B in the equations $\tilde{h}_k = 0$ we obtain a quotient of a polynomial of degree $N+4$ by a polynomial of degree $N+3$ in the variables $(A^2, C_5, \ldots, C_m)$.

Therefore, by applying Bézout's theorem we have that the maximum number of possible roots $(A^2, C_5, \ldots, C_m)$ of the numerator of $(\tilde{h}_4, \tilde{h}_5, \ldots, \tilde{h}_m) = 0$ is $2(N+3)(N+4)^{m-4}$. For each solution $(A_0^2, C_{50}, \ldots, C_{m0})$ we have at most one $B_0 = B(A_0^2, C_{50}, \ldots, C_{m0})$ and one pair

$$(z_0, w_0) = (z(A_0^2, C_{50}, \ldots, C_{m0}), w(A_0^2, C_{50}, \ldots, C_{m0})).$$

For each pair (z_0, w_0) we can find $s_1, \ldots, s_N \in [0, 2\pi)$ such that $\sin(Ns_i) = z_0$ and $\cos(Ns_i) = w_0$ for $i = 1, \ldots, N$. So in this case the maximum number of zeros of system (27) is at most $2N(N+3)(N+4)^{m-4}$.

Now we put together the results of the three cases. By Theorem 3 the maximum number of limit cycles obtained via averaging theory for system (2) is

$$2^m + 2N^{m-2}(N+1)^2 + 2N(N+3)(N+4)^{m-4} = 2^m + 2^{m-1}3^2 + 2^{m-2}3^{m-4}5$$

if $N = 2$, or

$$2N^{m-2}(N+1)^2 + 2N(N+3)(N+4)^{m-4},$$

if $N > 3$. This completes the proof of the theorem. □

5. Some improvements for $N = 2$ and $N = 3$ with $m = 5$

In this section we prove Theorem 2.

Proof of statement (a) of Theorem 2. We can compute explicitly system (27) for $N = 2$ and $N = 3$ when $m = 5$. In particular for $N = 2$ and $m = 5$ system (2) is of the form

$$\begin{aligned} h_1 &= r\,(a_1 + \rho(b_1 z + c_1 w) + d_1 y_5) = 0, \\ h_2 &= a_2\rho + r^2(b_2 z + c_2 w) + d_2\rho y_5 = 0, \\ h_3 &= a_3 - 2\rho(-c_1 z + b_1 w) - r^2\rho^{-1}(-c_2 z + b_2 w) + d_3 y_5 = 0, \\ h_4 &= z^2 + w^2 - 1 = 0, \\ h_5 &= \lambda_5 y_5 + d_4 r^2 + d_5\rho^2 + d_6 y_5^2 = 0, \end{aligned} \tag{29}$$

where the constants a_i for $i = 1, 2, 3$, b_1, b_2, c_1, c_2 and d_j for $j = 1, \ldots, 6$ are arbitrary. Here $z = \sin(2s)$ and $w = \cos(2s)$. After doing the explicit computations many terms of system (27) become zero, and consequently we can improve the general results for system (27), studying the particular system (29) for $N = 2$ and $m = 5$. We distinguish the same cases as in the proof of Theorem 1.

Case 1: $r = 0$ and $\rho \neq 0$. Then system (29) reduces to

$$\begin{aligned} g_2 &= a_2 + d_2 y_5 = 0, \\ g_3 &= a_3 - 2\rho(-c_1 z + b_1 w) + d_3 y_5 = 0, \\ g_4 &= z^2 + w^2 - 1 = 0, \\ g_5 &= \lambda_5 y_5 + d_5\rho^2 + d_6 y_5^2 = 0. \end{aligned}$$

From $g_2 = 0$ we get y_5 (if $d_2 \neq 0$). Substituting it in $g_5 = 0$ we obtain at most one $\rho > 0$. Substituting y_5 and ρ in $g_3 = g_4 = 0$, we get at most two solutions (z_0, w_0) for (z, w). Since for each solution $w_0 = \cos(2s)$ and $z_0 = \sin(2s)$ of $\cos^2(2s) + \sin^2(2s) = 1$ we can find $s_1, s_2 \in [0, 2\pi)$ such that $\sin(2s_i) = z_0$ and $\cos(2s_i) = w_0$ for $i = 1, 2$, we get that the total number of solutions of system (27) is at most 4. In the proof of Theorem 1 for the general case the upper bound obtained in this case was 2^5.

