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Uniqueness of Limit Cycles For Liénard Differential Equations of Degree Four

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Abstract

We prove that any classical Liénard differential equation of degree four has at most one limit cycle, and the limit cycle is hyperbolic if it exists. This result gives a positive answer to the conjecture by A. Lins, W. de Melo and C. C. Pugh [4] in 1977 about the number of limit cycles for polynomial Liénard differential equations for $n = 4$.

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1 Introduction

The study of Liénard differential equations has a long history and a lot of results were obtained, see [8] for example. A classical polynomial Liénard differential equation can be written as a planar system

$$\begin{aligned} \dot{x} &= y - F(x), \\ \dot{y} &= -x, \end{aligned} \tag{1.1}$$

where $F(x)$ is a polynomial of degree n. In 1977 A. Lins, W. de Melo and C. C. Pugh conjectured in [4] that the classical Liénard differential equation of degree $n \geq 3$ has at most $\left[\frac{n-1}{2}\right]$ limit cycles, where $\left[\frac{n-1}{2}\right]$ means the largest integer less than or equal to $\frac{n-1}{2}$. They also proved that the conjecture is true for $n = 3$. In 2007 F. Dumortier, D. Panazzolo and R. Roussarie [3] gave a counterexample to this conjecture for $n = 7$ and they mentioned that it can be extended to $n \geq 7$ odd. Recently, P. De Maesschalck and F. Dumortier proved in [1] that the classical Liénard differential equation of degree $n \geq 6$ can have $\left[\frac{n-1}{2}\right] + 2$ limit cycles. In the last two papers the discussions are based on singular perturbation theory, and the authors used relaxation oscillation solutions to study the number of limit cycles.

In 1982 Xianwu Zeng gave some results about the uniqueness of limit cycle for Liénard differential equations in [6, 7]. As an application he found a sufficient condition to guarantee the uniqueness of limit cycles for a subclass of classical Liénard differential equations of degree 4. In Remark 3.9 we will precisely explain how this condition can be applied to partial cases in our study. Some techniques in this paper are stimulated or borrowed from Zeng's work. We will prove the following theorem.

Theorem 1.1. *Any classical Liénard differential equation of degree four has at most one limit cycle, and the limit cycle is hyperbolic, if it exists.*

Theorem 1.1 shows that the Lins-de Melo-Pugh's conjecture is also true for $n = 4$. So at this moment only remains open the conjecture for $n = 5$. We give a setting of the equation and some lemmas in Section 2, then prove Theorems 1.1 in Section 3.

2 Preliminaries

Consider a classical Liénard Equation of the form (1.1), where $F(x)$ is a polynomial of degree four.

Lemma 2.1. *Without loss of generality, we can transform* (1.1) *to*

$$\begin{aligned}\dot{x} &= y - F(x),\\ \dot{y} &= -(x-\lambda),\end{aligned} \tag{2.1}$$

where λ is a constant, and the function F has the form

$$F(x) = \frac{a}{2}x^2 + \frac{b}{3}x^3 + \frac{1}{4}x^4, \tag{2.2}$$

satisfying $a \geq 0, b \geq 0$ and $a \geq \frac{2}{9}b^2$. Moreover, the shape of the curve $C_F := \{(x,y) : y = F(x)\}$ has only 4 cases, shown in Figure 1. The shape looks as in case (A) if $a > \frac{1}{4}b^2$; in case (B) if $a = \frac{1}{4}b^2$, where $x' = -\frac{1}{2}b$ corresponding to the inflection point; in case (C) if $\frac{2}{9}b^2 < a < \frac{1}{4}b^2$, where

$$x_m = \frac{1}{2}(-b - \sqrt{b^2 - 4a}) < x_M = \frac{1}{2}(-b + \sqrt{b^2 - 4a}), \tag{2.3}$$

corresponding to the left local minimum and the local maximum respectively; and in case (D) if $a = \frac{2}{9}b^2$ where $x_m = -\frac{2b}{3}$ and $x_M = -\frac{b}{3}$. In the last case $F(x) = \frac{1}{4}x^2(x + \frac{2b}{3})^2$.

