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ON THE NUMBER OF LIMIT CYCLES OF A CLASS OF POLYNOMIAL DIFFERENTIAL SYSTEMS

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Abstract. We study the number of limit cycles of the polynomial differential systems of the form

$$\dot{x} = y - g_1(x) - f_1(x)y, \quad \dot{y} = -x - g_2(x) - f_2(x)y,$$

where g_1, f_1, g_2 and f_2 are polynomials of a given degree. Note that when $g_1(x) = f_1(x) = 0$ we obtain the generalized polynomial Liénard differential systems. We provide an accurate upper bound of the maximum number of limit cycles that the above system can have bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of first and second order.

1. Introduction

The second part of the 16–th Hilbert's problem wants to find an upper bound on the maximum number of limit cycles that a polynomial vector field of a fixed degree can have. In this paper we will try to give a partial answer to this problem for the class of polynomial differential systems given by

$$\dot{x} = y - g_1(x) - f_1(x)y, \quad \dot{y} = -x - g_2(x) - f_2(x)y. \tag{1}$$

Note that when $g_1(x) = f_1(x) = 0$ coincide with the generalized polynomial Liénard differential systems. The classical polynomial Liénard differential systems are

$$\dot{x} = y, \quad \dot{y} = -x - f(x)y, \tag{2}$$

where $f(x)$ is a polynomial in the variable x of degree n. For these systems in 1977 Lins, de Melo and Pugh [13] stated the conjecture that if $f(x)$ has degree $n \geq 1$ then system (2) has at most $[n/2]$ limit cycles. They prove this conjecture for $n = 1, 2$. The conjecture for $n \in \{3, 4, 5\}$ is still open. For $n \geq 6$ Dumortier, Panazzolo and Roussarie in [6] showed that the conjecture is not true.

Many of the results on the limit cycles of polynomial differential systems have been obtained by considering limit cycles which bifurcate from a single degenerate singular point (i.e. from a Hopf bifurcation), that are called *small amplitude limit cycles*, see for instance [16]. There are partial results concerning the maximum number of small amplitude limit cycles for Liénard polynomial differential systems. The number of small amplitude limit cycles gives a lower bound for the maximum number of limit cycles that a polynomial differential system can have.

There are many results concerning the existence of small amplitude limit cycles for the following generalization of the classical Liénard polynomial differential

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system (2)

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)y, \tag{3}$$

where $f(x)$ and $g(x)$ are polynomials in the variable x of degrees n and m, respectively. We denote by $H(m,n)$ the maximum number of limit cycles that systems (3) can have. This number is usually called the *Hilbert number* for systems (3).

(i) In 1928 Liénard [12] proved that if $m = 1$ and $F(x) = \int_0^x f(s)ds$ is a continuous odd function, which has a unique root at $x = a$ and is monotone increasing for $x \geq a$, then equation (3) has a unique limit cycle.
(ii) In 1973 Rychkov [22] proved that if $m = 1$ and $F(x)$ is an odd polynomial of degree five, then equation (3) has at most two limit cycles.
(iii) In 1977 Lins, de Melo and Pugh [13] proved that $H(1,1) = 0$ and $H(1,2) = 1$.
(iv) In 1998 Coppel [5] proved that $H(2,1) = 1$.
(v) Dumortier, Li and Rousseau in [9] and [7] proved that $H(3,1) = 1$.
(vi) In 1997 Dumortier and Chengzhi [8] proved that $H(2,2) = 1$.

Up to now and as far as we know only for these four cases ((iii)-(vi)) the Hilbert number for systems (3) has been determined.

The maximum number of small amplitude limit cycles for systems (3) is denoted by $\hat{H}(m,n)$. Blows, Lloyd and Lynch, [2], [17] and [18] have used inductive arguments in order to prove the following results.

(I) If g is odd then $\hat{H}(m,n) = [n/2]$.
(II) If f is even then $\hat{H}(m,n) = n$, whatever g is.
(III) If f is odd then $\hat{H}(m, 2n+1) = [(m-2)/2] + n$.
(IV) If $g(x) = x + g_e(x)$, where g_e is even then $\hat{H}(2m,2) = m$.

Christopher and Lynch [4], [19], [20], [21] have developed a new algebraic method for determining the Liapunov quantities of systems (3) and proved some other bounds for $\hat{H}(m,n)$ for different m and n.

(V) $\hat{H}(m,2) = [(2m+1)/3]$.
(VI) $\hat{H}(2,n) = [(2n+1)/3]$.
(VII) $\hat{H}(m,3) = 2[(3m+2)/8]$ for all $1 < m \leq 50$.
(VIII) $\hat{H}(3,n) = 2[(3n+2)/8]$ for all $1 < m \leq 50$.
(IX) $\hat{H}(4,k) = \hat{H}(k,4)$, $k = 6,7,8,9$ and $\hat{H}(5,6) = \hat{H}(6,5)$.

In 1998 Gasull and Torregrosa [10] obtained upper bounds for $\hat{H}(7,6)$, $\hat{H}(6,7)$, $\hat{H}(7,7)$ and $\hat{H}(4,20)$.