Case 2: $b_2 = c_2 = 0$, $\rho = 0$ and $r \neq 0$. Now system (29) reduces to

$$\begin{aligned} g_1 &= a_1 + d_1 y_5 = 0, \\ g_2 &= b_2 z + c_2 w = 0, \\ g_3 &= a_3 + d_3 y_5 = 0, \\ g_4 &= z^2 + w^2 - 1 = 0, \\ g_5 &= \lambda_5 y_5 + d_4 r^2 + d_6 y_5^2 = 0. \end{aligned}$$

We assume that the possible solution of $g_1 = 0$ and $g_3 = 0$ coincides. Then substituting it in $g_5 = 0$ we obtain at most one $r > 0$. Substituting y_5 and r in $g_2 = g_4 = 0$, we get at most two solutions (z_0, w_0) for (z, w). As in the previous case $w_0 = \cos(2s)$ and $z_0 = \sin(2s)$, and consequently the total number of solutions of system (27) is at most 4. In the proof of Theorem 1 for the general case the upper bound obtained in this case was $9 \cdot 2^4$.

Case 3: $r\rho \neq 0$. Doing the same changes as in Case 3 of the proof of Theorem 1 we get that system (28) becomes

$$\begin{aligned}
\tilde{h}_1 &= a_1 + AB(b_1 z + c_1 w) + Bd_1 C_5,\\
\tilde{h}_2 &= a_2 A + B(b_2 z + c_2 w) + ABd_2 C_5,\\
\tilde{h}_3 &= a_3 A - 2BA^2(-c_1 z + b_1 w) - B(c_2 z + b_2 w) + ABd_3 C_5,\\
\tilde{h}_4 &= z^2 + w^2 - 1,\\
\tilde{h}_5 &= \lambda_5 C_5 + B(d_4 + d_5 A^2 + d_6 C_5^2).
\end{aligned}$$

Solving $\tilde{h}_1 = \tilde{h}_2 = \tilde{h}_3 = 0$ with respect to the variables z, w and B, and substituting these into $\tilde{h}_4 = \tilde{h}_5 = 0$, we obtain

$$\begin{aligned}
&\frac{A^2 C_5^2 (K_1^2 + K_2^2)}{D^2} - 1 = 0,\\
&\lambda_5 C_5 + \frac{(d_4 + d_5 A^2 + d_6 C_5^2) D}{C_5 E} = 0,
\end{aligned} \tag{30}$$

where

$$\begin{aligned}
K_1 = {} & \left(c_2 - 2A^2 c_1\right)(a_2 d_1 - a_1 d_2) + a_3\left(b_2 d_1 - A^2 b_1 d_2\right) + \left(A^2 a_2 b_1 - a_1 b_2\right) d_3,\\
K_2 = {} & (2a_2 b_1 d_1 - 2a_1 b_1 d_2 - a_3 c_1 d_2 + a_2 c_1 d_3)A^2 + a_2 b_2 d_1 + a_3 c_2 d_1 -\\
& a_1(b_2 d_2 + c_2 d_3),\\
D = {} & 2a_2\left(b_1^2 + c_1^2\right)A^4 + (a_2 b_1 b_2 - a_3 c_1 b_2 + a_3 b_1 c_2 - a_2 c_1 c_2 -\\
& 2a_1(b_1 b_2 + c_1 c_2))A^2 + a_1\left(c_2^2 - b_2^2\right),\\
E = {} & 2\left(b_1^2 + c_1^2\right) d_2 A^4 - (c_1(c_2(2d_1 + d_2) + b_2 d_3) +\\
& b_1(2b_2 d_1 - b_2 d_2 - c_2 d_3))A^2 + \left(c_2^2 - b_2^2\right) d_1.
\end{aligned}$$

System (30) reduces to

$$\begin{aligned}
&A^2 C_5^2 (K_1^2 + K_2^2) - D^2 = 0,\\
&\lambda_5 C_5^2 E + (d_4 + d_5 A^2 + d_6 C_5^2) D = 0.
\end{aligned}$$

Substituting C_5^2, obtained from the first equation, into the second one we obtain

$$D\left(\lambda_5 DE + (d_4 + d_5 A^2) A^2 (K_1^2 + K_2^2) + d_6 D^2\right) = 0,$$

a polynomial equation of degree 12 in the variable A^2, which can have at most 6 positive real solutions for A. Each of these possible solutions for A will provide at most 1 positive solution for C_5. Finally each of these at most 6 solutions for (A, C_5) provide one solution for (z, w, B). As before every one of these possible 6 solutions for $w = \cos(2s)$ and $z = \sin(2s)$ can provide two solutions for s, and consequently the total number of solutions of system (27) is at most 12, instead of the $2^3 \cdot 3 \cdot 5 = 120$ estimated in the general case for $N = 2$ and $m = 5$.

