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GLOBAL LINEARIZATION OF PERIODIC DIFFERENCE EQUATIONS

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Abstract. We deal with m-periodic, n-th order difference equations and study whether they can be globally linearized. We give an affirmative answer when $m = n + 1$ and for most of the known examples appearing in the literature. Our main tool is a refinement of the Montgomery-Bochner Theorem.

1. **Introduction.** In this paper we investigate the linearization of periodic difference equations defined on an open subset of $\mathbb{R}$. A difference equation or a recurrence of order n on $\mathbb{R}$ of class $\mathcal{C}^k$ ($k \in \mathbb{N} \cup \{0\} \cup \{\infty\} \cup \{\omega\}$) is an equation of the form:

$$x_{j+n} = f(x_j, x_{j+1}, \ldots, x_{j+n-1}) \tag{1}$$

where $x_i \in \mathbb{R}$ for all $i \in \mathbb{N}$, and f is a $\mathcal{C}^k$ map from an open subset of $\mathbb{R}^n$ into $\mathbb{R}$.

The study of the dynamics of the recurrence (1) is given by the dynamics of the associated map $F : U \to U$, where U is an open subset of $\mathbb{R}^n$, not necessarily connected, and F is given by

$$F(x_1, \ldots, x_n) = (x_2, \ldots, x_n, f(x_1, \ldots, x_n)). \tag{2}$$

We will say that equation (1) is m-periodic if $F^m = \mathrm{Id}$ and m is the smallest natural with this property. Clearly, in this case, $m \geq n$ and the only possibility for $m = n$ is when $f(x_1, \ldots, x_n) = x_1$.

Recall that it is said that a map $F : U \to U$, $\mathcal{C}^k$-*linearizes* on an open set $U \subset \mathbb{R}^n$ if there exists a $\mathcal{C}^k$-homeomorphism, $\psi : U \to \psi(U) \subset \mathbb{R}^n$, for which $G := \psi \circ F \circ \psi^{-1}$ is the restriction of a linear map to $\psi(U)$. The map ψ is called a *linearization of F on U*. When the map F is of the form (2) and the linearized map G is as well of the form (2) then we will say that the associated recurrence (1) linearizes in the corresponding domain.

Notice that real periodic difference equations are a particular case of periodic maps on subsets of $\mathbb{R}^n$. These maps have been largely studied. In order to have a better understanding of our goal when we restrict our interest to maps of the form (2), first we give a brief summary of some of the most relevant results on general periodic maps on $\mathbb{R}^n$.

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It is a well-known result that every periodic $\mathcal{C}^k$ map on $\mathbb{R}$ is either the identity, or 2-periodic and that in this later case it $\mathcal{C}^k$-linearizes (notice that in this situation this is equivalent to say that it is $\mathcal{C}^k$-conjugated to $-\mathrm{Id}$), see for instance [18]. From a classical result of Kerékjártó, see [13] we also know that any $\mathcal{C}^k, k \geq 0$, m-periodic map on $\mathbb{R}^2$ is $\mathcal{C}^0$-linearizable on $\mathbb{R}^2$. This situation changes when $n \geq 3$. In a series of papers, Bing shows that for any $m \geq 2$ there are continuous m-periodic maps in $\mathbb{R}^3$ which are not linearizable, see [6, 7]. On the other hand Montgomery and Bochner prove a local result saying that a $\mathcal{C}^k, k \geq 1$, m-periodic map having a fixed point p is locally $\mathcal{C}^k$-linearizable in a neighborhood of p, see [21] or Theorem 2.1 below. However in [12, 16, 17] it is shown that for $n \geq 7$ there are continuous and also differentiable periodic maps on $\mathbb{R}^n$ without fixed points.

In this context it is natural to wonder whether real m-periodic difference equations linearize. The main result of this paper answers this question when $m = n+1$. We prove:

Theorem A. *Let $U \subset \mathbb{R}^n$ be an open connected set and let $F : U \to U$ given by $F(x_1, \ldots, x_n) = (x_2, \ldots, x_n, f(x_1, \ldots, x_n))$ be a $(n+1)$-periodic recurrence of class $\mathcal{C}^k$. Then F is $\mathcal{C}^k$-linearizable on U. More precisely, f is either increasing or decreasing with respect x_1. If f is decreasing with respect x_1 then F is $\mathcal{C}^k$-conjugated to $L_1(x_1, \ldots, x_n) = (x_2, \ldots, x_n, -\sum_{i=1}^n x_i)$ on $\psi(U)$, where ψ is the $\mathcal{C}^k$-linearization. Otherwise, n is odd and F is $\mathcal{C}^k$-conjugated to the linear map $L_2(x_1, \ldots, x_n) = (x_2, \ldots, x_n, \sum_{i=1}^n (-1)^{i+1} x_i)$ also on $\psi(U)$. Moreover when $U = \mathbb{R}^n$ the recurrence is $\mathcal{C}^k$-linearizable on the whole $\mathbb{R}^n$ and hence it has a fixed point.*

For the particular case that $U = (\mathbb{R}^+)^n$ this result is also proved in the recent paper [8] using different tools and an equivalent notation.

It is also natural to check whether the examples of periodic difference equations appearing in the literature linearize or not. This is the second goal of this paper.

As we will see, many periodic difference equations correspond to maps F of the form (2), defined only on some proper subset $U \subset \mathbb{R}^n$. Let us see which are some natural properties that these open sets U should satisfy. The special form of the map F and its periodicity implies that $F(U) = U$ and imposes restrictions on its shape. In particular if F is periodic we get that $\pi_i(U) = \pi_j(U)$ for any $i, j = 1, \ldots, n$ where π_i denotes the i-th projection. The characterization of the structure of these periodic difference equations is perhaps a too general problem because, for example when U is not connected the map F can permute the different components, and moreover since an iterate of F is not a difference equation we cannot reduce the problem to the study of the difference equation on a connected open set. So it seems that the question of the linearization of a recurrence must be formulated when U is homeomorphic to $\mathbb{R}^n$. Notice that in this case all the projections $\pi_i(U), i = 1, \ldots, n$ have to be equal to the same open interval $I \subset \mathbb{R}$.

Now we can state with more precision the question we deal with:

Question: Consider a couple U and F such that:

- The set $U \subset \mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$.
- The map $F : U \to U$ is of the form (2), is of class $\mathcal{C}^k$ and is m-periodic on U.

Is is true that F linearizes? If yes, is it possible to get a linearized map of the form (2)?

Under this point of view we collect from the literature as many as possible couples U and F under the above hypotheses and we prove that they linearize as difference equations. It is worth to comment here that while in this paper our approach to the periodicity problem is through the existence of a linearization, there are different points of view. For instance in [9] it is proved that most periodic maps are completely integrable.

We need to introduce some preliminary definitions. In the particular case where the linearization of a recurrence is given by a conjugacy ψ of the form

$$\psi(x_1,\ldots,x_n)=(\tilde{\psi}(x_1),\ldots,\tilde{\psi}(x_n)), \tag{3}$$

being $\tilde{\psi}: I=\pi_1(U)\to\mathbb{R}$, we will say that the recurrence *trivially linearizes* on U.

Notice that each $r\in\mathbb{N}$ and each m-periodic difference equation of order n of the form (1) generates a new rm-periodic difference equation of order rn given by

$$x_{j+rn}=f(x_j,x_{j+r},\ldots,x_{j+r(n-1)}), \tag{4}$$

or equivalently a periodic map of the form (2) with

$$F(x_1,x_2,\ldots,x_{rn})=(x_2,x_3,\ldots,x_{rn},f(x_1,x_{1+r},x_{1+2r}\ldots,x_{1+r(n-1)})),$$

defined on some open subset of $\mathbb{R}^{rn}$. We will say that (4) is a difference equation *derived* from (1). If there is no any difference equation such that (1) is derived from it, then we will say that (1) is *minimal*. For instance, from this point of view the well-known Lyness periodic difference equations

$$x_{j+2}=\frac{x_{j+1}+1}{x_j}\quad\text{(5-periodic)},\qquad x_{j+3}=\frac{x_{j+1}+x_{j+2}+1}{x_j}\quad\text{(8-periodic)}, \tag{5}$$

are clearly minimal, while the derived periodic difference equations

$$x_{j+2r}=\frac{x_{j+r}+1}{x_j}\quad(5r\text{-periodic}),\qquad x_{j+3r}=\frac{x_{j+r}+x_{j+2r}+1}{x_j}\quad(8r\text{-periodic}),$$

with $1<r\in\mathbb{N}$, are not.

It is not difficult to prove that if a difference equation of the form (1) is $\mathcal{C}^k$-linearizable on U then all the difference equations (4) derived from it are as well $\mathcal{C}^k$-linearizable on the corresponding domains. Hence, from now on we will only center our attention on periodic minimal difference equations.