In 2006 Yu and Han in [24] give some accurate values of $\hat{H}(m,n) = \hat{H}(n,m)$, for $n = 4$, $m = 10,11,12,13$; $n = 5$, $m = 6,7,8,9$; $n = 6$, $m = 5,6$, see also [15] for a table with all the specific values.

In 2010 Llibre, Mereu and Teixeira [15] compute the maximum number of limit cycles $\tilde{H}_k(m,n)$ of systems (3) which bifurcate from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$, using the averaging theory of order k, for $k = 1,2,3$.

In this paper first we consider the system

$$\begin{aligned} \dot{x} &= y - \varepsilon(g_{11}(x) + f_{11}(x)y), \\ \dot{y} &= -x - \varepsilon(g_{21}(x) + f_{21}(x)y), \end{aligned} \tag{4}$$

where g_{11}, f_{11}, g_{21}, f_{21} have degree l, k, m and n respectively, and ε is a small parameter.

Theorem 1. *For $|\varepsilon|$ sufficiently small the maximum number of limit cycles of the generalized Liénard polynomial differential systems* (4) *bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of first order is*

$$\lambda_1 = \max\{[n/2], [(k-1)/2]\}. \tag{5}$$

The proof of Theorem 1 is given in Section 3.

Now we consider the system

$$\begin{aligned} \dot{x} &= y - \varepsilon(g_{11}(x) + f_{11}(x)y) - \varepsilon^2(g_{12}(x) + f_{12}(x)y), \\ \dot{y} &= -x - \varepsilon(g_{21}(x) + f_{21}(x)y) - \varepsilon^2(g_{22}(x) + f_{22}(x)y), \end{aligned} \tag{6}$$

where g_{11} and g_{12} have degree l; f_{11} and f_{12} have degree k; g_{21} and g_{22} have degree m; and f_{21}, f_{22} have degree n. Furthermore ε is a small parameter.

Theorem 2. *For $|\varepsilon|$ sufficiently small the maximum number of limit cycles of the generalized Liénard polynomial differential systems* (6) *bifurcating from the periodic orbits of the linear center $\dot{x} = y$, $\dot{y} = -x$ using the averaging theory of second order is $\lambda_3 = \max\{\lambda_1, \lambda_2\}$ where*

$$\begin{aligned} \lambda_2 = \max\big\{&\mu + [(m-1)/2], \mu + [l/2], [(n-1)/2] + [m/2], [k/2] + [m/2] - 1, \\ &[(n-1)/2] + [(l-1)/2] + 1, [k/2] + [(l-1)/2]\big\}, \end{aligned} \tag{7}$$

with $\mu = \min\{[n/2], [(k-1)/2]\}$.

The proof of Theorem 2 is given in Section 4.

The results that we shall use from the averaging theory of first and second order for computing limit cycles are presented in Section 2.

2. The averaging theory of first and second order

The averaging theory for studying specifically limit cycles up to first order in ε was developed many years ago, and can be found in [23], [11] and [3]. The averaging theory for computing limit cycles up to second order in ε was developed in [14] and [3]. It is summarized as follows.

Consider the differential system

$$\dot{x} = \varepsilon F_1(t, x) + \varepsilon^2 F_2(t, x) + \varepsilon^3 R(t, x, \varepsilon), \tag{8}$$

where $F_1, F_2 \colon \mathbb{R} \times D \to \mathbb{R}$, $R \colon \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following conditions hold.

(i) $F_1(t, \cdot) \subset C^2(D)$, $F_2(t, \cdot) \subset C^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, R are locally Lipschitz with respect to x, and R is twice differentiable with respect to ε.

We define $F_{k0} \colon D \to \mathbb{R}$ for $k = 1, 2$ as

$$\begin{aligned} F_{10}(z) &= \frac{1}{T} \int_0^T F_1(s, z)\, ds, \\ F_{20}(z) &= \frac{1}{T} \int_0^T \big[D_z F_1(s, z) \cdot y_1(s, z) + F_2(s, z)\big]\, ds \end{aligned}$$

where

$$y_1(s,z) = \int_0^s F_1(t,z)\,dt.$$

(ii) For $V \subset D$ an open and bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, there exists $a_\varepsilon \in V$ such that $F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) = 0$ and $d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0$.

Then for $|\varepsilon| > 0$ sufficiently small there exists a T-periodic solution $\phi(\cdot, \varepsilon)$ of the system such that $\phi(0, a_\varepsilon) \to a_\varepsilon$ when $\varepsilon \to 0$.

The expression $d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} + \varepsilon F_{20} \colon V \to \mathbb{R}^n$ at the fixed point a_ε is not zero. A sufficient condition in order that this inequality holds is that the Jacobian of the function $F_{10} + \varepsilon F_{20}$ at a_ε is not zero.