In short the maximum number of solutions of system (29) is bounded by $4 + 4 + 12 = 20$. □

Proof of statement (b) of Theorem 2. Now we shall improve the upper estimate on the number of limit cycles when $N = 3$ and $m = 5$. In this case system (2) after direct computations is of the form

$$\begin{aligned}
h_1 &= r\left(a_1 + r\rho(b_1 z + c_1 w) + d_1 r^2 + d_2\rho^2 + d_3 y_5^2\right) = 0,\\
h_2 &= a_2\rho + r^3(b_2 z + c_2 w) + \rho(d_4 r^2 + d_5\rho^2 + d_6 y_5^2) = 0,\\
h_3 &= a_3 + 3r\rho(-c_1 z + b_1 w) + r^3\rho^{-1}(-c_2 z + b_2 w) + d_7 r^2 + d_8\rho^2 + d_9 y_5^2 = 0,\\
h_4 &= z^2 + w^2 - 1 = 0,\\
h_5 &= \lambda_5 y_5 + y_5(d_{10} r^2 + d_{11}\rho^2 + d_{12} y_5^2) = 0,
\end{aligned} \tag{31}$$

where the constants a_i for $i = 1, 2, 3$, b_1, b_2, c_1, c_2 and d_j for $j = 1, \ldots, 12$ are arbitrary. As for the case $N = 2$ and $m = 5$ we distinguish the following three cases.

Case 1: $r = 0$ and $\rho \neq 0$. Then system (29) reduces to

$$\begin{aligned}
g_2 &= a_2 + d_5\rho^2 + d_6 y_5^2 = 0,\\
g_3 &= a_3 + d_8\rho^2 + d_9 y_5^2 = 0,\\
g_4 &= z^2 + w^2 - 1 = 0,\\
g_5 &= \lambda_5 y_5 + y_5(d_{11}\rho^2 + d_{12} y_5^2) = 0,
\end{aligned}$$

From $g_2 = 0$ we get y_5 (if $d_2 \neq 0$). Substituting it in $g_5 = 0$ we obtain at most one $\rho > 0$. Substituting y_5 and ρ in $g_3 = g_4 = 0$, we get at most two solutions (z_0, w_0) for (z, w). Since for each solution $w_0 = \cos(2s)$ and $z_0 = \sin(2s)$ of $\cos^2(2s)+\sin^2(2s) = 1$ we can find $s_1, s_2 \in [0, 2\pi)$ such that $\sin(2s_i) = z_0$ and $\cos(2s_i) = w_0$ for $i = 1, 2$, we get that the total number of solutions of system (27) is at most 4. In the proof of Theorem 1 for the general case the upper bound obtained in this case was $2 \cdot 3^3 \cdot 4^2 = 864$.

Case 2: $b_2 = c_2 = 0$, $\rho = 0$ and $r \neq 0$. Now system (29) reduces to

$$\begin{aligned}
g_1 &= a_1 + d_1 y_5 = 0,\\
g_2 &= b_2 z + c_2 w = 0,\\
g_3 &= a_3 + d_3 y_5 = 0,\\
g_4 &= z^2 + w^2 - 1 = 0,\\
g_5 &= \lambda_5 y_5 + d_4 r^2 + d_6 y_5^2 = 0.
\end{aligned}$$

In the case that $g_1 = 0$ and $g_2 = 0$ share some solution, we shall get a continuum of solutions for (z, w) and consequently the Jacobian of the system at these solutions will be zero, and we cannot apply the averaging theory for obtaining limit cycles in this case.

Case 3: $r\rho \neq 0$. Doing the same changes as in Case 3 of the proof of Theorem 1 we get that system (28) becomes

$$\begin{aligned}
\tilde h_1 &= a_1 + AB(b_1 z + c_1 w) + B(d_1 + d_2 A^2 + d_3 C_5^2),\\
\tilde h_2 &= a_2 A + B(b_2 z + c_2 w) + AB(d_4 + d_5 A^2 + d_6 C_5^2),\\
\tilde h_3 &= a_3 A - 2BA^2(-c_1 z + b_1 w) - B(c_2 z + b_2 w) + AB(d_7 + d_8 A^2 + d_9 C_5^2),\\
\tilde h_4 &= z^2 + w^2 - 1,\\
\tilde h_5 &= \lambda_5 C_5 + B(d_{10} + d_{11} A^2 + d_{12} C_5^2) C_5.
\end{aligned}$$