Many periodic examples exhibited in the literature consist on difference equations defined on $U=(\mathbb{R}^+)^n$, where $\mathbb{R}^+=\{x\in\mathbb{R},\ x>0\}$, and they are trivially C^ω linearizable. This is the case for the ones appearing in [1, 3, 4, 5, 20],

$$\begin{aligned}
x_{j+n}&=\frac{C}{x_jx_{j+1}\cdots x_{j+n-1}},\quad C>0,\quad (n+1)\text{-periodic},\\
x_{j+n}&=\frac{x_jx_{j+2}\cdots x_{j+n-1}}{x_{j+1}x_{j+3}\cdots x_{j+n-2}},\quad n\text{ odd},\quad (n+1)\text{-periodic},\\
x_{j+3}&=x_j\left(\frac{x_{j+2}}{x_{j+1}}\right)^{\phi},\ \text{where }\phi^2=\phi+1,\quad 5\text{-periodic}.
\end{aligned}$$

To see this it suffices to consider the linearization (3) given by the analytic map $\tilde{\psi}:(0,\infty)\to\mathbb{R}$, where $\tilde{\psi}(x)=\ln x$. We want to comment that the goal of these papers is not to prove that the above difference equations are periodic, because as we have seen this is quite easy, but to prove that they are the only ones once some special form of the difference equation and some period are fixed.

On the other hand periodic recurrences which seem not to be trivially linearizable are given by the Lyness equations (5) and in the papers [2, 3, 10, 14, 15]. We will study them in Section 4.

In Subsection 4.1 we prove that the Lyness recurrences (5) are not trivially linearizable but they are $\mathcal{C}^\omega$ linearizable. We point out that we do not study the 8-periodic recurrence given in [15],

$$x_{j+3} = \frac{x_{j+1} - x_{j+2} - 1}{x_j},$$

because it can be seen that it has no connected invariant regions.

In Subsection 4.2 we also prove global linearization results for the max-type periodic recurrences, like the 5-periodic one,

$$x_{j+2} = \max(x_{j+1}, 0) - x_j,$$

defined on the whole $\mathbb{R}^2$.

That all the periodic recurrences given in [2], constructed by using symmetric functions, are linearizable is proved in Subsection 4.3.

Finally, in Subsection 4.4 we make some comments on the periodic recurrences given by Coxeter in [14].

2. **Preliminary results.** In this section we recall three classical results, the Montgomery-Bochner Theorem about local linearization of periodic maps with a fixed point, the Kerékjártó Theorem about the linearization of planar periodic maps and a theorem that gives a standard way for proving that a local homeomorphism is a global one: the properness of the homeomorphism. We also prove the first one, because it inspired some of our proofs.

Theorem 2.1. *(Montgomery-Bochner Theorem, see [21]) Let $\mathcal{U} \subset \mathbb{R}^n$ be an open set and let $F : \mathcal{U} \to \mathcal{U}$ be a map of class $\mathcal{C}^r(\mathcal{U})$, $r \geq 1$, such that $F^m = Id$ for some integer number $m \geq 1$. If $p \in \mathcal{U}$ is a fixed point of F then there is a neighborhood of p in $\mathcal{U}$ where the dynamical system generated by F is $\mathcal{C}^k$ linearizable. Moreover the conjugated linear system is given by the linear map $L(x) := d(F)_p\, x$.*

Proof. Consider the map from $\mathcal{U}$ into $\mathbb{R}^n$, defined as

$$\psi = \frac{1}{m} \sum_{i=0}^{m-1} L^{-i} \circ F^i.$$

Note that since F is m-periodic then L is also m-periodic and then $L \circ \psi = \psi \circ F$. That ψ is locally invertible and has the same regularity as F follows by applying the Inverse Function Theorem, because $d(\psi)_p = Id$. □

Remark 2.2. *Notice that from the proof of the above Theorem it is easy to get the classification of $\mathcal{C}^k$-periodic maps in $\mathbb{R}$ as either the identity or globally $\mathcal{C}^k$-conjugated to $-Id$.*

Theorem 2.3. *(Kerékjártó's Theorem, see [13]) Let $U \subset \mathbb{R}^2$ be homeomorphic to $\mathbb{R}^2$ and let $F : U \to U$ be a $\mathcal{C}^k$, m-periodic map, $k \geq 0$. Then F is $\mathcal{C}^0$-linearizable.*

Corollary 2.4. *Consider a second order $\mathcal{C}^k$-periodic recurrence. Let $U \subset \mathbb{R}^2$ be the open set where the map F, given in (2) and associated to it, is defined and assume that U is homeomorphic to $\mathbb{R}^2$. Then the recurrence is linearizable in U.*

Proof. From Kerékjártó's Theorem it is clear that F is linearizable. We only need to prove that it is always possible to make a further linear change of variables such that this linear map is of the form (2). Since we are in $\mathbb{R}^2$ the only situations where this is not possible would be the ones where the linear map L is either $L = \mathrm{Id}$ or $L = -\mathrm{Id}$.

Clearly the first case is impossible because the only map conjugated to the identity is the identity itself, which is not a difference equation. Similarly, in the second case the recurrence would be 2-periodic, but the only 2-periodic recurrence is the one given by the map $F(x,y)=(y,x)$ which is clearly not conjugated to $-\mathrm{Id}$, because the dimension of the corresponding spaces of fixed points do not coincide. $\square$

Recall that a continuous map $F:\mathbb{R}^n\to\mathbb{R}^n$ is proper if and only if $F^{-1}(C)$ is a compact set whenever C is a compact set. Sometimes is useful to use the following characterization for the properness of a map F : for any sequence $\{x_n\}_n$ leaving any compact of $\mathbb{R}^n$, the sequence $\{F(x_n)\}_n$ also leaves any compact of $\mathbb{R}^n$.

The next result implies that in $\mathbb{R}^n$ a local homeomorphism having this property is indeed a global homeomorphism.

Theorem 2.5. *([22]) Consider $F:\mathbb{R}^n\to\mathbb{R}^n$. Then F is a homeomorphism of $\mathbb{R}^n$ onto $\mathbb{R}^n$ if and only if F is a local homeomorphism and it is a proper map.*

As a corollary of the above result it is not difficult to get a more general one that we will use in Section 4.

Corollary 2.6. *Let U and V open subsets of $\mathbb{R}^n$. Assume that both sets are homeomorphic to $\mathbb{R}^n$ and $F:U\to V$. Then F is a homeomorphism of U into V if and only if it is a local homeomorphism and for any sequence $\{x_n\}_n$ leaving any compact set of U, the sequence $\{F(x_n)\}_n$ leaves also any compact set of V.*

3. **Proof of Theorem A.**

We start by proving some preliminary results. The first one is already known, see [15, 19], but we include a proof for the sake of completeness.

Lemma 3.1. *Let $L:\mathbb{R}^n\to\mathbb{R}^n$ be a periodic linear map be such that $L(x_1,\ldots,x_n)=(x_2,\ldots,x_n,l(x_1,x_2,\ldots,x_n))$. Then the characteristic polynomial of L has no multiple roots.*

Proof. Since L is linear we can consider $L:\mathbb{C}^n\to\mathbb{C}^n$. Since for any $\lambda\in\mathbb{C}$, for any Jordan block and for any $m\in\mathbb{N}$,

$$\begin{pmatrix} \lambda & 1 & 0 & \dots & 0 & 0\\ 0 & \lambda & 1 & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & \lambda & 1\\ 0 & 0 & 0 & \dots & 0 & \lambda \end{pmatrix}^m \neq \mathrm{Id}$$

we get that L is diagonalizable. So to prove the result it suffices to show that for any λ eigenvalue of L the corresponding space of eigenvectors is one dimensional. Let λ be an eigenvalue of L and $0\neq e\in\mathbb{C}^n$ such that $L(e)=\lambda e$. Then $e=(x_1,\lambda x_1,\ldots,\lambda^{n-1}x_1)=x_1(1,\lambda,\ldots,\lambda^{n-1})$ and the result follows. $\square$

Note that the decomposition of x^m-1, $m\geq 1$ in real factors is

$$x^m-1=\begin{cases}(x+1)(x-1)\prod_{j=1}^{((m-2)/2}(x^2+s_jx+1), & \text{when } m \text{ is even},\\ (x-1)\prod_{j=1}^{((m-1)/2}(x^2+s_jx+1), & \text{when } m \text{ is odd},\end{cases} \tag{6}$$

where $s_j=-2\mathrm{Re}(x_j)=-2\mathrm{Re}(\overline{x}_j)$ being x_j and $\overline{x}_j$ all the couples of non-real conjugated m-roots of the unity.

Lemma 3.2. *Any m-periodic, n-th order real linear difference equation writes as*

$$x_{n+j} = -a_{n-1}x_{n+j-1} - a_{n-2}x_{n+j-2} - \cdots - a_2x_{j+2} - a_1x_{j+1} - a_0x_j,$$

with corresponding linear map in $\mathbb{R}^n$

$$L(x_1, x_2, \ldots, x_n) = (x_2, x_3, \ldots, x_n, -a_{n-1}x_n - a_{n-2}x_{n-1} - \cdots - a_2x_3 - a_1x_2 - a_0x_1),$$

where

$$\frac{x^m - 1}{P_{m-n}(x)} = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0 := Q_n(x),$$

and $P_{m-n}(x)$ is any polynomial of degree $m-n$ constructed by taking different real factors of the decomposition of $x^m - 1$ given in (6).