If F_{10} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are mainly the zeros of F_{10} for ε sufficiently small. In this case the previous result provides the *averaging theory of first order.*

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are mainly the zeros of F_{20} for ε sufficiently small. In this case the previous result provides the *averaging theory of second order.*

3. Proof of Theorem 1

We shall need the first order averaging theory to prove Theorem 1. We write system (4) in polar coordinates (r, θ) where

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r > 0.$$

In this way system (4) will become written in the standard form for applying the averaging theory. If we write

(9)

$$f_{11}(x) = \sum_{i=0}^{l} a_{i,1} x^i, \quad f_{21}(x) = \sum_{i=0}^{n} a_{i,2} x^i, \quad g_{11}(x) = \sum_{i=0}^{k} b_{i,1} x^i, \quad g_{21}(x) = \sum_{i=0}^{m} b_{i,2} x^i,$$

then system (4) becomes

$$\begin{aligned}
\dot r &= -\varepsilon\Big(\sum_{i=0}^{n} a_{i,2} r^{i+1} \cos^i\theta \sin^2\theta + \sum_{i=0}^{m} b_{i,2} r^i \cos^i\theta \sin\theta \\
&\quad + \sum_{i=0}^{l} a_{i,1} r^{i+1} \cos^{i+1}\theta \sin\theta + \sum_{i=0}^{k} b_{i,1} r^i \cos^{i+1}\theta \Big), \\
\dot\theta &= -1 - \frac{\varepsilon}{r}\Big(\sum_{i=0}^{n} a_{i,2} r^{i+1} \cos^{i+1}\theta \sin\theta + \sum_{i=0}^{m} b_{i,2} r^i \cos^{i+1}\theta \\
&\quad - \sum_{i=0}^{l} a_{i,1} r^{i+1} \cos^i\theta \sin^2\theta - \sum_{i=0}^{k} b_{i,1} r^i \cos^i\theta \sin\theta \Big).
\end{aligned} \tag{10}$$

Now taking θ as the new independent variable, system (10) becomes

$$\begin{aligned}\frac{dr}{d\theta} = \varepsilon\Big(&\sum_{i=0}^{n} a_{i,2} r^{i+1}\cos^i\theta\sin^2\theta + \sum_{i=0}^{m} b_{i,2} r^i \cos^i\theta\sin\theta \\ &+ \sum_{i=0}^{l} a_{i,1} r^{i+1}\cos^{i+1}\theta\sin\theta + \sum_{i=0}^{k} b_{i,1} r^i\cos^{i+1}\theta\Big) + O(\varepsilon^2),\end{aligned}$$

and

$$\begin{aligned}F_{10}(r) = \frac{1}{2\pi}\int_0^{2\pi}\Big(&\sum_{i=0}^{n} a_{i,2} r^{i+1}\cos^i\theta\sin^2\theta + \sum_{i=0}^{m} b_{i,2} r^i \cos^i\theta\sin\theta \\ &+ \sum_{i=0}^{l} a_{i,1} r^{i+1}\cos^{i+1}\theta\sin\theta + \sum_{i=0}^{k} b_{i,1} r^i\cos^{i+1}\theta\Big)\, d\theta.\end{aligned}$$

Now using the expressions for the integrals in the Appendix (note that $\alpha_{k+1} = (2k+1)\alpha_k$), we get

$$\begin{aligned}F_{10}(r) &= \frac{1}{2\pi}\sum_{i=0}^{[n/2]} a_{2i,2} r^{2i+1}\int_0^{2\pi}\cos^{2i}\theta\sin^2\theta\, d\theta \\ &\quad + \frac{1}{2\pi}\sum_{i=0}^{[(k-1)/2]} b_{2i+1,1} r^{2i+1}\int_0^{2\pi}\cos^{2i+2}\theta\, d\theta \\ &= r\sum_{i=0}^{[n/2]}\frac{a_{2i,2}\alpha_i}{2^{i+1}(i+1)!} r^{2i} + r\sum_{i=0}^{[(k-1)/2]}\frac{b_{2i+1,1}\alpha_{i+1}}{2^{i+1}(i+1)!} r^{2i} \\ &= r\sum_{i=0}^{[n/2]}\frac{a_{2i,2}\alpha_i}{2^{i+1}(i+1)!} r^{2i} + r\sum_{i=0}^{[(k-1)/2]}\frac{b_{2i+1,1}(2i+1)\alpha_i}{2^{i+1}(i+1)!} r^{2i}.\end{aligned} \tag{11}$$

Then the polynomial $F_{10}(r)$ has at most λ_1 (see (5)) positive roots, and we can choose the coefficients $a_{2i,2}$ and $b_{2i+1,1}$ in such a way that $F_{10}(r)$ has exactly λ_1 simple positive roots. Hence Theorem 1 is proved.