Solving $\tilde h_1 = \tilde h_2 = \tilde h_3 = 0$ with respect to the variables z, w and B, and substituting these into $\tilde h_4 = \tilde h_5 = 0$, we obtain

$$\begin{aligned}
&\frac{A^2(K_1^2 + K_2^2)}{D^2} - 1 = 0,\\
&\lambda_5 C_5 + \frac{C_5(d_{10} + d_{11} A^2 + d_{12} C_5^2) D}{E} = 0,
\end{aligned} \tag{32}$$

where

$$\begin{aligned} K_1 = \quad & (-2a_2c_1d_2 + 2a_1c_1d_5 + a_2b_1d_8)A^4 + \big(a_1\left(-c_2d_5 + 2c_1\left(d_6C_5^2 + d_4\right) - \right.\\ & \left. b_2d_8\right) + a_2\left(b_1d_9C_5^2 + c_2d_2 - 2c_1\left(d_3C_5^2 + d_1\right) + b_1d_7\right)\big) A^2 + \\ & a_2c_2\left(d_3C_5^2 + d_1\right) + a_3\big(b_2\left(d_2A^2 + d_1 + C_5^2d_3\right) - \\ & A^2b_1\left(d_5A^2 + d_4 + C_5^2d_6\right)\big) - a_1\left(c_2\left(d_6C_5^2 + d_4\right) + b_2\left(d_9C_5^2 + d_7\right)\right), \end{aligned}$$

$$\begin{aligned} K_2 = \quad & (-2a_2b_1d_2 + 2a_1b_1d_5 + a_3c_1d_5 - a_2c_1d_8)A^4 + \\ & \big(a_3\left(c_1\left(d_6C_5^2 + d_4\right) - c_2d_2\right) + a_1\left(b_2d_5 + 2b_1\left(d_6C_5^2 + d_4\right) + c_2d_8\right) - \\ & a_2\left(c_1d_9C_5^2 + b_2d_2 + 2b_1\left(d_3C_5^2 + d_1\right) + c_1d_7\right)\big) A^2 - a_2b_2\left(d_3C_5^2 + d_1\right) - \\ & a_3c_2\left(d_3C_5^2 + d_1\right) + a_1\left(b_2\left(d_6C_5^2 + d_4\right) + c_2\left(d_9C_5^2 + d_7\right)\right), \\ D = \quad & 2a_2\left(b_1^2 + c_1^2\right)A^4 + (a_2b_1b_2 - a_3c_1b_2 + a_3b_1c_2 - a_2c_1c_2 - \\ & 2a_1(b_1b_2 + c_1c_2))A^2 + a_1\left(c_2^2 - b_2^2\right), \\ E = \quad & 2\left(b_1^2 + c_1^2\right)d_5A^6 + \big(2\left(d_6C_5^2 + d_4\right)b_1^2 + (b_2(d_5 - 2d_2) + c_2d_8)b_1 + \\ & c_1\left(-c_2(2d_2 + d_5) + 2c_1\left(d_6C_5^2 + d_4\right) - b_2d_8\right)\big) A^4 - \\ & \big(\left(b_2^2 - c_2^2\right)d_2 + c_1\left(c_2\left((2d_3 + d_6)C_5^2 + 2d_1 + d_4\right) + b_2\left(d_9C_5^2 + d_7\right)\right) + \\ & b_1\left(b_2\left((2d_3 - d_6)C_5^2 + 2d_1 - d_4\right) - c_2\left(d_9C_5^2 + d_7\right)\right)\big) A^2 + \left(c_2^2 - b_2^2\right) \\ & \left(d_3C_5^2 + d_1\right). \end{aligned}$$

Since D cannot be zero system (30) reduces to

$$\begin{aligned} & A^2(K_1^2 + K_2^2) - D^2 = 0, \\ & C_5\left(\lambda_5E + (d_{10} + d_{11}A^2 + d_{12}C_5^2)D\right) = 0. \end{aligned}$$

Substituting $C_5 = 0$, obtained from the second equation, into the first one we obtain a polynomial of degree 10 in the variable A^2, which can have at most 5 positive real solutions for A.

Substituting C_5^2, obtained from the second factor of the second equation, into the first one we obtain a rational function in the variable A^2 whose numerator is a polynomial of degree 18, which can have at most 9 positive real solutions for A.