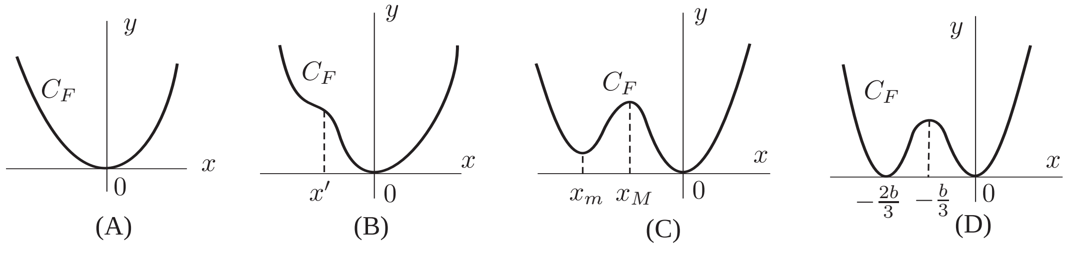

Figure 1. The different shapes of C_F

Proof. Since F in (1.1) is a polynomial of degree 4, its graph has at least one local extreme point. We shift the origin to this point, then (1.1) has the form (2.1) with

$$F(x) = \alpha x^2 + \beta x^3 + \gamma x^4,$$

where $\gamma \neq 0$. Doing the change of variables and parameter $(x, y, \lambda) \mapsto ((4\gamma)^{-\frac{1}{3}}x, (4\gamma)^{-\frac{1}{3}}y, (4\gamma)^{-\frac{1}{3}}\lambda)$, equation (2.1) keeps the same form with F as in (2.2).

Since $F'(x) = x(a + bx + x^2)$, C_F has one or two local minimum point(s). If the local minimum is unique, then it is the origin, and we have $a \geq 0$. If $b < 0$, then doing the change of variables and parameter $(x, t, \lambda) \mapsto (-x, -t, -\lambda)$, the form of the equation and the shape of C_F do not change, but in the new variables the equation has $b > 0$.

If C_F has two local minimum points and the minimum values are different, then we put the origin at the lower one. If the other minimum point is located right to the origin, then doing the change of variables and parameter $(x, t, \lambda) \mapsto (-x, -t, -\lambda)$ we move it to the left. If the two local minimum points have the same minimum value, we put the origin at the right one. Thus the left local minimum and the unique local maximum appear at x_m and x_M respectively, with the expressions given in (2.3). In this case, since $x_m < x_M < 0$ and $a \geq 0$, we certainly have $b > 0$.

By using $F(x) = x^2(\frac{a}{2} + \frac{b}{3}x + \frac{1}{4}x^2) \geq 0$ we have $a \geq \frac{2}{9}b^2$. The classification of shapes for C_F is easily obtained from $F'(x) = x(a + bx + x^2)$. Note that in case (C) we have the estimates $-\frac{2b}{3} < x_m < -\frac{b}{2} < x_M < -\frac{b}{3} < 0$. □

Lemma 2.2. *Under the assumptions of Lemma 2.1 if system (2.1) has a limit cycle, then* $a > 0$, $b > 0$, *and the value of* λ *satisfies one of the following necessary conditions:*

(i) $\lambda \in (-\frac{b}{3}, 0)$ *in cases (A) and (B);*

(ii) $\lambda \in (x_m, x_M) \cup (-\frac{b}{3}, 0)$ *in case (C);*

(iii) $\lambda \in (-\frac{2b}{3}, -\frac{b}{3}) \cup (-\frac{b}{3}, 0)$ *in case (D).*

Proof. If system (2.1) has a limit cycle, it must surround the unique singular point $M_\lambda = (\lambda, F(\lambda))$. We do the change of variables $z = x - \lambda, w = y - F(\lambda)$, which moves the origin to M_λ. Denote the part of C_F for $z \geq 0$ by $\mathcal{F}_\lambda^+$ and reverse the left part (for $z \leq 0$) to right by symmetry with respect to w-axis, and denote it by $\mathcal{F}_\lambda^-$. It is well known that, see for instance the Exercise 1 of Chapter 4 in [8], a necessary condition for the existence of a limit cycle is $\mathcal{F}_\lambda^+ \cap \mathcal{F}_\lambda^- \setminus \{O\} \neq \emptyset$, see Figure 2.