4. Proof of Theorem 2

We write f_{11}, f_{21}, g_{11} and g_{21} as in (9) and

$$f_{12}(x) = \sum_{i=0}^{l} c_{i,1}x^i, \quad f_{22}(x) = \sum_{i=0}^{n} c_{i,2}x^i, \quad g_{12}(x) = \sum_{i=0}^{k} d_{i,1}x^i, \quad g_{22}(x) = \sum_{i=0}^{m} d_{i,2}x^i.$$

Then system (4) in polar coordinates (r,θ) with $r>0$ becomes

$$
\begin{aligned}
\dot r = &-\varepsilon\Big(\sum_{i=0}^{n} a_{i,2}r^{i+1}\cos^i\theta\sin^2\theta+\sum_{i=0}^{m} b_{i,2}r^{i}\cos^i\theta\sin\theta\\
&+\sum_{i=0}^{l} a_{i,1}r^{i+1}\cos^{i+1}\theta\sin\theta+\sum_{i=0}^{k} b_{i,1}r^{i}\cos^{i+1}\theta\Big)\\
&-\varepsilon^2\Big(\sum_{i=0}^{n} c_{i,2}r^{i+1}\cos^i\theta\sin^2\theta+\sum_{i=0}^{m} d_{i,2}r^{i}\cos^i\theta\sin\theta\\
&+\sum_{i=0}^{l} c_{i,1}r^{i+1}\cos^{i+1}\theta\sin\theta+\sum_{i=0}^{k} d_{i,1}r^{i}\cos^{i+1}\theta\Big),\\
\dot\theta = &-1-\frac{\varepsilon}{r}\Big(\sum_{i=0}^{n} a_{i,2}r^{i+1}\cos^{i+1}\theta\sin\theta+\sum_{i=0}^{m} b_{i,2}r^{i}\cos^{i+1}\theta\\
&-\sum_{i=0}^{l} a_{i,1}r^{i+1}\cos^{i}\theta\sin^2\theta-\sum_{i=0}^{k} b_{i,1}r^{i}\cos^{i}\theta\sin\theta\Big)\\
&-\frac{\epsilon^2}{r}\Big(\sum_{i=0}^{n} c_{i,2}r^{i+1}\cos^{i+1}\theta\sin\theta+\sum_{i=0}^{m} d_{i,2}r^{i}\cos^{i+1}\theta\\
&-\sum_{i=0}^{l} c_{i,1}r^{i+1}\cos^{i}\theta\sin^2\theta-\sum_{i=0}^{k} d_{i,1}r^{i}\cos^{i}\theta\sin\theta\Big).
\end{aligned}
\tag{12}
$$

Taking θ as the new independent variable, system (12) becomes

$$\frac{dr}{d\theta}=\varepsilon F_1(\theta,r)+\varepsilon^2F_2(\theta,r)+O(\varepsilon^3),$$

where

$$
\begin{aligned}
F_1(\theta,r)=&\sum_{i=0}^{n} a_{i,2}r^{i+1}\cos^i\theta\sin^2\theta+\sum_{i=0}^{m} b_{i,2}r^{i}\cos^i\theta\sin\theta\\
&+\sum_{i=0}^{l} a_{i,1}r^{i+1}\cos^{i+1}\theta\sin\theta+\sum_{i=0}^{k} b_{i,1}r^{i}\cos^{i+1}\theta,
\end{aligned}
$$

and

$$F_2(\theta,r)=\mathcal{I}(r,\theta)+r\mathcal{II}(r,\theta), \tag{13}$$

where

$$
\begin{aligned}
\mathcal{I}(r,\theta)=&\sum_{i=0}^{n} c_{i,2}r^{i+1}\cos^i\theta\sin^2\theta+\sum_{i=0}^{m} d_{i,2}r^{i}\cos^i\theta\sin\theta\\
&+\sum_{i=0}^{l} c_{i,1}r^{i+1}\cos^{i+1}\theta\sin\theta+\sum_{i=0}^{k} d_{i,1}r^{i}\cos^{i+1}\theta,
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{II}(r,\theta) = -\Big(&\sum_{i=0}^{n} a_{i,2} r^i \cos^i\theta \sin^2\theta + \sum_{i=0}^{m} b_{i,2} r^{i-1}\cos^i\theta\sin\theta \\
&+ \sum_{i=0}^{l} a_{i,1} r^i \cos^{i+1}\theta\sin\theta + \sum_{i=0}^{k} b_{i,1} r^{i-1}\cos^{i+1}\theta\Big) \\
&\cdot\Big(\sum_{i=0}^{n} a_{i,2} r^i \cos^{i+1}\theta\sin\theta + \sum_{i=0}^{m} b_{i,2} r^{i-1}\cos^{i+1}\theta \\
&- \sum_{i=0}^{l} a_{i,1} r^i \cos^i\theta\sin^2\theta - \sum_{i=0}^{k} b_{i,1} r^{i-1}\cos^i\theta\sin\theta\Big).
\end{aligned}
$$

In order to compute $F_{20}(r)$ we need that F_{10} be identically zero. Then from (11)

$$
\begin{aligned}
b_{2i+1,1} &= -\frac{a_{2i,2}}{2i+1}, \quad i = 0,1,\ldots,\mu, \\
b_{2i+1,1} &= a_{2i,2} = 0, \quad i = \mu+1,\ldots,\lambda_1.
\end{aligned} \tag{14}
$$