Each of these possible solutions for A will provide at most $5 + 9 = 14$ solutions for C_5. Finally each of these at most 14 solutions for (A, C_5) provide one solution for (z, w, B). As before every one of these possible 14 solutions for $w = \cos(3s)$ and $z = \sin(3s)$ can provide three solutions for s, and consequently the total number of solutions of system (32) is at most 42, instead of the $6^2 \cdot 7 = 252$ estimated in the general case for $N = 3$ and $m = 5$.

In short the maximum number of solutions of system (31) is bounded by $4 + 0 + 42 = 46$. □

Acknowledgements

We thank to the referee some comments which allow us to improve the presentation of this paper.

The first and third authors are partially supported by FCT through CAMGSD, Lisbon. The second author is supported by the grants MCYT/FEDER MTM 2008–03437, Generalitat de Catalunya 2009SGR410 and ICREA Academia.

References

[1] A. Buică and J. Llibre, *Averaging methods for finding periodic orbits via Brouwer degree*, Bull. Sci. Math. **128** (2004), 7–22.

[2] C. A. Buzzi, J. Llibre, J. C. Medrado and J. Torregrosa, *Bifurcation of limit cycles from a centre in* $\mathbb{R}^4$ *in resonance* $1 : N$, Dyn. Syst. **24** (2009), 123–137.

[3] M. Carbonell, B. Coll and J. Llibre, *Limit cycles of polynomial systems with homogeneous nonlinearities*, J. of Math. Anal. and Appl. **142** (1989), 573–590.

[4] M. Carbonell and J. Llibre, *Hopf bifurcation, averaging methods and Liapunov quantities for polynomial systems with homogeneous nonlinearities*, Proc. European Conference on Iteration Theory, ECIT87, World Scientific, Singapore, 1989, pp. 145–160.
[5] J. Chavarriga and J. Giné, *Integrability of a linear center perturbed by fifth degree homogeneous polynomial*, Publicacions Matemàtiques **41** (1997), 335–356.
[6] J. Chavarriga, J. Giné and I. García, *Isochronous centers of a linear system perturbed by fourth degree homogeneous polynomial*, Bull. Sciences Math. **123** (1999), 77–97.
[7] L.A. Cherkas, *Number of limit cycles of an autonomous second–order system*, Differential Equations **5** (1976), 666–668.
[8] C. Christopher and C. Li, *Limit cycles of differential equations*, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser Verlag, Basel, 2007.
[9] D. Hilbert, *Mathematische Probleme, Lecture*, Second Internet. Congr. Math. (Paris, 1900), Nachr. Ges. Wiss. Göttingen Math. Phys. KL. (1900), 253–297; English Transl: Bull. Amer. Math. Soc. **8** (1902), 437-479.
[10] Yu. Ilyashenko, *Centennial history of Hilbert's 16th problem*, Bull. Amer. Math. Soc. **39** (2002), 301–354.
[11] J. Li, *Hilbert's 16th problem and bifurcations of planar polynomial vector fields*, Internat. J. Bifur. Chaos Appl. Sci. Engrg. **13** (2003), 47–106.
[12] C. Li, W. Li, J. Llibre and Z. Zhang, *On the limit cycles of polynomial differential systems with homogeneous nonlinearities*, Proc. of the Edinburgh Math. Soc. **43** (2000), 529–543.
[13] J. Llibre, A.C. Mereu and M.A. Teixeira, *Limit cycles of the generalized polynomial Liénard differential equations*, Math. Proceed. Camb. Phyl. Soc. **148** (2009), 363–383.
[14] J. Llibre, A.C. Mereu and M.A. Teixeira, *Limit cycles of invariant four-dimensional resonant systems*, Dynamical Systems **17** (2010), 1–14.
[15] N. G. Lloyd, *Degree Theory*, Cambridge University Press, Cambridge, New York, Melbourne, 1978.
[16] N.G. Lloyd, *Limit cycles of certain polynomial differential systems*, in Nonlinear Functional Analysis and its Applications, S.P. Singh, ed NATO AS1 Series C, Vol. **173**, Reidel, Dordrecht, The Netherlands, 1986, pp. 317–326.
[17] I. R. Shafarevich, *Basic Algebraic Geometry*, Springer, 1974.
[18] K.S. Sibirskii, *Conditions for a center in the case of homogeneous nonlinearities of third degree* (russian), Differential Equations **1** (1965), 1482–1487.

Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal
E-mail address: barreira@math.ist.utl.pt

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
E-mail address: jllibre@mat.uab.cat

Departamento de Matemática, Instituto Superior Técnico, 1049-001 Lisboa, Portugal
E-mail address: cvalls@math.ist.utl.pt