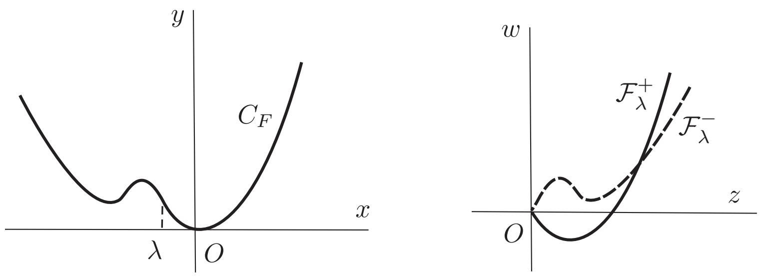

Figure 2. Comparing $\mathcal{F}_\lambda^+$ with $\mathcal{F}_\lambda^-$

Let $F_1(z) = F(z + \lambda)$, then it is easy to find that $F_1(z) - F_1(-z) = 0$ is equivalent to

$$z\left\{\left(\frac{b}{3} + \lambda\right) z^2 + \lambda\left(\lambda^2 + b\lambda + a\right)\right\} = 0, \tag{2.4}$$

where $a \geq 0$ and $b \geq 0$, see Lemma 2.1. The solution $z = 0$ corresponds to the intersection of $\mathcal{F}_\lambda^+$ and $\mathcal{F}_\lambda^-$ at the origin. It is clear that if $b = 0$, then (2.4) has no any positive solution for z, so we have $b > 0$ and $a \geq \frac{2}{9}b^2 > 0$.

In cases (A) and (B), by Lemma 2.1 we have $b^2 - 4a \le 0$, if (2.4) has a positive solution in z then $\lambda \in (-\frac{b}{3}, 0)$.

In case (C) we have $(\lambda^2 + b\lambda + a) = (\lambda - x_m)(\lambda - x_M)$ and $x_m < x_M < -\frac{b}{3} < 0$, if (2.4) has a positive solution in z then $\lambda \in (x_m, x_M) \cup (-\frac{b}{3}, 0)$.

Finally, $a = \frac{2}{9}b^2$ in case (D), and $(\lambda^2 + b\lambda + a) = (\lambda + \frac{2b}{3})(\lambda + \frac{b}{3})$, we can similarly find the necessary condition $\lambda \in (-\frac{2b}{3}, -\frac{b}{3}) \cup (-\frac{b}{3}, 0)$. □

From the proof of Lemma 2.2 we have the following result.

Corollary 2.3. *For system* (2.1)*, the set* $\mathcal{F}_\lambda^+ \cap \mathcal{F}_\lambda^- \setminus \{O\}$ *consists of at most one point.*

To study the properties of the curve C_F, we define a function $\tilde{x} = \tilde{x}(x)$ by $F(x) = F(\tilde{x})$ for $\tilde{x} < 0 < x$. Note that $y = F(x)$ has an inverse function $x = g(y)$ for $x > 0$. In cases (A) and (B) the function $\tilde{x}(x)$ is single-valued. In case (C) it is single-valued for $x \in (0, \bar{x}) \cup (\hat{x}, +\infty)$, where $\bar{x} = g(F(x_m))$ and $\hat{x} = g(F(x_M))$, and three-valued for $x \in (\bar{x}, \hat{x})$. Let

$$u(x) = x + \tilde{x}(x), \;\; v(x) = x\,\tilde{x}(x) < 0, \;\; \text{for } x > 0. \tag{2.5}$$

Lemma 2.4. *In cases (A) - (C) with* $a > 0$ *and* $b > 0$ *we have* $-\frac{2b}{3} < u(x) < 0$ *for* $x \in (0, +\infty)$*, where* $u(x)$ *is defined for any branch of* $\tilde{x}$ *in case (C).*

