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$\mathcal{C}^1$ SELF–MAPS ON $\mathbb{S}^n$, $\mathbb{S}^n \times \mathbb{S}^m$, $\mathbb{CP}^n$ AND $\mathbb{HP}^n$ WITH ALL THEIR PERIODIC ORBITS HYPERBOLIC

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Abstract. We study in its homological class the periodic structure of the $\mathcal{C}^1$ self–maps on the manifolds $\mathbb{S}^n$ (the n–dimensional sphere), $\mathbb{S}^n \times \mathbb{S}^m$ (the product space of the n–dimensional with the m–dimensional spheres), $\mathbb{CP}^n$ (the n–dimensional complex projective space) and $\mathbb{HP}^n$ (the n–dimensional quaternion projective space), having all their periodic orbits hyperbolic.

1. Introduction and statement of the main results

Let $\mathbb{M}$ be topological space and $f : \mathbb{M} \to \mathbb{M}$ be a continuous map. A point x is called *fixed* if $f(x) = x$, and *periodic* of *period* k if $f^k(x) = x$ and $f^i(x) \neq x$ if $0 \leq i < k$. By $\mathrm{Per}(f)$ we denote the *set of periods* of all the periodic points of f.

If $x \in \mathbb{M}$ the set $\{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$ is called the *orbit* of the periodic point x. Here f^n means the composition of n times f with itself. To study the dynamics of the map f is to study all the different kind of orbits of f. Of course if x is a periodic point of f of period k, then its orbit is $\{x, f(x), f^2(x), \ldots, f^{k-1}(x)\}$, and it is called a *periodic orbit*.

In this paper we study the periodic dynamics of $\mathcal{C}^1$ self–maps f defined on a given compact manifold $\mathbb{M}$ without boundary. Often the periodic orbits play an important role in the general dynamics of a map, for studying them we can use topological information. Perhaps the best known example in this direction are the results contained in the seminal paper entitle *Period three implies chaos* for continuous self–maps on the interval, see [11].

For continuous self–maps on compact manifolds one of the most useful tools for proving the existence of fixed points and in general of periodic points, is the *Lefschetz Fixed Point Theorem* and its improvements, see for instance [1, 2, 3, 4, 6, 8, 10, 13, 14]. The *Lefschetz zeta function* $\mathcal{Z}_f(t)$ simplifies the study of the periodic points of f. This is a generating function for the Lefschetz numbers of all iterates of f, see section 2 for a precise definition and results about it. There we also define the *minimal Lefschetz set of periods*, denoted by $\mathrm{MPer}_L(f)$.

Let f be a $\mathcal{C}^1$ map defined on a compact manifold $\mathbb{M}$ without boundary. Recall that if $f(x) = x$ and the Jacobian matrix of f at x, $Df(x)$, has all its eigenvalues

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disjoint from the unit circle, then x is called a *hyperbolic fixed point*. Moreover if y is a periodic point of period p, then y is a *hyperbolic periodic point* if y is a hyperbolic fixed point of f^p. If the points of a periodic orbit are hyperbolic we say that the periodic orbit is *hyperbolic*. Of course if a point of a periodic orbit is hyperbolic, then all the points of the orbit are hyperbolic.

In this work we put our attention on the periodic dynamics of the $\mathcal{C}^1$ self–maps having all their periodic orbits hyperbolic, and such that they are defined on the compact manifolds without boundary $\mathbb{S}^n$, $\mathbb{S}^n \times \mathbb{S}^m$, $\mathbb{C}\mathrm{P}^n$ and $\mathbb{H}\mathrm{P}^n$. Moreover if such maps have finitely many periodic orbits, then we say that they are of *finite type*. For precise definitions see section 2.

We summarize our results in the following four theorems.

Theorem 1. *For $n \geq 1$ let f be a $\mathcal{C}^1$ self-map defined on $\mathbb{S}^n$ of degree D with all its periodic orbits hyperbolic. Then the following statement hold.*

(a) *If $D \notin \{-1, 0, 1\}$, then f has infinitely many periodic orbits.*
(b) *If either $D = -1$ and n odd, or $D = 0$, or $D = 1$ and n even, then $1 \in \mathrm{Per}(f)$. Moreover if f is of finite type, then $\mathrm{MPer}_L(f) = \{1\}$.*
(c) *If $D = -1$ and n even, then $\{1, 2\} \cap \mathrm{Per}(f) \neq \emptyset$. Moreover if f is of finite type, then $\mathrm{MPer}_L(f) = \emptyset$.*
(d) *If $D = 1$, n odd and f is of finite type, then $\mathrm{MPer}_L(f) = \emptyset$.*

Theorem 1 is proved in section 3. As we shall see in its proof its statements (b) and (c) also hold for continuous self–maps on $\mathbb{S}^n$.