Proof. From Lemma 3.1 the characteristic polynomial of L, which is $Q_n(\lambda)$, can not have multiple roots. On the other hand it is a real polynomial with degree n and all its roots must be m-roots of the unity. So the result follows. □

Notice that some of the difference equations generated in the above lemma can be m'-periodic with m' a divisor of m.

Corollary 3.3. *There are only two different $(n+1)$-periodic linear recurrences of order n with real coefficients. If we write them as*

$$L(x_1, \ldots, x_n) = (x_2, \ldots, x_n, l(x_1, x_2, \ldots))$$

then either n is arbitrary and

$$l(x_1, \ldots, x_n) = -\sum_{i=1}^{n} x_i$$

or n is odd and

$$l(x_1, \ldots, x_n) = \sum_{i=1}^{n} (-1)^{i+1} x_i.$$

Proof. From Lemma 3.2 it suffices to study how many polynomials of degree n can be constructed from the decomposition of $x^{n+1} - 1$ given in (6). When n is even case this implies that $Q_n(x) = \frac{x^{n+1}-1}{x-1} = \sum_{i=0}^{n} x^i$. In the odd case there is another possibility, namely $Q_n(x) = \frac{x^{n+1}-1}{x+1} = \sum_{i=0}^{n} (-1)^{i+1} x^i$. □

Let $F : U \to U$ given by $F(x_1, \ldots, x_n) = (x_2, \ldots, x_n, f(x_1, \ldots, x_n))$ be a continuous $(n+1)$-periodic recurrence, where U is an open connected subset of $\mathbb{R}^n$ and let the open interval $\pi_j(U) = I \subset \mathbb{R}$ be any of its projections. Also for any $i \in \{1, \ldots, n\}$ let $\pi^i : U \longrightarrow \mathbb{R}^{n-1}$ the projection that eliminates the i-coordinate. For each $i \in \{1, \ldots, n\}$ and for any $(x_1, \ldots, x_{n-1}) \in \pi^i(U)$ set

$$I^i_{(x_1,\ldots,x_{n-1})} = \{t \in I : (x_1, \ldots, t, \ldots, x_{n-1}) \in U\}.$$

Clearly $I^i_{(x_1,\ldots,x_{n-1})}$ is and open subset of I and hence it is a countable union of open intervals which are its connected components.

Now we define $f_i^{(x_1,\ldots,x_{n-1})} : I^i_{(x_1,\ldots,x_{n-1})} \to I$ by

$$f_i^{(x_1,\ldots,x_{n-1})}(x) = f(x_1, \ldots, x_{i-1}, x, x_i, \ldots, x_{n-1}).$$

Notice that $f_i^{(x_1,\ldots,x_{n-1})}$ is a continuous map for any $i \in \{1, \ldots, n\}$ and for all $(x_1, \ldots, x_{n-1}) \in \pi^i(U)$.

Lemma 3.4. *Let $F : U \to U$ be a continuous $(n+1)$-periodic recurrence*

$$F(x_1, \ldots, x_n) = (x_2, \ldots, x_n, f(x_1, \ldots, x_n)),$$

where U is an open connected subset of $\mathbb{R}^n$ and $\pi(U) = I$. Then for any $i \in \{1, \ldots, n\}$ and for all $(x_1, \ldots, x_{n-1}) \in \pi^i(U)$, $f_i^{(x_1,\ldots,x_{n-1})}$ is an homeomorphism of $I^i_{(x_1,\ldots,x_{n-1})}$ into its image. Moreover either $f_i^{(x_1,\ldots,x_{n-1})}$ is decreasing for all $i \in \{1, \ldots, n\}$ and for all $(x_1, \ldots, x_{n-1}) \in \pi^i(U)$ or n is odd and $(-1)^i f_i^{(x_1,\ldots,x_{n-1})}$ is decreasing for all $i \in \{1, \ldots, n\}$ and for all $(x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$.

Proof. From the fact that F is $(n+1)$-periodic it follows that for any $i \in \{1, \ldots, n\}$ and for all $(x_1, \ldots, x_{n-1}) \in \pi^i(U)$, and $x \in I^i_{(x_1,\ldots,x_{n-1})}$ we have

$$f(x_i, \ldots, x_{n-1}, f(x_1, \ldots, x_{i-1}, x, x_i, \ldots, x_{n-1}), x_1, \ldots, x_{i-1}) = x, \tag{7}$$

which can be written as

$$f_{n-i+1}^{(x_i,\ldots,x_{n-1},x_1,\ldots,x_{i-1})}(f_i^{(x_1,\ldots,x_{n-1})}(x)) = x.$$

Thus we obtain that

$$f_{n-i+1}^{(x_i,\ldots,x_{n-1},x_1,\ldots,x_{i-1})} \circ f_i^{(x_1,\ldots,x_{n-1})} = \mathrm{Id}.$$

Applying the above equality to $n-i+1$ instead of i and $(x_i, \ldots, x_{n-1}, x_1, \ldots, x_{i-1})$ instead of $(x_1, \ldots, x_{n-1})$ we get

$$f_i^{(x_1,\ldots,x_{n-1})} \circ f_{n-i+1}^{(x_i,\ldots,x_{n-1},x_1,\ldots,x_{i-1})} = \mathrm{Id}.$$

This proves that $f_i^{(x_1,\ldots,x_{n-1})}$ is a homeomorphism of $I^i_{(x_1,\ldots,x_{n-1})}$ into its image. Since $I^i_{(x_1,\ldots,x_{n-1})}$ is a countable union of open intervals this implies that $f_i^{(x_1,\ldots,x_{n-1})}$ is monotone on each of the connected components of $I^i_{(x_1,\ldots,x_{n-1})}$.

For any fixed $i \in \{1, \ldots, n\}$ we claim that either for all $(x_1 \ldots, x_{n-1}) \in \pi^i(U)$, $f_i^{(x_1,\ldots,x_{n-1})}$ is increasing on all the connected components of $I^i_{(x_1,\ldots,x_{n-1})}$ or for all $(x_1 \ldots, x_{n-1}) \in \pi^i(U)$, $f_i^{(x_1,\ldots,x_{n-1})}$ is decreasing on all the connected components of $I^i_{(x_1,\ldots,x_{n-1})}$. To do this consider

$$V = \{(x_1, \ldots, x_n) \in U : \ f_i^{(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)} \text{ is increasing at } x_i\}.$$

If $V = \emptyset$ the claim is proved. So assume that $x \in V$ and set $A = (a_1, b_1) \times \ldots \times (a_n, b_n)$ be an open neighborhood of x in U. Then, for any $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \in \pi^i(A)$, $f^i_{(y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_n)}$ is defined in (a_i, b_i). Thus the family

$$f^i_{(y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_n)}\Big|_{(a_i,b_i)}, \quad \text{with} \quad (y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \in \pi^i(A),$$

is a continuous family of monotone homeomorphisms on (a_i, b_i). Since $x \in A$ and $x \in V$ we have that $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \pi^i(A)$ and $f^i_{(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)}$ is increasing on (a_i, b_i). By continuity arguments this implies that $f^i_{(y_1,\ldots,y_{i-1},y_{i+1},\ldots,y_n)}$ is increasing in (a_i, b_i) for all $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n) \in \pi^i(A)$. Hence $A \subset V$ and V is a open subset of U. The same argument can be used to show that V is closed in U. Since U is connected we obtain $V = U$ and the claim is proved.

Since from the above observation the character (increasing or decreasing) of the maps $f_i^{(x_1,\ldots,x_{n-1})}$ is independent of $(x_1, \ldots, x_{n-1})$ from now on we will speak about the character of the maps f_i

Consider now the case when f_1 is decreasing. Note that in this case f_n is also decreasing. We will prove by induction that f_i is decreasing for all $i = 1,\ldots,n$. For $i = 1$ there is nothing to prove. Assume now that f_i is decreasing and we will prove that f_{i+1} is also decreasing. Suppose to arrive a contradiction that f_{i+1} is increasing and consider the equality

$$f(x_2,\ldots,x_n,f(x_1,\ldots,x_n)) = x_1.$$

When x_{i+1} increases, the i and n coordinates of $y = (x_2,\ldots,x_n,f(x_1,\ldots,x_n))$ also increase. Since by induction hypothesis f_i and f_n are decreasing it follows that the left part of this equation decreases when the $(i+1)$-th coordinate increases. This contradicts the fact that the right part of the equation is independent of x_{i+1}. This proves that f_{i+1} is decreasing.