Proof. By using $F(\tilde{x}(x)) = F(x)$ it is easy to find that

$$u(x)\,(3u(x)^2 + 4bu(x) + 6a) = 2(3u(x) + 2b)\,v(x). \tag{2.6}$$

Note that $a > 0, b > 0$, $v(x) < 0$, and $u(x) \to 0$ as $x \to 0$ (in case (C) this is true for the nearest branch of $\tilde{x}$ to the y-axis). Hence $u(x) < 0$ as $0 < x \ll 1$. For the other branches of $\tilde{x}$ in case (C), $|\tilde{x}|$ is bigger, hence we still have $u(x) < 0$. On the other hand,

$$3u(x)^2 + 4bu(x) + 6a = 3\left(u(x) + \frac{2}{3}b\right)^2 + 6\left(a - \frac{2}{9}b^2\right) > 0,$$

because $a > \frac{2}{9}b^2$ (see Lemma 2.1). Hence from (2.6) we find $u(x) < 0$ and $3u(x) + 2b > 0$ for all $x > 0$, because $v(x) < 0$ for all $x > 0$. □

3 Proof of Theorem 1.1

By Lemmas 2.1 and 2.2 we only need to consider the uniqueness of limit cycles for system (2.1) under the conditions described in these lemmas. More precisely we will prove the following result.

Theorem 3.1. *If system* (2.1) *has a limit cycle, then it is unique and hyperbolic. Moreover, the limit cycle is stable in cases (A) and (B), and in cases (C) and (D) with* $\lambda \in (-\frac{b}{3}, 0)$*; unstable in case (C) with* $\lambda \in (x_m, x_M)$ *and in case (D) with* $\lambda \in (-\frac{2b}{3}, -\frac{b}{3})$*.*

Any limit cycle of system (2.1) must surround the unique singular point $(\lambda, F(\lambda))$. For convenience we let $\bar{x} = x - \lambda$ and $\bar{y} = y - F(\lambda)$, then still use (x, y) instead of $(\bar{x}, \bar{y})$, system (2.1) becomes

$$\begin{aligned} \dot{x} &= y - E(x), \\ \dot{y} &= -x. \end{aligned} \tag{3.1}$$

where $E(x) = F(x+\lambda) - F(\lambda) = a_1 x + a_2 x^2 + a_3 x^3 + \frac{1}{4}x^4$, and

$$a_1 = \lambda(a + b\lambda + \lambda^2),\ \ a_2 = \frac{1}{2}(a + 2b\lambda + 3\lambda^2),\ \ a_3 = \frac{1}{3}(b + 3\lambda). \tag{3.2}$$

By using Lemmas 2.1 and 2.2 it is easy to verify the following result.

Lemma 3.2. *If system (3.1) has a limit cycle, then the following statements hold.*

(i) $a_1 < 0 < a_3$ in cases (A), (B), and in cases (C) and (D) with $\lambda \in (-\frac{b}{3}, 0)$.

(ii) $a_1 > 0 > a_3$ in case (C) with $\lambda \in (x_m, x_M)$, and in case (D) with $\lambda \in (-\frac{2b}{3}, -\frac{b}{3})$.

Lemma 3.3. *Suppose $a > 0$, $b > 0$. If $\lambda \in (-\frac{b}{3}, 0)$, then $\frac{d}{dx}\left(\frac{E'(x)}{x}\right)$ has exactly one zero point x^*, located in $(-\infty, 0)$. Moreover in cases (C) and (D)*

$$x^* \in (x_m - \lambda, x_M - \lambda). \tag{3.3}$$

In case (C) if $\lambda \in (x_m, x_M)$, let $G(x) = E(-x)$, then $\frac{d}{dx}\left(\frac{G'(x)}{x}\right)$ has exactly one zero point x^*, located in $(-\infty, 0)$, and more precisely

$$x^* \in (\lambda, \lambda - x_M). \tag{3.4}$$

Proof. Calculation shows

$$\frac{d}{dx}\left(\frac{E'(x)}{x}\right) = \frac{1}{x^2}[2x^3 + (b+3\lambda)x^2 - \lambda(\lambda^2 + b\lambda + a)]. \tag{3.5}$$