Let $P(t)/Q(t)$ be a rational function, we say that it is *irreducible* if the greatest common divisor of the polynomials $P(t)$ and $Q(t)$ in the ring of all polynomials with complex coefficients is 1.

Theorem 2. *For $n \geq 1$ let f be a $\mathcal{C}^1$ self-map defined on $\mathbb{S}^n \times \mathbb{S}^n$ of degree D with all its periodic orbits hyperbolic, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be the induced linear map on n–th dimensional homological space of $\mathbb{S}^n \times \mathbb{S}^n$ with rational coefficients. We define the polynomial $p(t) = 1 - (a + d)t + (ad - bc)t^2$. Then the following statement hold.*

(a) *If n is even and, either $D \notin \{-1, 0, 1\}$, or $D \in \{-1, 0, 1\}$ and $p(t) \notin \{1, 1 \pm t, (1 \pm t)^2, 1 \pm t^2\}$, then f has infinitely many periodic orbits.*
(b) *If n is odd and some factor of the numerator or of the denominator of the irreducible rational function $p(t)/((1 - t)(1 - Dt))$ is different from 1, $1 \pm t$ and $1 + t^2$, then f has infinitely many periodic orbits.*
(c) *Assume that n is even and $p(t) \in \{1, 1 \pm t, (1 \pm t)^2, 1 \pm t^2\}$. If $D = -1$ then $1 \in \mathrm{Per}(f)$ if $p(t) \neq 1$. Moreover if f is of finite type*
$$\mathrm{MPer}_L(f) = \begin{cases} \{1\} & p(t) \in \{1 \pm t, (1 \pm t)^2\}, \\ \emptyset & \textit{otherwise}. \end{cases}$$
If $D = 0$ then $1 \in \mathrm{Per}(f)$ if $p(t) \notin \{1 + t, 1 + t^2\}$. Moreover if f is of finite type
$$\mathrm{MPer}_L(f) = \begin{cases} \{1\} & p(t) \neq 1 + t, \\ \emptyset & \textit{otherwise}. \end{cases}$$
If $D = 1$ then $1 \in \mathrm{Per}(f)$ if $p(t) \neq (1 + t)^2$. Moreover if f is of finite type
$$\mathrm{MPer}_L(f) = \begin{cases} \{1\} & p(t) \neq (1 + t)^2, \\ \emptyset & \textit{otherwise}. \end{cases}$$

(d) *Suppose that n odd and all the factors of the numerator and of the denominator of the irreducible rational function $p(t)/((1-t)(1-Dt))$ are of the form 1, $1 \pm t$ and $1+t^2$. If $D=-1$ then $1 \in \mathrm{Per}(f)$ if $p(t) \neq 1$. Moreover if f is of finite type*

$$\mathrm{MPer}_L(f) = \begin{cases} \{1\} & p(t) \in \{1 \pm t, (1 \pm t)^2\}, \\ \emptyset & \textit{otherwise.} \end{cases}$$

If $D=0$ then $1 \in \mathrm{Per}(f)$ if $p(t) \notin \{1-t, 1-t^2\}$. Moreover if f is of finite type

$$\mathrm{MPer}_L(f) = \begin{cases} \{1\} & p(t) \neq 1-t, \\ \emptyset & \textit{otherwise.} \end{cases}$$

If $D=1$ then $1 \in \mathrm{Per}(f)$ if $p(t) \neq (1-t)^2$. Moreover if f is of finite type

$$\mathrm{MPer}_L(f) = \begin{cases} \{1\} & p(t) \neq (1-t)^2, \\ \emptyset & \textit{otherwise.} \end{cases}$$