Now it remains the case when f_1 is increasing. Remember that in this case f_n is also increasing. We must to prove that $(-1)^i f_i$ is decreasing for $i = 1,\ldots,n$. To do this it suffices to show that if f_i is increasing (respectively decreasing) then f_{i+1} is decreasing (respectively increasing). To do this consider the equation

$$f(x_2,\ldots,x_n,f(x_1,\ldots,x_n)) = x_1$$

and suppose for instance that f_i is increasing. Suppose to arrive a contradiction that f_{i+1} is also increasing. Then when x_{i+1} increases the i-th and n-th coordinates of $y = (x_2,\ldots,x_n,f(x_1,\ldots,x_n))$ increase and since f_i and f_n are increasing then the left part of the equation increases when x_{i+1} increases which contradicts the fact that the right part of the equation does not depends on x_{i+1}. The other case follows by the same argument.

Lastly, note that in this last case, since f_i and f_{n-i+1} must have the same character and $(-1)^i f_i$ must be increasing, this implies that n is odd. □

Lemma 3.5. *Let $F : U \to U$ be a $(n+1)$-periodic recurrence of class $\mathcal{C}^1$,*

$$F(x_1,\ldots,x_n) = (x_2,\ldots,x_n,f(x_1,\ldots,x_n)),$$

where U is an open connected subset of $\mathbb{R}^n$. Then $\frac{\partial f}{\partial x_i} \neq 0$ for all $i = 1,\ldots,n$. Moreover either $\frac{\partial f}{\partial x_i} < 0$ for all $i = 1,\ldots,n$ or n is odd and $(-1)^i \frac{\partial f}{\partial x_i} < 0$.

Proof. First of all note that since $F \circ F^n = \mathrm{Id}$ it follows that F is a diffeomorphism on U. Hence $(-1)^{n+1}\left(\frac{\partial f}{\partial x_1}\right)_x = \det\left(d(F)_x\right) \neq 0$. From the equality $f(F(x_1,\ldots,x_n)) = x_1$ we deduce that $\left(\frac{\partial f}{\partial x_n}\right)_{F(x_1,\ldots,x_n)} \cdot \left(\frac{\partial f}{\partial x_1}\right)_{(x_1,\ldots,x_n)} = 1$. This implies that $\frac{\partial f}{\partial x_n} \neq 0$ and has the same sign as $\frac{\partial f}{\partial x_1}$. We also get

$$\left(\frac{\partial f}{\partial x_i}\right)_{F(x_1,\ldots,x_n)} + \left(\frac{\partial f}{\partial x_n}\right)_{F(x_1,\ldots,x_n)} \left(\frac{\partial f}{\partial x_{i+1}}\right)_{(x_1,\ldots,x_n)} = 0 \tag{8}$$

for $i = 1,\ldots,n-1$.

Now assume that $\frac{\partial f}{\partial x_1} < 0$. Then $\frac{\partial f}{\partial x_n} < 0$ and from equation (8) for $i = 1$ we obtain that $\frac{\partial f}{\partial x_2} < 0$. Thus applying recursively equation (8) for $i = j$ we obtain that $\frac{\partial f}{\partial x_{j+1}} < 0$.

Lastly assume that $\frac{\partial f}{\partial x_1} > 0$. Then $\frac{\partial f}{\partial x_n} > 0$. Also applying equation (8) for $i = 1$ we obtain that $\frac{\partial f}{\partial x_2} < 0$. Thus applying recursively equation (8) for $i = j$ we obtain that $\frac{\partial f}{\partial x_j}\frac{\partial f}{\partial x_{j+1}} < 0$. So we get that $(-1)^i \frac{\partial f}{\partial x_i} < 0$ for $i = 1,\ldots,n$. In particular

$(-1)^n \frac{\partial f}{\partial x_n} < 0$ and since $\frac{\partial f}{\partial x_n} > 0$ we deduce that n must be odd. This finish the proof of the Lemma. □

Proof of Theorem A. From Lemma 3.4 we already know that f_1 is a homeomorphism of I into its image. So either f is increasing with respect x_1 or it is decreasing.

We consider first the case when f is increasing with respect x_1. From Lemma 3.4 it follows that n is odd and f is increasing with respect to the odd coordinates and decreasing with respect to the even coordinates.

First of all we will prove that F is $\mathcal{C}^0$-conjugated to L_2. Similarly that in the proof of the Montgomery-Bochner Theorem, consider the map $\varphi : U \to \varphi(U)$ given by $\varphi = \frac{1}{n+1}\sum_{i=0}^{n} L_2^{-i} \circ F^i$. We have that

$$\varphi \circ F = L_2 \circ \varphi$$

and we will prove that φ is an homeomorphism of U into $\varphi(U)$. Notice that, in contrast with the proof of Montgomery-Bochner Theorem, we are not assuming smoothness conditions on F. Using that F and L_2 are $(n+1)$-periodic, for $i = 1, \ldots, n-1$ we obtain

$$F^i(x_1, \ldots, x_n) = (x_{i+1}, \ldots, x_n, f(x_1, \ldots, x_n), x_1, \ldots, x_{i-1})$$

and

$$F^n(x_1, \ldots, x_n) = (f(x_1, \ldots, x_n), x_1, \ldots, x_{n-1}).$$

Also for $i = 2, \ldots, n$ we have

$$\begin{aligned} L_2^{-i}(x_1, \ldots, x_n) &= L_2^{n+1-i}(x_1, \ldots, x_n) \\ &= (x_{n+2-i}, \ldots, x_n, \sum_{j=1}^{n}(-1)^{j+1}x_j, x_1, \ldots, x_{n-i}), \end{aligned}$$

and

$$L_2^{-1}(x_1, \ldots, x_n) = L_2^{n}(x_1, \ldots, x_n) = \Big(\sum_{j=1}^{n}(-1)^{j+1}x_j, x_1, \ldots, x_{n-1}\Big).$$

Thus we obtain that

$$\begin{aligned} &L_2^{-i}(F^i(x_1, \ldots, x_n) = \\ &\quad \Big(x_1, \ldots, x_{i-1}, x_i + (-1)^{i+1}\Big(\sum_{j=1}^{n}(-1)^j x_j + f(x_1, \ldots, x_n)\Big), x_{i+1}, \ldots, x_n\Big) \end{aligned} \tag{9}$$

and if we denote by φ_j the j component of φ we get,

$$\varphi_j(x_1, \ldots, x_n) = \frac{1}{n+1}\Big((n+1)x_j + (-1)^{j+1}\Big(\sum_{i=1}^{n}(-1)^i x_i + f(x_1, \ldots, x_n)\Big)\Big).$$

To prove the invertibility of φ it suffices to show that for any $(u_1, \ldots, u_n) \in \varphi(U)$ the system

$$\varphi_j(x_1, \ldots, x_n) = u_j,\ j = 1, \ldots, n \tag{10}$$

has only one solution. To do this for any $j = 2, 4, \ldots, n-1$ we get

$$(n+1)(u_j + u_1) = (n+1)\left(\varphi_j(x_1 \ldots, x_n) + \varphi_1(x_1 \ldots, x_n)\right) = (n+1)(x_j + x_1).$$

Then $x_j = u_j + u_1 - x_1$. On the other hand for $j = 3, 5 \ldots, n$ we have

$$(n+1)(u_j - u_1) = (n+1)\left(\varphi_j(x_1 \ldots, x_n) - \varphi_1(x_1 \ldots, x_n)\right) = (n+1)(x_j - x_1),$$

and we obtain $x_j = u_j - u_1 + x_1$. Substituting in the first equation we get

$$(n+1)x_1 - nx_1 + (n-1)u_1 + \\ \sum_{i=2}^{n}(-1)^i u_i + f(x_1, -x_1+u_2+u_1, \dots, x_1+u_n-u_1) = (n+1)u_1.$$

Hence,

$$x_1 + f(x_1, -x_1+u_2+u_1, \dots, (-1)^{i+1}x_1 + u_i + (-1)^i u_1, \dots, x_1+u_n-u_1) \\ = 2u_1 - \sum_{i=2}^{n}(-1)^i u_i.$$

Now we consider the map $g_{(u_1,\dots u_n)} : J \subset \mathbb{R} \to \mathbb{R}$ defined by

$$g_{(u_1,\dots,u_n)}(x) = x + f(x, -x+u_2+u_1, \dots, (-1)^{i+1}x + u_i + (-1)^i u_1, \dots, x+u_n-u_1).$$

Since by Lemma 3.4, f is decreasing in its even variables and increasing in its odd variables it follows that $g_{(u_1,\dots u_n)}$ is increasing and so it is injective, as we wanted to see.

Finally, when $U = \mathbb{R}^n$ we have that $\lim_{x\to\pm\infty} g_{(u_1,\dots u_n)}(x) = \pm\infty$. Then $g_{(u_1,\dots u_n)}$ is a homeomorphism of $\mathbb{R}$. Hence φ has a global continuous inverse in the whole $\mathbb{R}^n$ given by

$$\varphi^{-1}(u_1, \dots, u_n) = (x_1, u_2+u_1+x_1, \dots, u_n-u_1+x_1)$$

where $x_1 = g^{-1}_{(u_1,\dots u_n)}(2u_1 - \sum_{i=2}^{n}(-1)^i u_i)$.