First we compute

$$
\begin{aligned}
\frac{d}{dr}F_1(\theta,r) &= \sum_{i=0}^{n}(i+1)a_{i,2}r^i\cos^i\theta\sin^2\theta + \sum_{i=0}^{m} i b_{i,2} r^{i-1}\cos^i\theta\sin\theta \\
&\quad + \sum_{i=0}^{l}(i+1)a_{i,1}r^i\cos^{i+1}\theta\sin\theta + \sum_{i=0}^{k} i b_{i,1} r^{i-1}\cos^{i+1}\theta \\
&= \sum_{i=0}^{\mu} a_{2i,2} r^{2i}\cos^{2i}\theta((2i+1)\sin^2\theta - \cos^2\theta) \\
&\quad + \sum_{i=0}^{[(n-1)/2]}(2i+2)a_{2i+1,2}r^{2i+1}\cos^{2i+1}\theta\sin^2\theta \\
&\quad + \sum_{i=0}^{m} i b_{i,2} r^{i-1}\cos^i\theta\sin\theta + \sum_{i=0}^{l}(i+1)a_{i,1}r^i\cos^{i+1}\theta\sin\theta \\
&\quad + \sum_{i=0}^{[k/2]} 2i b_{2i,1} r^{2i-1}\cos^{2i+1}\theta,
\end{aligned}
$$

and $y_1(\theta,r)=\displaystyle\int_0^\theta F_1(\psi,r)\,d\psi$. To do it we rewrite

$$\begin{aligned}
F_1(\theta,r) = & \sum_{i=0}^{[(n-1)/2]} a_{2i+1,2}r^{2i+2}\cos^{2i+1}\theta - \sum_{i=0}^{[(n-1)/2]} a_{2i+1,2}r^{2i+2}\cos^{2i+3}\theta \\
& + \sum_{i=0}^{[n/2]} a_{2i,2}r^{2i+1}\cos^{2i}\theta - \sum_{i=0}^{[n/2]} a_{2i,2}r^{2i+1}\cos^{2i+2}\theta \\
& + \sum_{i=0}^{m} b_{i,2}r^{i}\cos^{i}\theta\sin\theta + \sum_{i=0}^{l} a_{i,1}r^{i+1}\cos^{i+1}\theta\sin\theta \\
& + \sum_{i=0}^{[(k-1)/2]} b_{2i+1,1}r^{2i+1}\cos^{2i+2}\theta + \sum_{i=0}^{[k/2]} b_{2i,1}r^{2i}\cos^{2i+1}\theta \\
= & \sum_{i=0}^{[(n-1)/2]} a_{2i+1,2}r^{2i+2}\cos^{2i+1}\theta - \sum_{i=0}^{[(n-1)/2]} a_{2i+1,2}r^{2i+2}\cos^{2i+3}\theta \\
& + \sum_{i=0}^{\mu} a_{2i,2}r^{2i+1}\cos^{2i}\theta - \sum_{i=0}^{\mu} \frac{2i+2}{2i+1}a_{2i,2}r^{2i+1}\cos^{2i+2}\theta \\
& + \sum_{i=0}^{m} b_{i,2}r^{i}\cos^{i}\theta\sin\theta + \sum_{i=0}^{l} a_{i,1}r^{i+1}\cos^{i+1}\theta\sin\theta \\
& + \sum_{i=0}^{[k/2]} b_{2i,1}r^{2i}\cos^{2i+1}\theta.
\end{aligned}$$

Then taking into account that

$$\frac{1}{2^{2i}}\binom{2i}{i}\theta - \frac{2i+2}{2i+1}\frac{1}{2^{2i+2}}\binom{2i+2}{i+1}\theta = 0,$$

and using the integrals of the Appendix we obtain

$$\begin{aligned}
y_1(\theta,r) = & \sum_{i=0}^{[(n-1)/2]} a_{2i+1,2}r^{2i+2}\sum_{l=0}^{i+1}\tilde{\gamma}_{i,l}\sin((2l+1)\theta) \\
& + \sum_{i=0}^{\mu} a_{2i,2}r^{2i+1}\sum_{l=1}^{i+1}\tilde{\beta}_{i,l}\sin(2l\theta) \\
& + \sum_{i=0}^{m}\frac{b_{i,2}}{i+1}r^{i}(1-\cos^{i+1}\theta) + \sum_{i=0}^{l}\frac{a_{i,1}}{i+2}r^{i+1}(1-\cos^{i+2}\theta) \\
& + \sum_{i=0}^{[k/2]} b_{2i,1}r^{2i}\sum_{l=0}^{i}\gamma_{i,l}\sin((2l+1)\theta),
\end{aligned}$$

where

$$\tilde{\gamma}_{i,l} = \begin{cases}\gamma_{i,l}-\gamma_{i+1,l}, & 0\le l\le i,\\ -\gamma_{i+1,i+1}, & l=i+1\end{cases}, \quad \tilde{\beta}_{i,l} = \begin{cases}\beta_{i,l}-2(i+1)\beta_{i+1,l}/(2i+1), & 0\le l\le i,\\ -2(i+1)\beta_{i+1,i+1}/(2i+1), & l=i+1.\end{cases}$$