Theorem 3. *For $n \neq m$ let f be a $\mathcal{C}^1$ self-map defined on $\mathbb{S}^n \times \mathbb{S}^m$ of degree D with all its periodic orbits hyperbolic, and let (a) and (b) be the induced linear maps on its homological spaces of dimension n and m with coefficients in $\mathbb{Q}$, respectively. Then the following statement hold.*

(a) *If n and m are even and $\{a, b, D\} \not\subset \{-1, 0, 1\}$, then f has infinitely many periodic orbits.*
(b) *If n and m are odd, and some factor of the numerator or of the denominator of the irreducible rational function $(1-at)(1-bt)/((1-t)(1-Dt))$ is different from 1 and $1 \pm t$, then f has infinitely many periodic orbits.*
(c) *If n is even and m is odd, and some factor of the numerator or of the denominator of the irreducible rational function $(1-bt)(1-Dt)/((1-t)(1-at))$ is different from 1 and $1 \pm t$, then f has infinitely many periodic orbits.*
(d) *If n is odd and m is even, and some factor of the numerator or of the denominator of the irreducible rational function $(1-at)(1-Dt)/((1-t)(1-bt))$ is different from 1 and $1 \pm t$, then f has infinitely many periodic orbits.*
(e) *Assume that n and m are even and $\{a, b, D\} \subset \{-1, 0, 1\}$. Then $1 \in \mathrm{Per}(f)$ if $1+a+b+D \neq 0$. Moreover if f is of finite type*

$$\mathrm{MPer}_L(f) = \begin{cases} \{1\} & \textit{if } 1+a+b+D \neq 0, \\ \emptyset & \textit{otherwise.} \end{cases}$$

(f) *Assume that n and m are odd, and all the factors of the numerator and of the denominator of the irreducible rational function $\mathcal{Z}_f(t) = (1-at)(1-bt)/((1-t)(1-Dt))$ are 1 or $1 \pm t$. Then $1 \in \mathrm{Per}(f)$ if $1+D \neq a+b$. Moreover if f is of finite type*

$$\mathrm{MPer}_L(f) = \begin{cases} \{1\} & \textit{if } \mathcal{Z}_f(t) \neq 1, \\ \emptyset & \textit{otherwise.} \end{cases}$$

(g) *Assume that n is even and m is odd, and all the factors of the numerator and of the denominator of the irreducible rational function $\mathcal{Z}_f(t) = (1-bt)(1-Dt)/((1-t)(1-at))$ are 1 or $1 \pm t$. Then $1 \in \mathrm{Per}(f)$ if $1+a \neq b+b$. Moreover if f is of finite type*

$$\mathrm{MPer}_L(f) = \begin{cases} \{1\} & \textit{if } \mathcal{Z}_f(t) \neq 1, \\ \emptyset & \textit{otherwise.} \end{cases}$$

(h) *Assume that n is odd and m is even, and all the factors of the numerator and of the denominator of the irreducible rational function $\mathcal{Z}_f(t) = (1-at)(1-Dt)/((1-t)(1-bt))$ are 1 or $1 \pm t$. Then $1 \in \mathrm{Per}(f)$ if $1+b \neq a+D$. Moreover if f is of finite type*

$$\mathrm{MPer}_L(f) = \begin{cases} \{1\} & \text{if } \mathcal{Z}_f(t) \neq 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Theorems 2 and 3 are proved in section 4.

Theorem 4. *For $n \geq 1$ let f be a $\mathcal{C}^1$ self-map defined on either $\mathbb{C}P^n$ or $\mathbb{H}P^n$ with generator $a \in \mathbb{Z}$ and all its periodic orbits hyperbolic. Then the following statement hold.*

(a) *If $a \notin \{-1, 0, 1\}$, then f has infinitely many periodic orbits.*
(b) *Assume that $a \in \{-1, 0, 1\}$. Then $1 \in \mathrm{Per}(f)$ except perhaps if $a = -1$ and n is odd. Moreover if f is of finite type*

$$\mathrm{MPer}_L(f) = \begin{cases} \emptyset & \text{if } a = -1 \text{ and } n \text{ is odd,} \\ \{1\} & \text{otherwise.} \end{cases}$$

Theorem 4 is proved in section 5.

In fact our results are in some sense inspired in the seminal paper in this direction of Franks [7] where he studied similar problems for C^1 maps on the circle, the 2–dimensional sphere and the 2–dimensional disk.