Until now we have proved that φ is a $\mathcal{C}^0$-conjugation between F and L_2. Now assume that F is of class $\mathcal{C}^k$ with $k \geq 1$. Then by construction φ is also of class $\mathcal{C}^k$. So to prove the theorem in this case it only remains to see that φ^{-1} also is of class $\mathcal{C}^k$. By the Inverse Function Theorem it suffices to show that $\det(d(\varphi)_x) \neq 0$, for all $x \in \mathbb{R}^n$. Differentiating the expression of φ we obtain

$$d(\varphi)_x = \frac{1}{n+1}\begin{pmatrix} n+a_1(x) & 1+a_2(x) & -1+a_3(x) & \dots & -1+a_n(x) \\ 1-a_1(x) & n-a_2(x) & 1-a_3(x) & \dots & 1-a_n(x) \\ -1+a_1(x) & 1+a_2(x) & n+a_3(x) & \dots & -1+a_n(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1+a_1(x) & 1+a_2(x) & -1+a_3(x) & \dots & n+a_n(x) \end{pmatrix},$$

where for the sake of simplicity we write $a_i(x) = \left(\frac{\partial f}{\partial x_i}\right)(x), i = 1, 2, \dots, n$.

In order to compute the determinant of the above matrix we first consider the simplest one

$$A_n(x) := \begin{pmatrix} -1+a_1(x) & 1+a_2(x) & -1+a_3(x) & \dots & -1+a_n(x) \\ 1-a_1(x) & -1-a_2(x) & 1-a_3(x) & \dots & 1-a_n(x) \\ -1+a_1(x) & 1+a_2(x) & -1+a_3(x) & \dots & -1+a_n(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -1+a_1(x) & 1+a_2(x) & -1+a_3(x) & \dots & -1+a_n(x) \end{pmatrix}.$$

Notice that $d(\varphi)_x = \dfrac{1}{n+1}\big(A_n(x) + (n+1)\mathrm{Id}\big)$. Clearly $A_n(x)$ has $n-1$ null eigenvalues and hence its characteristic polynomial is

$$P_n(\lambda, x) := \det(A_n(x) - \lambda \mathrm{Id}) = -\lambda^n + \Big(-n + \sum_{i=1}^{n} (-1)^{i+1} a_i(x)\Big)\lambda^{n-1}.$$

Since

$$\det\left(d(\varphi)_x\right) = \left(\frac{1}{n+1}\right)^n P_n(-(n+1), x) = \frac{1}{n+1}\left(1 + \sum_{i=1}^{n} (-1)^{i+1} a_i(x)\right),$$

we get that

$$\det(d(\varphi)_x) = \frac{1}{n+1}\left(1 + \sum_{i=1}^{n} (-1)^{i+1} \left(\frac{\partial f}{\partial x_i}\right)(x)\right).$$

Remember that in our case f is increasing with respect the first coordinate, so $\frac{\partial f}{\partial x_1} > 0$. Then by Lemma 3.5 we get that $(-1)^{i+1}\frac{\partial f}{\partial x_i} > 0$ for $i = 1, \ldots, n$. Thus $\det\left(d(\varphi)_x\right) > 0$ and this finishes the proof of this case.

Now assume that f is decreasing with respect to x_1. As in the previous case, we first prove that F is $\mathcal{C}^0$-conjugated to L_1. Again the conjugation that we consider is similar to the one used in the proof of the Montgomery-Bochner Theorem. Consider $\psi : U \to \psi(U)$ defined as $\psi = \frac{1}{n+1}\sum_{i=0}^{n} L_1^{-i} \circ F^i$. In this case we get that

$$\psi_j(x_1, \ldots, x_n) = \frac{1}{n+1}\left((n+1)x_j - \sum_{i=1}^{n} x_i - f(x_1, \ldots, x_n)\right),$$

where ψ_j is the j component of ψ.

Arguing as is in the previous case we obtain that the inverse of ψ is

$$\psi^{-1}(u_1, \ldots, u_n) = (x_1, u_2 - u_1 + x_1, \ldots, u_n - u_1 + x_1),$$

where $x_1 = f^{-1}_{(u_1,\ldots,u_n)}(2u_1 + \sum_{i=2}^{n} u_i)$ and $f_{(u_1,\ldots u_n)} : J \subset \mathbb{R} \to \mathbb{R}$ is defined as

$$f_{(u_1,\ldots,u_n)}(x) = x - f(x, x + u_2 - u_1, \ldots, x + u_n - u_1).$$

Notice that again by Lemma 3.4, f is decreasing in all its variables and so it follows that $f_{(u_1,\ldots,u_n)}$ is increasing giving the injectivity of the map, as we wanted to prove.

When $U = \mathbb{R}^n$ we have that $\lim_{x\to\pm\infty} f_{(u_1,\ldots,u_n)}(x) = \pm\infty$, proving that $f_{(u_1,\ldots,u_n)}$ is a homeomorphism of $\mathbb{R}$.

To show that ψ is a $\mathcal{C}^k$-conjugation between F and L_1 when F is of class $\mathcal{C}^k$ with $k \geq 1$, we will show that $\det\left(d(\psi)_x\right) \neq 0$ for $x \in \mathbb{R}$. After some computations we have that

$$d(\psi)_x = \frac{1}{n+1}\begin{pmatrix} n - \left(\frac{\partial f}{\partial x_1}\right)(x) & -1 - \left(\frac{\partial f}{\partial x_2}\right)(x) & \ldots & -1 - \left(\frac{\partial f}{\partial x_n}\right)(x) \\ -1 - \left(\frac{\partial f}{\partial x_1}\right)(x) & n - \left(\frac{\partial f}{\partial x_2}\right)(x) & \ldots & -1 - \left(\frac{\partial f}{\partial x_n}\right)(x) \\ \vdots & \vdots & \vdots & \vdots \\ -1 - \left(\frac{\partial f}{\partial x_1}\right)(x) & -1 - \left(\frac{\partial f}{\partial x_2}\right)(x) & \ldots & n - \left(\frac{\partial f}{\partial x_n}\right)(x) \end{pmatrix}$$

and

$$\det\left(d(\psi)_x\right) = \frac{(-1)^n}{n+1}\left(1 - \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i}\right)(x)\right).$$

Since in our case f is decreasing with respect the first coordinate, we get that $\frac{\partial f}{\partial x_1} < 0$. Then by Lemma 3.5 we have that $\frac{\partial f}{\partial x_i} < 0$ for $i = 1, \ldots, n$. Thus $\det\left(d(\psi)_x\right) \neq 0$ as we wanted to see. This ends the proof of the theorem. $\square$

4. **Non–trivial linearizations.** In this section we give non–trivial linearizations for most of the known periodic recurrences.

4.1. **Lyness type maps.** This subsection is devoted to study the linearizations of the well-known Lyness maps. Recall that they are given by $G(x,y) = (y, \frac{1+y}{x})$ which is 5-periodic and $H(x,y,z) = (y, z, \frac{1+y+z}{x})$ which is 8-periodic.

First we consider the map G. Easy computations show that G and its iterates are defined in $\mathbb{R}^2 \setminus \mathcal{L}$ where $\mathcal{L}$ is the union of the straight lines $x = 0, x = -1, y = 0, y = -1, x + y = -1$. Clearly $\mathbb{R}^2 \setminus \mathcal{L}$ has twelve connected components and G fixes two components and permutes the rest. We denote by $A = \{(x,y) \in \mathbb{R}^2 : \ x > 0, y > 0\}$ and by B the interior of the triangle with vertices $(-1,0), (-1,-1)$ and $(0,-1)$ which are the two components invariants by G. The next Lemma shows that $G|_A$ and $G|_B$ are not trivially linearizable.

Lemma 4.1. *The maps $G|_A$ and $G|_B$ are not trivially linearizable.*

Proof. Along this proof we will denote by C any of the two sets A or B, indistinctly. Also, we set $p = (c,c)$ for the corresponding fixed point, $(a,a) = \frac{1+\sqrt{5}}{2}(1,1)$ or $(b,b) = \frac{1-\sqrt{5}}{2}(1,1)$.

We will prove our result by contradiction. Assume that $G|_C$ is trivially linearizable. The point $p = (c,c)$ with $c^2 = c+1$ belongs to C and it is a fixed point of G. The Montgomery-Bochner Theorem implies that G is conjugated in a neighborhood of p to the linear recurrence given by $(dG)_p$ that is $L(x,y) = (y, -x + y/c)$. Then it must exists a homeomorphism $\varphi : \pi(C) \to \varphi(\pi(C))$, where $\pi(C)$ is the projection of C into any of the coordinate axis, such that

$$\varphi\left(\frac{1+y}{x}\right) = -\varphi(x) + \frac{\varphi(y)}{c} \tag{11}$$

for all $(x,y) \in C$. Note also that φ must satisfy that $\varphi\left(c\right) = 0$ because the map φ has to send the fixed point of $G|_C$ to the fixed point of L. Thus putting $x = c$ in the above equality we obtain

$$\varphi\left(\frac{1+y}{c}\right) = \frac{\varphi(y)}{c}, \tag{12}$$

for all $y \in C \cap \{x = c\}$.