Again using the integrals of the Appendix we conclude that

$$\mathcal{III} = \int_0^{2\pi} \frac{d}{dr} F_1(\theta, r) y_1(\theta, r)\, d\theta = rP_1(r^2)$$

where $P_1(r^2)$ is equal to

$$\begin{aligned}
&\sum_{i=0}^{\mu} \sum_{j=0}^{[(m-1)/2]} A_{i,j}(r^2)^{i+j} + \sum_{i=0}^{\mu} \sum_{j=0}^{[l/2]} B_{i,j}(r^2)^{i+j} + \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[m/2]} C_{i,j}(r^2)^{i+j} \\
&+ \sum_{i=0}^{[(n-1)/2]} \sum_{j=0}^{[(l-1)/2]} D_{i,j}(r^2)^{i+j+1} + \sum_{i=0}^{[k/2]} \sum_{j=0}^{[m/2]} E_{i,j}(r^2)^{i+j-1} + \sum_{i=0}^{[k/2]} \sum_{j=0}^{[(l-1)/2]} F_{i,j}(r^2)^{i+j} \\
&+ \sum_{i=0}^{[m/2]} \sum_{j=0}^{[(n-1)/2]} G_{i,j}(r^2)^{i+j} + \sum_{i=0}^{[(m-1)/2]} \sum_{j=0}^{\mu} H_{i,j}(r^2)^{i+j} + \sum_{i=0}^{[m/2]} \sum_{j=0}^{[k/2]} I_{i,j}(r^2)^{i+j-1} \\
&+ \sum_{i=0}^{[(l-1)/2]} \sum_{j=0}^{[(n-1)/2]} J_{i,j}(r^2)^{i+j+1} + \sum_{i=0}^{[l/2]} \sum_{j=0}^{\mu} K_{i,j}(r^2)^{i+j} + \sum_{i=0}^{[(l-1)/2]} \sum_{j=0}^{[k/2]} L_{i,j}(r^2)^{i+j}
\end{aligned}$$

with

$$\begin{aligned}
A_{i,j} &= \frac{\pi a_{2i,2} b_{2j+1,2} \alpha_{i+j+1}}{2^{i+j+1}(i+j+2)!}, \quad B_{i,j} = \frac{\pi a_{2i,2} a_{2j,1} \alpha_{i+j+1}}{2^{i+j+1}(i+j+2)!}, \\
C_{i,j} &= -\frac{\pi(2i+2) a_{2i+1,2} b_{2j,2} \alpha_{i+j+1}}{(2j+1) 2^{i+j+1}(i+j+2)!}, \quad D_{i,j} = -\frac{\pi(2i+2) a_{2i+1,2} a_{2j+1,1} \alpha_{i+j+2}}{(2j+3) 2^{i+j+2}(i+j+3)!}, \\
E_{i,j} &= \frac{\pi 2i b_{2i,1} b_{2j,2} \alpha_{i+j+1}}{(2j+1) 2^{i+j}(i+j+1)!}, \quad F_{i,j} = \frac{\pi 2i b_{2i,1} a_{2j+1,1} \alpha_{i+j+2}}{(2j+3) 2^{i+j+1}(i+j+2)!}, \\
G_{i,j} &= \pi \sum_{s=0}^{j+1} 2i b_{2i,2} a_{2j+1,2} \tilde{\gamma}_{j,s} C_{i,s}, \quad H_{i,j} = \pi \sum_{s=1}^{j+1} (2i+1) b_{2i+1,2} a_{2j,2} \tilde{\beta}_{j,s} K_{i,s}, \\
I_{i,j} &= \pi \sum_{s=0}^{j} 2i b_{2i,2} b_{2j,1} \gamma_{j,s} C_{i,s}, \quad J_{i,j} = \pi \sum_{s=0}^{j+1} (2i+2) a_{2i+1,1} a_{2j+1,2} \tilde{\gamma}_{j,s} C_{i,s}, \\
K_{i,j} &= \pi \sum_{s=1}^{j+1} (2i+1) a_{2i,1} a_{2j,2} \tilde{\beta}_{j,s} K_{i,s}, \quad L_{i,j} = \pi \sum_{s=0}^{j} (2i+2) a_{2i+1,1} b_{2j,1} \gamma_{j,s} C_{i,s}.
\end{aligned}$$

Then $P_1(r^2)$ is a polynomial in the variable r^2 which has degree λ_2, see (7). In the notation of (13) we have that

$$\begin{aligned}
&\int_0^{2\pi} \mathcal{I}(r, \theta)\, d\theta \\
&= \sum_{i=0}^{[n/2]} c_{2i,2} r^{2i+1} \int_0^{2\pi} \cos^{2i}\theta \sin^2\theta\, d\theta + \sum_{i=0}^{[(k-1)/2]} d_{2i+1,1} r^{2i+1} \int_0^{2\pi} \cos^{2i+2}\theta\, d\theta \\
&= \pi r \sum_{i=0}^{[n/2]} \frac{c_{2i,2}\alpha_i}{2^i(i+1)!} r^{2i} + \pi r \sum_{i=0}^{[(k-1)/2]} \frac{d_{2i+1,1}\alpha_i(2i+1)}{2^i(i+1)!} r^{2i} = rP_2(r^2),
\end{aligned}$$

where P_2 is a polynomial in the variable r^2 of degree λ_1.