2. Frank's Theorem

Probably the main contribution of the Lefschetz's work in 1920's was to link the homology class of a given map with an earlier work on the indices of Brouwer on the continuous self–maps on compact manifolds. These two notions provide equivalent definitions for the Lefschetz numbers, and from their comparison, can be obtained information about the existence of fixed points.

Let $\mathbb{M}$ be an n–dimensional manifold. We denote by $H_k(\mathbb{M}, \mathbb{Q})$ for $k = 0, 1, ..., n$ the homological groups with coefficients in $\mathbb{Q}$. Each of these groups is a finite linear space over $\mathbb{Q}$.

Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ there exist $n+1$ induced linear maps $f_{*k} : H_k(\mathbb{M}, \mathbb{Q}) \to H_k(\mathbb{M}, \mathbb{Q})$ for $k = 0, 1, \ldots, n$ by f. Every linear map f_{*k} is given by an $n_k \times n_k$ matrix with integer entries, where n_k is the dimension of $H_k(\mathbb{M}, \mathbb{Q})$.

Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ on a compact n–dimensional manifold $\mathbb{M}$, its *Lefschetz number* $L(f)$ is defined as

$$L(f) = \sum_{k=0}^{n} (-1)^k \mathrm{trace}(f_{*k}). \tag{1}$$

One of the main results connecting the algebraic topology with the fixed point theory is the *Lefschetz Fixed Point Theorem* which establishes the existence of a fixed point if $L(f) \neq 0$, see for instance [3].

Our aim is to obtain information on the set of periods of f. To this purpose it is useful to have information on the whole sequence $\{L(f^m)\}_{m=0}^{\infty}$ of the Lefschetz numbers of all iterates of f. Thus we define the *Lefschetz zeta function* of f as

$$\mathcal{Z}_f(t) = \exp\left(\sum_{m=1}^{\infty} \frac{L(f^m)}{m} t^m \right).$$

This function generates the whole sequence of Lefschetz numbers, and it may be independently computed through

$$\mathcal{Z}_f(t) = \prod_{k=0}^{n} \det(I_{n_k} - tf_{*k})^{(-1)^{k+1}}, \tag{2}$$

where I_{n_k} is the $n_k \times n_k$ identity matrix, and we take $\det(I_{n_k} - tf_{*k}) = 1$ if $n_k = 0$. Note that the expression (2) is a rational function in t. So the information on the infinite sequence of integers $\{L(f^m)\}_{m=0}^{\infty}$ is contained in two polynomials with integer coefficients, for more details see [8].

If γ is a hyperbolic periodic orbit of period p, then for each $x \in \gamma$ let E_x^u denotes the subspace of the tangent space $T_x\mathbb{M}$ generated by the eigenvectors of $Df^p(x)$ corresponding to the eigenvalues whose moduli are greater than one. Let E_x^s be the subspace of $T_x\mathbb{M}$ generated by the remaining eigenvectors. We define the *orientation type* Δ of γ to be $+1$ if $Df^p(x) : E_x^u \to E_x^u$ preserves orientation, and -1 if it reverses orientation. The *index* u of γ is the dimension of E_x^u for some $x \in \gamma$. We note that the definitions of Δ and u do not depend on the point x, only depend of the periodic orbit γ. Finally we associated the triple (p, u, Δ) to the periodic orbit γ.

For f the *periodic data* is defined as the collection composed by all triples (p, u, Δ), where a same triple can occur more than once provided it corresponds to different periodic orbits. Franks in [8] proved the following result which will play a key role for proving our results.

Theorem 5. *Let f be a C^1 self–map defined on a compact manifold without boundary having finitely many periodic orbits all of them hyperbolic, and let Σ be the period data of f. Then the Lefschetz zeta function of f satisfies*

$$\mathcal{Z}_f(t) = \prod_{(p,u,\Delta)\in\Sigma} (1 - \Delta t^p)^{(-1)^{u+1}}. \tag{3}$$

The statement of Theorem 5 allows us to define the minimal Lefschetz set of periods for a C^1 map on a compact manifold without boundary having finitely many periodic points all of them hyperbolic. Such a map has a Lefschetz zeta function of the form (3).