In the case $C = A, c = a$ equation (12) implies that φ can be extended continuously to 0 putting $\varphi(0) = a\varphi\left(1/a\right)$. Since $a^{-2} = 1 - a^{-1}$, by using again (12) we get

$$\varphi(1) = \varphi(\frac{1}{a^2} + \frac{1}{a}) = \varphi\left(\frac{1+1/a}{a}\right) = \frac{\varphi(1/a)}{a}.$$

On the other hand, by (11),

$$\varphi(1) = -\varphi(1) + \frac{\varphi(0)}{a} = -\varphi(1) + \frac{1}{a}a\varphi(\frac{1}{a}) = -\varphi(1) + \varphi(\frac{1}{a})$$

which gives that $\varphi(1) = \varphi(1/a)/2$. The above two equations imply that $\varphi(1/a) = 0$ and since $\varphi(a) = 0$ this contradicts the fact that φ is a homeomorphism.

Consider now the case $C = B, c = b$. In this situation, equation (12) implies that φ can be extended continuously to -1 putting $\varphi(-1) = \varphi(-1-b)/b$. Then, by (11), we get

$$\varphi(-1) = \frac{\varphi(-1-b)}{b} = \frac{1}{b}\varphi\left(\frac{1+b}{-1}\right) = \frac{1}{b}\left(-\varphi(-1) + \frac{\varphi(b)}{b}\right) = \frac{1}{b}\left(-\varphi(-1)\right)$$

which implies $\varphi(-1) = 0 = \varphi(b)$. This equality is again in contradiction with the fact that φ is a homeomorphism. So the result follows. □

Despite the fact that $G|_A$ and $G|_B$ are not trivially linearizable we prove that they are $\mathcal{C}^\omega$-linearizable. It is worth to comment that the fact that they are $\mathcal{C}^0$-linearizable follows from Kerékjártó's Theorem, see Corollary 2.4.

Theorem 4.2. *The maps $G|_A$ and $G|_B$ are $\mathcal{C}^\omega$-linearizable.*

Proof. As in the proof of Lemma 4.1, we denote by C anyone of the two sets A or B, indistinctly and by $p = (c, c)$, with $c^2 = c + 1$, the corresponding fixed point, (a, a) or (b, b), where $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$.

To prove the theorem we will use Corollary 2.6 to show that the local $\mathcal{C}^\omega$-linearization φ at the fixed point given in the proof of the Montgomery-Bochner Theorem is indeed a global $\mathcal{C}^\omega$-linearization from C into $\mathbb{R}^2$.

As a first step we compute φ and prove that in both cases it is a locally $\mathcal{C}^\omega$ invertible map.

By the proof of Montgomery-Bochner Theorem we know that the linearization near p is given by the map

$$\varphi_c(z) = \frac{1}{5}\sum_{i=0}^{4} L^{-i}(G|_C^i(z)),$$

being $z = (x, y)$ and $L(x, y) = (y, -x + y/c)$. After some computations we get that $\varphi_c(x, y) = (\varphi_c^1(x, y), \varphi_c^2(x, y))$, where $\varphi_c : C \to \varphi_c(C)$

$$\varphi_c^1(x, y) = \frac{1}{5}\left(2x + (c-1)\left(y + \frac{x+1}{y}\right) - c\left(\frac{1+y}{x} + \frac{1+x+y}{xy}\right)\right), \qquad (13)$$

$$\varphi_c^2(x, y) = \varphi_c^1(y, x) = \varphi_c^1\left(y, \frac{1+y}{x}\right), \qquad (14)$$

and moreover that

$$\begin{aligned}\det\left(d(\varphi_c)_{(x,y)}\right) = \frac{1}{25x^3y^3}\Big(&(2+c)\left(x^3y^3 + x^4y + y^4x + x^3 + y^3 + 2x^2 + 2y^2 + x + y\right)\\ &+ (-1+2\,c)\,x^2y^2\,(x+y) + (1+3\,c)\,xy\left(x^2+y^2\right)\\ &+ 5cxy\,(x+y) + (3+4\,c)\,xy\Big) := \frac{\Phi_c(x,y)}{25x^3y^3}.\end{aligned} \qquad (15)$$

Note also that when $c = a = \frac{1+\sqrt{5}}{2}$, then $2 + c, -1 + 2c, 1 + 3c, c$ and $3 + 4c$ are all positive. Hence $\det\left(d(\varphi_a)\right)|_A > 0$ and we have proved that $\varphi_a : A \to \mathbb{R}^2$ is a local $\mathcal{C}^\omega$ diffeomorphism.

We claim that $\det\left(d(\varphi_b)\right)|_B$ does not vanish. Since the proof of this fact requires some more computations we postpone it for the moment. From the claim we get that $\varphi_b : B \to \mathbb{R}^2$ is also a local diffeomorphism.

Let us prove now that in both cases $\varphi_c : C \to \varphi_c(C)$ is a proper map.

When $C = A$, consider the map $\varphi_a : A \to \mathbb{R}^2$. To see that it is proper it suffices to prove that given a sequence $\{p_n = (x_n, y_n)\}_n$ of points contained in A, that approaches either to its boundary $\partial A = \{(x, y) : x = 0, y \geq 0\} \cup \{(x, y) : x \geq 0, y = 0\}$ or to infinity, then $\{\varphi_a(p_n)\}_n$ tends to infinity.

The above assertion is easy to prove by doing a case by case study. For instance, assume that $\{x_n\}_n$ tends to infinity and $\{y_n\}_n$ remains bounded. Then, by (13) we get that $\{\varphi_a^1(p_n)\}_n$ tends to $+\infty$ because the negative terms remain bounded. The other cases follow similarly.

To prove the properness of $\varphi_b : B \to \varphi_b(B)$ is a little bit more complicated. Clearly φ_b extends continuously to the boundary of the triangle B except at the points $(-1, 0)$ and $(0, -1)$. Some computations show the following facts:

(a) The image by φ_b of the segment given by the equation $x + y + 1 = 0$ with $x \in (-1, 0)$ is the segment ℓ_1 with endpoints $(-1, 2 - b)$ and $(2 - b, -1)$.
(b) The image by φ_b of the segment given by the equation $x + 1 = 0$ with $y \in [-1, 0)$ is the segment ℓ_2 with endpoints $(2b - 2, 2 - b)$ and $(-1, -1)$.
(c) The image by φ_b of the segment given by the equation $y + 1 = 0$ with $x \in [-1, 0)$ is the segment ℓ_3 with endpoints $(2 - b, 2b - 2)$ and $(-1, -1)$.
(d) The set of accumulation points of $\varphi_b(z)$ when z tends to $(-1, 0)$ and $z \in B$ is the segment ℓ_4 with endpoints $(2b - 2, 2 - b)$ and $(-1, 2 - b)$.
(e) The set of accumulation points of $\varphi_b(z)$ when z tends to $(0, -1)$ and $z \in B$ is the segment ℓ_5 with endpoints $(2 - b, 2b - 2)$ and $(2 - b, -1)$.

The five segments described in the above list determine a pentagone in $\mathbb{R}^2$. We denote by P the open bounded component of $\mathbb{R}^2 \setminus \cup_{i=1}^5 \ell_i$. Clearly $\varphi(B) \subset P$ and for all $p_n \in B$ with $p_n \to \partial(B)$ we get that $\varphi_b(p_n) \to \cup_{i=1}^5 \ell_i = \partial(P)$. Thus $\varphi_b : B \to P = \varphi_b(B)$ is a proper map.

In short, we have proved that for C either A or B, the map $\varphi_c : C \to P = \varphi_c(C)$ given in (13) is proper and a local $\mathcal{C}^\omega$ diffeomorphism. Hence by Corollary 2.6 the theorem follows.

To end the proof we have to prove the above claim. Concretely we will prove that the function $\Phi_b(x, y)$ given in (15) does not vanish on B. To do this it is more convenient to use the new coordinates u, v given by $x = u - 1, y = v - u - 1$. By using them we write

$$\Phi_b(x, y) = \Phi_b(u - 1, v - u - 1) := D(u, v),$$

where $D(u, v)$ is a polynomial of degree 6. It is clear that it suffices to prove that $D(u, v)$ does not vanishes on the interior of a new triangle, B' that in the (u, v) coordinates has the boundary given by the straight lines $u = 0, v = 1$ and $u - v = 0$. In Figure 1 we illustrate the situation by plotting the triangle B' and the algebraic curve $D(u, v) = 0$. Let us prove this fact.

Our proof follows by showing that for each $v \in (0, 1)$, the equation $D(u, v) = 0$ has always six simple real solutions and that none of the corresponding points is inside the triangle B'. This result will be a consequence of the following facts:

(a) The resultant of $D(u, v)$ with respect to v does not vanish when $u \in (0, 1)$.
(b) The functions $D(0, v)$ and $D(v, v)$ do not vanish when $v \in [0, 1)$.
(c) The function $D(u, 1)$ does not vanish when $u \in (0, 1)$.
(d) The function $D(u, 0)$ has six real roots, all of them simple and the corresponding points are outside B'.
(f) The algebraic curve D has two branches at each of the points $(0, 1)$ and $(1, 1)$ and none of them enters in the triangle B'.