Furthermore

$$
\begin{aligned}
\int_0^{2\pi} \mathcal{II}(r,\theta)\,d\theta = & -2\sum_{i=0}^{n}\sum_{j=0}^{m} a_{i,2}b_{j,2}r^{i+j-1}\int_0^{2\pi}\cos^{i+j+1}\theta\sin^2\theta\,d\theta \\
& + \sum_{i=0}^{n}\sum_{j=0}^{l} a_{i,2}a_{j,1}r^{i+j}\int_0^{2\pi}\cos^{i+j}\theta\sin^4\theta\,d\theta \\
& + \sum_{i=0}^{m}\sum_{j=0}^{k} b_{i,2}b_{j,1}r^{i+j-2}\int_0^{2\pi}\cos^{i+j}\theta\sin^2\theta\,d\theta \\
& - \sum_{i=0}^{l}\sum_{j=0}^{n} a_{i,1}a_{j,2}r^{i+j}\int_0^{2\pi}\cos^{i+j+2}\theta\sin^2\theta\,d\theta \\
& + 2\sum_{i=0}^{l}\sum_{j=0}^{k} a_{i,1}b_{j,1}r^{i+j-1}\int_0^{2\pi}\cos^{i+j+1}\theta\sin^2\theta\,d\theta \\
& - \sum_{i=0}^{k}\sum_{j=0}^{m} b_{i,1}b_{j,2}r^{i+j-2}\int_0^{2\pi}\cos^{i+j+2}\theta\,d\theta.
\end{aligned} \tag{15}
$$

Using relation (14) we get

$$
\begin{aligned}
\int_0^{2\pi} \mathcal{II}(r,\theta)\,d\theta = & -2\sum_{i=0}^{\mu}\sum_{j=0}^{[(m-1)/2]} a_{2i,2}b_{2j+1,2}r^{2i+2j}\int_0^{2\pi}\cos^{2i+2j+2}\theta\sin^2\theta\,d\theta \\
& - 2\sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[m/2]} a_{2i+1,2}b_{2j,2}r^{2i+2j}\int_0^{2\pi}\cos^{2i+2j+2}\theta\sin^2\theta\,d\theta \\
& - \sum_{i=0}^{\mu}\sum_{j=0}^{[l/2]} a_{2i,2}a_{2j,1}r^{2i+2j}\int_0^{2\pi}\cos^{2i+2j}\theta\sin^2\theta\cos(2\theta)\,d\theta \\
& - \sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[(l-1)/2]} a_{2i+1,2}a_{2j+1,1}r^{2i+2j+2}\int_0^{2\pi}\cos^{2i+2j+2}\theta\sin^2\theta\cos(2\theta)\,d\theta \\
& - \sum_{i=0}^{[m/2]}\sum_{j=0}^{[k/2]} b_{2i,2}b_{2j,1}r^{2i+2j-2}\int_0^{2\pi}\cos^{2i+2j}\theta\cos(2\theta)\,d\theta \\
& - \sum_{i=0}^{[(m-1)/2]}\sum_{j=0}^{[(k-1)/2]} b_{2i+1,2}b_{2j+1,1}r^{2i+2j}\int_0^{2\pi}\cos^{2i+2j+2}\theta\cos(2\theta)\,d\theta \\
& - 2\sum_{i=0}^{[l/2]}\sum_{j=0}^{\mu} a_{2i,1}\frac{a_{2j,2}}{2j+1}r^{2i+2j}\int_0^{2\pi}\cos^{2i+2j+2}\theta\sin^2\theta\,d\theta \\
& + 2\sum_{i=0}^{[(l-1)/2]}\sum_{j=0}^{[k/2]} a_{2i+1,1}b_{2j,1}r^{2i+2j}\int_0^{2\pi}\cos^{2i+2j+2}\theta\sin^2\theta\,d\theta,
\end{aligned}
$$

where the third and fourth (resp. fifth and sixth) lines come from taking together the second and fourth (resp. third and sixth) lines of (15). We write

$\displaystyle\int_0^{2\pi} \mathcal{II}(r,\theta)\,d\theta = P_3(r^2)$ where P_3 is equal to

$$
\begin{aligned}
P_3(r^2) = &\sum_{i=0}^{\mu}\sum_{j=0}^{[(m-1)/2]} \bar{A}_{i,j}(r^2)^{i+j} + \sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[m/2]} \bar{B}_{i,j}(r^2)^{i+j} + \sum_{i=0}^{\mu}\sum_{j=0}^{[l/2]} \bar{C}_{i,j}(r^2)^{i+j}\\
&+ \sum_{i=0}^{[(n-1)/2]}\sum_{j=0}^{[(l-1)/2]} \bar{D}_{i,j}(r^2)^{i+j+1} + \sum_{i=0}^{[m/2]}\sum_{j=0}^{[k/2]} \bar{E}_{i,j}(r^2)^{i+j-1} + \sum_{i=0}^{[(l-1)/2]}\sum_{j=0}^{[k/2]} \bar{F}_{i,j}(r^2)^{i+j}
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{A}_{i,j} &= \frac{\pi a_{2i,2} b_{2j+1,2}\alpha_{i+j+1}}{2^{i+j}(i+j+2)!}\Big(-1+\frac{i+j+1}{2i+1}\Big), \quad \bar{B}_{i,j} = -\frac{\pi a_{2i+1,2} b_{2j,2}\alpha_{i+j+1}}{2^{i+j}(i+j+2)!},\\
\bar{C}_{i,j} &= -\frac{\pi a_{2i,2} a_{2j,1}\alpha_{i+j}}{2^{i+j}(i+j+2)!}\Big(i+j-1+\frac{2(i+j)+1}{2i+1}\Big),\\
\bar{D}_{i,j} &= -\frac{\pi(i+j) a_{2i+1,2} a_{2j+1,1}\alpha_{i+j+1}}{2^{i+j+1}(i+j+3)!}, \quad \bar{E}_{i,j} = -\frac{\pi(i+j) b_{2i,2} b_{2j,1}\alpha_{i+j}}{2^{i+j-1}(i+j+1)!},\\
\bar{F}_{i,j} &= \frac{\pi a_{2i+1,1} b_{2j,1}\alpha_{i+j+1}}{2^{i+j}(i+j+2)!}.
\end{aligned}
$$