Note that in general the expression of one of these Lefschetz zeta functions is not unique as product of the elements of the form $(1 \pm t^p)^{\pm 1}$. For instance the following Lefschetz zeta function can be written in four different ways in the form given by (3):

$$\mathcal{Z}_f(t) = \frac{(1-t^3)^2(1+t^3)}{(1-t)^6(1+t)^3} = \frac{(1-t^3)(1-t^6)}{(1-t)^6(1+t)^3} = \frac{(1-t^3)(1-t^6)}{(1-t)^3(1-t^2)^3} = \frac{(1-t^3)^2(1+t^3)}{(1-t)^3(1-t^2)^3}.$$

According with Theorem 5, the first expression will provide the periods $\{1, 3\}$ for f, the second the periods $\{1, 3, 6\}$, the third the period $\{1, 2, 3, 6\}$, and finally the fourth the periods $\{1, 2, 3\}$. Then for this Lefschetz zeta function $\mathcal{Z}_f(t)$ we will define its *minimal Lefschetz set of periods* as

$$\mathrm{MPer}_L(f) = \{1, 3\} \cap \{1, 3, 6\} \cap \{1, 2, 3, 6\} \cap \{1, 2, 3\} = \{1, 3\}.$$

In general for the Lefschetz zeta function $\mathcal{Z}_f(t)$ of a C^1 map f on a compact manifold without boundary having finitely many periodic points all of them hyperbolic, we define its *minimal Lefschetz set of periods* as the intersection of all sets of periods forced by the finitely many different representations of $\mathcal{Z}_f(t)$ as products of the form $(1 \pm t^p)^{\pm 1}$.

Note that always

$$\mathrm{MPer}_L(f) \subseteq \mathrm{Per}(f).$$

In general it is unknown when $\mathrm{MPer}_L(f) = \mathrm{Per}(f)$ for some f.

3. $\mathcal{C}^1$ self–maps on $\mathbb{S}^n$

For $n \geq 1$ let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a $\mathcal{C}^1$ map. The homological groups of $\mathbb{S}^n$ over $\mathbb{Q}$ are of the form

$$H_q(\mathbb{S}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, n\}, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore the induced linear maps are $f_{*0} = (1)$, $f_{*i} = (0)$ for $i = 1, ..., n-1$ and $f_{*n} = (D)$ where D is the degree of the map f, see for more details [5] or [15]. From (2) the Lefschetz zeta function of f is

$$\mathcal{Z}_f(t) = \frac{(1 - Dt)^{(-1)^{n+1}}}{1 - t}. \tag{4}$$

Proof of Theorem 1. Assume that we are under the assumptions of Theorem 1 and $D \notin \{-1, 0, 1\}$. Then, since expressions (3) and (4) are not compatible, the map f has infinitely many hyperbolic periodic points. Consequently statement (a) is proved.

From (1) we get that $L(f) = 1 + (-1)^n D$, and $L(f^2) = 1 + (-1)^n D^2$. So under the assumptions of statement (b) we have that $L(f) \neq 0$, and under the hypotheses of statement (c) we obtain that $L(f^2) = 2$. Hence, by the Lefschetz fixed point Theorem, if $L(f) \neq 0$ we obtain that $1 \in \mathrm{Per}(f)$, and if $L(f^2) \neq 0$ we have that $1 \in \mathrm{Per}(f^2)$, i.e. $\{1, 2\} \cap \mathrm{Per}(f) \neq \emptyset$. Therefore, without assuming that f is of finite type statements (b) and (c) are proved.

Assume that f is of finite type, so the Lefschetz zeta function is given by (3) and (4). Then, if $D = -1$ and n odd, then $Z_f(t) = (1 + t)/(1 - t)$. If $D = 0$, then $Z_f(t) = 1/(1 - t)$. If $D = 1$ and n even, then $Z_f(t) = 1/(1 - t)^2$. If $D = -1$ and n even, then $Z_f(t) = 1/((1 + t)(1 - t)) = 1/(1 - t^2)$. Finally, if $D = 1$ and n odd, then $Z_f(t) = 1$. Therefore, from Theorem 5 and the definition of $\mathrm{MPer}_L(f)$, it follows easily the proofs of statements (b), (c) and (d). □

4. $\mathcal{C}^1$ self–maps on $\mathbb{S}^n \times \mathbb{S}^m$

We shall need the following lemma proved in [12], but since its proof is shorter we add it here for completeness.