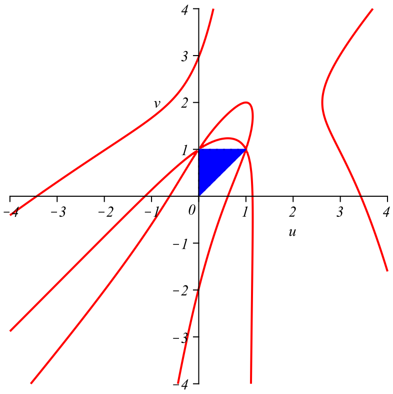

Figure 1. Triangle B' and the algebraic curve $D(u,v)=0$.

Before proving the above statements let us see how our result follows from them. By item (d) we know that $D(u,0)$ has six simple real roots, all outside B'. First of all, let us see that for all fixed $v\in[0,1)$ the polynomial $D_v : u\to D(u,v)$ has always six simple real roots. Since $D_v(u)=(\sqrt{5}-5)/2u^6+\cdots$ and by item (a), $\operatorname{Res}(D(u,v),v)$ is not zero in our region, we know that there are no multiple roots. Since the roots depend continuously (even in the complex) on v and the coefficients of D_v are real, the number of real roots has to be always six. Again by the continuity of the roots with respect to v, the only way for changing the number of real roots of D_v in B' is that for some $v\in(0,1)$ some root passes through its boundary. Precisely, items (a)–(c) and (f) prevent this fact. Thus the result follows.

Let us prove statements (a)–(f). Straightforward computations give that

$$\begin{aligned}\operatorname{Res}(D(u,v),v) = \left(\frac{500\sqrt{5}-1125}{256}\right) u^2(u-1)^2\left(8u^2+(3\sqrt{5}-7)u+15-5\sqrt{5}\right)\times\\ \left(2u^3+(\sqrt{5}-5)u^2+(9-3\sqrt{5})u-2\right)^2\left(2u-3-\sqrt{5}\right)^3\times\\ \left(-2u^3+(9+3\sqrt{5})u^2-(25+13\sqrt{5})u+22+12\sqrt{5}\right)^2.\end{aligned}$$

From the above expression it is easy to prove statement (a).

Simple computations give that $D(0,v)=D(v,v)=p(v)$ and $D(u,1)=u^2p(u)$, where

$$p(v)=\left(\frac{\sqrt{5}-5}{4}\right)\left(2v^2-2v-5-3\sqrt{5}\right)(v-1)^2$$

and that

$$D(u,0)=\left(\frac{\sqrt{5}-5}{16}\right)\left(2u^4-(15+5\sqrt{5})u^2+15+7\sqrt{5}\right)\left(4u^2+2\sqrt{5}-6\right).$$

Hence we can easily check that items (b), (c) and (d) follow.

Finally, to prove statement (f), observe that near $(0,1)$ we have

$$D(u,v)=\left(\frac{5+5\sqrt{5}}{2}\right)\left[u^2+(v-1)^2\right]-\left(\frac{15+9\sqrt{5}}{2}\right)u\,(v-1)+O_3(u,v-1),$$

and near $(1,1)$,

$$D(u,v)=\left(\frac{5+5\sqrt{5}}{2}\right)(u-1)^2 + \left(\frac{5-\sqrt{5}}{2}\right)\left[(u-1)(v-1)-(v-1)^2\right]+O_3(u-1,v-1).$$

Hence in both cases the two lines given by the quadratic part of D near the points does not enter inside B'. This fact implies that locally the same result holds for the curve $D(u,v)=0$. Hence the claim follows. $\square$

We could also consider the same questions that we have developed in this section for the two dimensional Lyness map, but for the three dimensional Lyness map, $H(x,y,z)=(y,z,\frac{1+y+z}{x})$. All that we have done in this direction seems to indicate that the results would be the same that in the two dimensional case, and that the techniques to prove them are also the same, but with much more computational effort. For instance it can be proved that H and its iterates are defined in $\mathbb{R}^3\setminus\mathcal{L}$ where $\mathcal{L}$ is now the union of the surfaces $x=0, y=0, z=0, y=-(x+1)(z+1), x+y=-1$ and $z+y=-1$. Then $\mathbb{R}^3\setminus\mathcal{L}$ is divided into a finite number of connected components and only two of them are fixed by H. They are $A=\{(x,y,z)\in\mathbb{R}^3 : x>0, y>0, z>0\}$ and the bounded region B limited by the two planes $x+y=-1$, $z+y=-1$ and the surface $y=-(x+1)(z+1)$. Moreover each one of them contains a fixed point (c,c,c) with $c^2=2c+1$. In both cases it is also not difficult to get the candidate φ_c to be the linearization, given similarly that in the proof of the Montgomery-Bochner Theorem. Moreover when $C=A$ it is easy to see that φ_a is a local diffeomorphism. On the other hand, at least at the level of numerical experiments, it also seems true that when $C=B$, the determinant of $d(\varphi_b)$ on B does not vanish, and so, that φ_b is also a local diffeomorphism. We have not done all the details but we believe that $H|_C$, for C either A or B, is linearizable and the linearization is given by φ_c.

4.2. **Max-type equations.** By Corollary 2.4 it is clear that the well-known examples

$$x_{j+2}=\max(x_{j+1},0)-x_j \quad \text{(5-periodic)}, \qquad x_{j+2}=|x_{j+1}|-x_j \quad \text{(9-periodic)}, \tag{16}$$

appearing for instance in [15], $\mathcal{C}^0$-linearize. This subsection is devoted to find a piecewise linear linearization of the 5-periodic example and indicate the main steps to perform a similar study for the third order Lyness recurrence,

$$x_{j+3}=\max(x_{j+2},x_{j+1},0)-x_j, \quad \text{(8-periodic)}. \tag{17}$$

The first equation of (16) and equation (17) are often also called Lyness equations because they can be seen as the limit when λ tends to infinity of the difference equations

$$z_{n+2}=\log_\lambda\left(1+\lambda^{z_{n+1}}\right)-z_n$$

and

$$z_{n+3}=\log_\lambda\left(1+\lambda^{z_{n+1}}+\lambda^{z_{n+2}}\right)-z_n,$$

which (for $\lambda>0$ and positive initial conditions) are clearly trivially conjugated to the two and three dimensional Lyness equations, respectively, by means of $\widetilde{\psi}(x)=\lambda^x$.

Proposition 4.3. *The map*

$$F(x,y) = (y, \max(0,y) - x) \tag{18}$$

is globally conjugated to the linear 5-periodic map

$$L(x,y) = (y, -x + \frac{\sqrt{5}-1}{2}y),$$

by means of a piecewise linear conjugation.

Proof. Using that our map $F(x,y)$ is the limit when λ tends to infinity of the map $(y\,,\,\log_\lambda(\lambda^0+\lambda^y)-x)$ it is easily seen that each five-cycle is given by

$$x\,,\,y\,,\,\max(0,y)-x\,,\,\max(0,x,y)-x-y\,,\,\max(0,x)-y.$$

A simple calculation shows that the two components of the map $\psi = \frac{1}{5}\sum_{i=0}^{4}(L^{-i}\circ F^i)$ are

$$\psi_1(x,y) = \frac{1}{5}\big((4+2k)x + (k+1)(y - \max(0,y) - \max(0,x,y)) + k\max(0,x)\big)$$

and $\psi_2(x,y) = \psi_1(y,x)$, where $k = (\sqrt{5}-1)/2$. The map $\psi(x,y)$ is piecewise linear. In fact, it can be written as

$$\psi(x,y) = \begin{cases} ((2k+3)x\,,\,-(k+1)x+(3k+4)y)\,/5 & \text{if} \quad (x,y)\in R_1, \\ ((2k+3)x+(k+1)y\,,\,-(k+1)x+(2k+4)y)\,/5 & \text{if} \quad (x,y)\in R_2, \\ ((2k+4)x-(k+1)y\,,\,(k+1)x+(2k+3)y)\,/5 & \text{if} \quad (x,y)\in R_3, \\ ((2k+4)x+(k+1)y\,,\,(k+1)x+(2k+4)y)\,/5 & \text{if} \quad (x,y)\in R_4, \\ ((3k+4)x-(k+1)y\,,\,(2k+3)y)\,/5 & \text{if} \quad (x,y)\in R_5, \end{cases}$$

where the regions R_i are defined by

$$\begin{aligned} &R_1 = \{(x,y)\in\mathbb{R}^2 : y\geq 0, x-y\geq 0\}, && R_2 = \{(x,y)\in\mathbb{R}^2 : x\geq 0, y\leq 0\},\\ &R_3 = \{(x,y)\in\mathbb{R}^2 : x\leq 0, y\geq 0\}, && R_4 = \{(x,y)\in\mathbb{R}^2 : x\leq 0, y\leq 0\},\\ &R_5 = \{(x,y)\in\mathbb{R}^2 : x\geq 0, y-x\geq 0\}. \end{aligned}$$