Then $P_3(r^2)$ is a polynomial in the variable r^2 with degree λ_2, see (7). Then

$$2\pi F_{20} = r(P_1(r^2) + P_2(r^2) + P_3(r^2)).$$

Then to find the real positive roots of F_{20} we must find the zeros of a polynomial in r^2 of degree equal to λ_3. This yields that F_{20} has at most λ_3 real positive roots. Moreover we can choose the coefficients $a_{i,1}, a_{i,2}, b_{i,1}, b_{i,2}, c_{i,1}, c_{i,2}, d_{i,1}, d_{i,2}$ in such a way that F_{20} has exactly λ_3 real positive roots. Hence the theorem is proved.

Appendix: Formulae

In this appendix we recall some formulae that will be used during the paper, see for more details [1]. For $i \geq 0$ we have

$$\int_0^{2\pi} \cos^{2i+1}\theta \sin^2\theta\, d\theta = 0, \quad \int_0^{2\pi} \cos^{i}\theta \sin\theta\, d\theta = 0, \quad \int_0^{2\pi} \cos^{2i+1}\theta\, d\theta = 0,$$

$$\int_0^{2\pi} \cos^{2i}\theta\, d\theta = \frac{\pi\alpha_i}{2^{i-1}i!}, \quad \int_0^{2\pi} \cos^{2i}\theta \sin^2\theta\, d\theta = \frac{\pi\alpha_i}{2^{i}(i+1)!}, \quad \alpha_i = 1{\cdot}3{\cdot}5\cdots(2k-1),$$

$$\int_0^{2\pi} \cos^{2i}\theta \cos(2\theta)\, d\theta = \frac{\pi i\alpha_i}{2^{i-1}(i+1)!}, \quad \int_0^{2\pi} \cos^{2i}\theta \sin^2\theta \cos(2\theta)\, d\theta = \frac{\pi(i-1)\alpha_i}{2^{i}(i+2)!},$$

$$\int_0^{\theta} \cos^{2i}\phi\, d\phi = \frac{1}{2^{2i}}\binom{2i}{i}\theta + \frac{1}{2^{2i}}\sum_{l=1}^{i}\binom{2i}{i+l}\frac{1}{l}\sin(2l\theta) = \frac{1}{2^{2i}}\binom{2i}{i}\theta + \sum_{l=1}^{i}\beta_{i,l}\sin(2l\theta),$$

$$\int_0^{\theta} \cos^{2i+1}\phi\, d\phi = \frac{1}{2^{2i}}\sum_{l=0}^{i}\binom{2i+1}{i-l}\frac{1}{2l+1}\sin((2l+1)\theta) = \sum_{l=0}^{i}\gamma_{i,l}\sin((2l+1)\theta),$$

$$\int_0^{\theta} \cos^{i}\phi \sin\phi\, d\phi = \frac{1}{i+1}(1-\cos^{i+1}\theta),$$

$$\int_0^{\theta} \cos^{2i+1}\theta \sin\theta \sin((2l+1)\theta)\, d\theta = \int_0^{\theta} \cos^{2i}\theta \sin\theta \sin(2l\theta)\, d\theta = 0, \quad l \geq 0,$$

$$\int_0^{2\pi} \cos^{i}\theta \sin((2l+1)\theta)\, d\theta = \int_0^{2\pi} \cos^{i}\theta \sin(2l\theta)\, d\theta = 0, \quad l \geq 0,$$

$$\int_0^{2\pi} \cos^{i}\theta \sin^2\theta \sin((2l+1)\theta)\, d\theta = \int_0^{2\pi} \cos^{i}\theta \sin^2\theta \sin(2l\theta)\, d\theta = 0, \quad l \geq 0,$$

$$\int_0^{2\pi} \cos^{2i}\theta \sin\theta \sin((2l+1)\theta)\, d\theta = \pi C_{i,l}, \quad l \geq 0,$$

$$\int_0^{2\pi} \cos^{2i+1}\theta \sin\theta \sin(2l\theta)\, d\theta = \pi K_{i,l}, \quad l \geq 1,$$

where $C_{i,j}$ and $K_{i,l}$ are nonzero constants.

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