Lemma 6. *There are no even numbers in* $\mathrm{MPer}_L(f)$.

Proof. If the number $2d$ is in $\mathrm{MPer}_L(f)$ then $(1 \pm t^{2d})^m$ is a factor of the Lefschetz zeta function $\mathcal{Z}_f(t)$, for some $m \neq 0$. So if the factor is $(1 - t^{2d})^m$ it can be written as $(1 - t^d)^m(1 + t^d)^m$, since the intersection of the exponents is taken over all possible expressions (3) of $\mathcal{Z}_f(t)$, the number $2d$ is not in $\mathrm{MPer}_L(f)$.

If the factor is $(1 + t^{2d})^m$, then it can be written as

$$(1 + t^{2d})^m = \frac{(1 + t^{2d})^m (1 - t^{2d})^m}{(1 - t^{2d})^m} = \frac{(1 - t^{4d})^m}{(1 - t^d)^m (1 + t^d)^m}.$$

Therefore $2d \notin \mathrm{MPer}_L(f)$. □

4.1. **Case $n = m$.** For $n \geq 1$ let $f : \mathbb{S}^n \times \mathbb{S}^n \to \mathbb{S}^n \times \mathbb{S}^n$ be a $\mathcal{C}^1$ map. The homological groups of $\mathbb{S}^n \times \mathbb{S}^n$ over $\mathbb{Q}$ are of the form

$$H_q(\mathbb{S}^n \times \mathbb{S}^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, 2n\}, \\ \mathbb{Q} \oplus \mathbb{Q} & \text{if } q = n, \\ 0 & \text{otherwise.} \end{cases}$$

The induced linear maps are $f_{*0} = (1)$, $f_{*n} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{Z}$, $f_{*2n} = (D)$ where D is the degree of the map f and $f_{*i} = (0)$ for $i \in \{0, ..., 2n\}$, $i \neq 0, n, 2n$ (see for more details [5]). From (2) the Lefschetz zeta function of f is

$$\mathcal{Z}_f(t) = \frac{p(t)^{(-1)^{n+1}}}{(1-t)(1-Dt)}. \tag{5}$$

where $p(t$ is defined in the statement of Theorem 2.

Proof of Theorem 2. Under the conditions stated in statements (a) or (b) the expressions of (3) and (5) are not compatible. Therefore, Theorem 5 forces that the map f has infinitely many hyperbolic periodic points.

>From (1) we have that $L(f) = 1 + D + (-1)^n(a + d)$. Since $a + d$ is equal to 0, ∓ 1, ∓ 2 and ∓ 1 if $p(t)$ is 1, $1 \pm t$, $(1 \pm t)^2$ and $1 \pm t^2$, respectively. From the Lefschetz fixed point theorem it follows the first part of statements (c) and (d).

When f if of finite type the second part of statements (c) and (d) follow easily from (5), Theorem 5 and Lemma 6. □

4.2. **Case $n \neq m$.** For $n, m \geq 1$, let $f : \mathbb{S}^n \times \mathbb{S}^m \to \mathbb{S}^n \times \mathbb{S}^m$ be a $\mathcal{C}^1$ map. The homological groups of $\mathbb{S}^n \times \mathbb{S}^m$ over $\mathbb{Q}$ are of the form

$$H_q(\mathbb{S}^n \times \mathbb{S}^m, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, n, m, n+m\}, \\ 0 & \text{otherwise.} \end{cases}$$

The induced linear maps are $f_{*0} = (1)$, $f_{*n} = (a)$, $f_{*m} = (b)$ with $a, b \in \mathbb{Z}$, $f_{*n+m} = (D)$, where D is the degree of the map f and $f_{*i} = (0)$ for $i \in \{0, ..., n + m\}$, $i \neq 0, n, m, n + m$ (see for more details [5]). From (2) the Lefschetz zeta function of f is of the form

$$\mathcal{Z}_f(t) = \frac{(1-at)^{(-1)^{n+1}}(1-bt)^{(-1)^{m+1}}(1-Dt)^{(-1)^{n+m+1}}}{1-t}. \tag{6}$$

Proof of Theorem 3. The proofs of statements (a), (b), (c) and (d) follow from the fact that under their assumptions the expression of the Lefschetz zeta function (6) is not compatible with (3), and consequently Theorem 5 implies that the map f has infinitely many hyperbolic periodic points.