These five regions are delimited by five half-straight lines, starting at the origin, and the map ψ sends these lines to other five different half-straight lines. Furthermore, the restriction of ψ at each R_i is a linear map with positive determinant, so each of these linear maps preserves the orientation. Hence, ψ is a homeomorphism from $\mathbb{R}^2$ to itself. □

In a similar manner, if we consider equation (17), for each $x,y,z\in\mathbb{R}$ the eight-cycle can be determined trough the expressions:

$$x, y, z, m_1 - x, m_2 - x - y, m_3 - x - y - z, m_2 - y - z, m_4 - z, \tag{19}$$

where

$$\begin{aligned} &m_1 = \max(0,y,z), && m_2 = \max(0,x,y,z,x+z),\\ &m_3 = \max(0,x,y,2y,z,x+y,x+z,y+z), && m_4 = \max(0,x,y). \end{aligned}$$

A computation similar to the one performed above and inspired again by the proof of the Montgomery-Bochner Theorem by taking $L(x,y,z) = (y,z,-x+ky+kz)$ with $k=\sqrt{2}-1$, gives the function $\psi(x,y,z)$ defined in components as:

$$\psi_1(x,y,z)=\frac{1}{8}\big(\,(8+2k)\,x+(2+2k)\,y+2z-(2+k)\,(m_1+m_3)+km_4\big),$$
$$\psi_2(x,y,z)=\frac{1}{8}\big(\,(2+2k)\,x+(8+2k)\,y+(2+2k)\,z+m_1-(4+2k)m_2-m_3+m_4\big),$$
$$\psi_3(x,y,z)=\frac{1}{8}\big(2x+(2+2k)\,y+(8+2k)\,z+km_1-(2+k)(m_3+m_4)\big).$$

Clearly $\psi(x,y,z)$ is again a piecewise linear map. To prove that it is bijective is a tedious task because the whole space $\mathbb{R}^3$ is divided into many pieces. Nevertheless on each of these pieces is a linear map. Although we have not performed the complete study we are convinced that it is a homeomorphism from $\mathbb{R}^3$ into itself.

4.3. **Periodic recurrences constructed from symmetric functions.** We recall the nice family of periodic recurrences introduced in [2]. Let $U\subset\mathbb{R}^{k+1}$ be an open set and $G:U\to\mathbb{R}$ a symmetric map (i.e. invariant with respect any permutation of the $(k+1)$ variables) and such that the function

$$\mu\longrightarrow G_{(y_1,y_2,\ldots,y_k)}(\mu):=G(y_1,y_2,\ldots,y_k,\mu)$$

is invertible for all points $(y_1,y_2,\ldots,y_k)$ such that $(y_1,y_2,\ldots,y_k,\lambda)\in U$ for some $\lambda\in\mathbb{R}$.

Theorem 4.4. *([2]) Let G and U be defined as above. Let p be a positive integer such that $p>n$ and λ any real fixed number. Suppose that $p-n$ divides n and set $\ell=n/(p-n)$. Then the recurrence associated to the map*

$$F(x_1,x_2,\ldots,x_n)=\Big(x_2,\ldots,x_{n-1},G^{-1}_{(x_1,x_{1+(p-n)},\ldots,x_{1+(\ell-1)(p-n)}),}(\lambda)\Big)$$

is periodic of period p.

Remark 4.5. *Observe that all the periodic recurrences given in Theorem 4.4 with $p-n>1$ are derived from the $(n+1)$-periodic recurrences corresponding to $p=n+1$, given by the map*

$$F(x_1,x_2,\ldots,x_n)=\Big(x_2,\ldots,x_{n-1},G^{-1}_{(x_1,x_2,\ldots,x_n)}(\lambda)\Big)\,. \tag{20}$$

By using this remark, Theorem A and the easy fact that when a difference equation is $\mathcal{C}^k$-linearizable then all the difference equations derived from it are as well $\mathcal{C}^k$-linearizable, we get the following result:

Theorem 4.6. *Consider a periodic difference equation of the ones given in Theorem 4.4 defined on some open connected set U. Then it is linearizable.*

Notice also, that from the symmetry of G it is easy to see that all the recurrences given in Theorem 4.4 linearize into only one of the two linear possible models given in Theorem A, the one given by the map L_1.

In order to see some concrete recurrences for which the above result works we give a family of examples. It includes some of the ones given in [2]. Consider the symmetric polynomials in k variables:

$$\begin{array}{rcl}
\sigma_0(x_1,\ldots,x_k) & = & 1,\\
\sigma_1(x_1,\ldots,x_k) & = & x_1+\cdots+x_k,\\
\sigma_2(x_1,\ldots,x_k) & = & x_1x_2+x_1x_3+\cdots+x_{k-1}x_k,\\
\vdots & \vdots & \vdots\\
\sigma_{k-1}(x_1,\ldots,x_k) & = & x_1x_2\ldots x_{k-1}+x_1x_2\ldots x_{k-2}x_k+\cdots+x_2x_3\ldots x_k,\\
\sigma_k(x_1,\ldots,x_k) & = & x_1x_2\ldots x_k.
\end{array}$$

Associated to them and for $k=n+1$ take the polynomial

$$G(x_1,x_2,\ldots,x_n,\mu)=\sum_{i=0}^{n+1}\alpha_i\,\sigma_i(x_1,\ldots,x_n,\mu),$$

where α_i are fixed arbitrary real numbers. Then if we solve $G(x_1,x_2,\ldots,x_n,\mu)=\lambda$ we get that

$$\mu=G^{-1}_{(x_1,x_2,\ldots,x_n)}(\lambda)=\frac{\lambda-\sum_{i=0}^{n}\alpha_i\,\sigma_i(x_1,\ldots,x_n)}{\sum_{i=1}^{n}\alpha_i\,\sigma_{i-1}(x_1,\ldots,x_n)}.$$

Hence, in a suitable open set U, the recurrence given by the map

$$F(x_1,x_2,\ldots,x_n)=\left(x_2,\ldots,x_{n-1},\frac{\lambda-\sum_{i=0}^{n}\alpha_i\,\sigma_i(x_1,\ldots,x_n)}{\sum_{i=1}^{n}\alpha_i\,\sigma_{i-1}(x_1,\ldots,x_n)}\right)$$

is $(n+1)$-periodic. In particular, when $\alpha_2\neq 0$, the second order 3-periodic recurrences generated by this method are

$$x_{j+2}=\frac{a-b(x_j+x_{j+1})-x_jx_{j+1}}{b+x_j+x_{j+1}},$$

where $a=(\lambda-\alpha_0)/\alpha_2$ and $b=\alpha_1/\alpha_2$ are arbitrary real parameters.

4.4. **Coxeter difference equations.** For each $n\in\mathbb{N}$, Coxeter gives in his nice paper [14] the following $(n+3)$-periodic recurrences:

$$x_{j+n}=1-\cfrac{x_{j+n-1}}{1-\cfrac{x_{j+n-2}}{1-\cfrac{x_{j+n-3}}{1-\cdots\cfrac{x_{j+1}}{1-x_j}}}}:=f_n(x_j,x_{j+1},\ldots,x_{j+n-1}).$$

For instance, for $n=2,3$, we get

$$x_{j+2}=1-\frac{x_{j+1}}{1-x_j},\quad\text{and}\quad x_{j+3}=\frac{1-x_j-x_{j+1}-x_{j+2}+x_jx_{j+2}}{1-x_j-x_{j+1}},$$

respectively. Observe that for $n=2$ the above recurrence corresponds, in the new variables $u_j=x_j-1$, to the 5-periodic Lyness equation. Coxeter's proof of the global periodicity is geometrical. In [11] we provide an algebraic proof together with some other properties of these recurrences. In particular we show that they have $[\frac{n+2}{2}]$ different fixed points.

We do not know a way of studying the problem of global linearization of the general Coxeter recurrences at each of the connected components which contains each of the $[\frac{n+2}{2}]$ fixed points. For instance, when $n=3$, the 6-periodic map associated to the recurrence is

$$F_3(x,y,z)=\left(y,z,\frac{1-x-y-z+xz}{1-x-y}\right).$$

It is well-defined in $\mathbb{R}^3 \setminus \mathcal{L}$ where $\mathcal{L}$ is the union of the surfaces $x = 0, y = 0, z = 0, 1-x-y-z+xz = 0, x+y-1$ and $y+z-1 = 0$. Then the fixed point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ is in the invariant bounded connected component limited by the planes $x = 0, y = 0$ and $z = 0$ and the surface $1 - x - y - z + xz = 0$. The other fixed point $(1, 1, 1)$ is in an unbounded invariant connected component. From this study and using similar tools that the ones introduced to study the Lyness type recurrences we are convinced that we could prove a global linearization result for this case. Nevertheless a new tool, maybe using the geometric flavor of the Coxeter recurrences, should be introduced to study the problem for arbitrary n.

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