By formula (1) the Lefschetz number of f is $L(f) = 1 + (-1)^n a + (-1)^m b + (-1)^{n+m} D$. Therefore, by the Lefschetz fixed point theorem it follows immediately the first part of the statements (e), (f), (g) and (h).

When f if of finite type the second part of statements (e), (f), (g) and (h) follow easily from (5) and Theorem 5. □

5. $\mathcal{C}^1$ SELF–MAPS ON $\mathbb{C}P^n$ AND $\mathbb{H}P^n$

5.1. **Case $\mathbb{C}P^n$.** For $n \geq 1$ let $f : \mathbb{C}P^n \to \mathbb{C}P^n$ be a $\mathcal{C}^1$ map. The homological groups of $\mathbb{C}P^n$ over $\mathbb{Q}$ are of the form

$$H_q(\mathbb{C}P^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, 2, 4, ..., 2n\}, \\ 0 & \text{otherwise.} \end{cases}$$

The induced linear maps are $f_{*q} = (a^{\frac{q}{2}})$ for $q \in \{0, 2, 4, ..., 2n\}$ with $a \in \mathbb{Z}$, and $f_{*q} = (0)$ otherwise (see for more details [15, Corollary 5.28]).

From (2) the Lefschetz zeta function of f has the form

$$\mathcal{Z}_f(t) = \left(\prod_q (1 - a^{q/2} t) \right)^{-1}, \tag{7}$$

where q runs over $\{0, 2, 4, ..., 2n\}$.

5.2. **Case $\mathbb{H}P^n$.** For $n \geq 1$ let $f : \mathbb{H}P^n \to \mathbb{H}P^n$ be a $\mathcal{C}^1$ map. The homological groups of $\mathbb{H}P^n$ over $\mathbb{Q}$ are of the form

$$H_q(\mathbb{H}P^n, \mathbb{Q}) = \begin{cases} \mathbb{Q} & \text{if } q \in \{0, 4, 8, ..., 4n\}, \\ 0 & \text{otherwise.} \end{cases}$$

The induced linear maps are $f_{*q} = (a^{\frac{q}{4}})$ for $q \in \{0, 4, 8, ..., 4n\}$ with $a \in \mathbb{Z}$, and $f_{*q} = (0)$ otherwise (see for more details [15, Corollary 5.33]).

From (2) the Lefschetz zeta function of f has the form

$$\mathcal{Z}_f(t) = \left(\prod_q (1 - a^{q/4} t) \right)^{-1}, \tag{8}$$

where q runs over $\{0, 4, 8, ..., 4n\}$.

Proof of Theorem 4. First we assume that $f : \mathbb{C}P^n \to \mathbb{C}P^n$. Suppose that $a \notin \{-1, 0, 1\}$, therefore the morphology of the Lefschetz zeta function given by (7) is not compatible with (3), and Theorem 5 implies that the map f has infinitely many hyperbolic periodic points. This proves statement (a).

By formula (1) the Lefschetz number of f is

$$L(f) = 1 + \sum_{q=2,\dots,2n} (-1)^q a^{q/2} = 1 + \sum_{k=1}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}.$$

If $a = -1$ then $L(f) = (1 - (-1)^{n+1})/2$ which is different from zero for n even. If $a = 0$ then $L(f) = 1$. If $a = 1$ then $L(f) = n + 1$. Therefore, by the Lefschetz fixed point theorem it follows the first part of statement (b).

Now assume that f is of finite type. From (7) and if $a = -1$ we obtain thaqt

$$\mathcal{Z}_f(t) = \begin{cases} (1 - t^2)^{1-n}(1 - t)^{-1} & \text{if } n \text{ is even,} \\ (1 - t^2)^{-n} & \text{if } n \text{ is odd.} \end{cases}$$

If $a = 0$ then $\mathcal{Z}_f(t) = 1 - t$. If $a = 1$ then $\mathcal{Z}_f(t) = (1 - t)^{-n}$. So, from Theorem 5 and the definition of $\mathrm{MPer}_L(f)$ we get the second part of statement (b).

The proof for a self–map defined on $\mathbb{H}P^n$ is analogous to the described one changing the role of $q/2$ by $q/4$, and the proof is over. $\square$

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