It is easy to check that if $\lambda \in (-\frac{b}{3}, 0)$ then $b + 3\lambda > 0$ and $-\lambda(\lambda^2 + b\lambda + a) > 0$, hence the above function has no real solution for $x > 0$. It is well-known that for a cubic equation $\alpha x^3 + \beta x^2 + \gamma = 0$, if $\Delta = \alpha\gamma(27\alpha^2\gamma + 4\beta^3) > 0$, then the equation has exactly one real solution. For the numerator of (3.5) we have $\alpha > 0, \beta > 0$ and $\gamma > 0$, hence $\Delta > 0$. So (3.5) has exactly one zero point for $x < 0$. Since in cases (C) and (D) $E'(x) = F'(x+\lambda)$ has two negative zeros at $x_m - \lambda < x_M - \lambda$, the unique zero of $\frac{d}{dx}\left(\frac{E'(x)}{x}\right)$ must between them, implying (3.3).

We next consider the case $\lambda \in (x_m, x_M)$. Let $G(x) = E(-x)$, then

$$\frac{d}{dx}\left(\frac{G'(x)}{x}\right) = \frac{1}{x^2}[2x^3 - (b+3\lambda)x^2 + \lambda(\lambda^2 + b\lambda + a)]. \tag{3.6}$$

In this case we have $b + 3\lambda < 0$ and $\lambda(\lambda^2 + b\lambda + a) > 0$, hence the above equality is positive for $x > 0$, and still has exactly one zero point for $x < 0$. (3.4) can be checked similarly. □

Lemma 3.4. *Suppose $a > 0$, $b > 0$ and $\lambda \in (-\frac{b}{3}, 0)$. Let $\tilde{x} = \tilde{x}(x)$ be defined by $E(x) = E(\tilde{x})$ for $\tilde{x} < -\lambda < x$. Then in cases (A) - (C) the function $\sigma(x) = \frac{E'(x)}{x} - \frac{E'(\tilde{x})}{\tilde{x}}$ has exactly one zero point $x_1 > -\lambda$, satisfying $(x - x_1)\sigma(x) > 0$. Moreover $\tilde{x}(x_1) > x_m - \lambda$ in case (C).*

Proof. It is convenient to use $\xi = x + \lambda$, and to prove that $\sigma(\xi) = \frac{F'(\xi)}{\xi - \lambda} - \frac{F'(\tilde{\xi})}{\tilde{\xi} - \lambda}$ has exactly one zero for $\tilde{\xi} < 0 < \xi$, where F is given in (2.2), and $\tilde{\xi} = \tilde{\xi}(\xi)$ is defined by $F(\xi) = F(\tilde{\xi})$. In fact ξ and $\tilde{\xi}$ here are the x and $\tilde{x}$ in Lemma 2.4. It is easy to find $\sigma(\xi) = -\frac{(\xi - \tilde{\xi})f(\xi, \tilde{\xi}(\xi))}{(\xi - \lambda)(\tilde{\xi} - \lambda)}$, where

$$\begin{aligned} f(\xi, \tilde{\xi}) &= -\xi\tilde{\xi}(\xi + \tilde{\xi} + b - \lambda) + \lambda(\xi^2 + \tilde{\xi}^2) + b\lambda(\xi + \tilde{\xi}) + \lambda a \\ &= -v(u + b + \lambda) + \lambda(u^2 + bu + a), \end{aligned} \tag{3.7}$$

$u=\xi+\tilde{\xi}$ and $v=\xi\,\tilde{\xi}$. By Lemma 2.4, in cases (A) - (C), $-\frac{2b}{3}<u<0$ for $\xi>0$. Solving v from (2.6) and substituting it in (3.7), we find $\sigma(\xi)=\frac{g(u)}{2(3u+2b)}$, where

$$g(u)=-3u^4+(3\lambda-7b)u^3-2(2b^2-3b\lambda+3a)u^2+2b(2b\lambda-3a)u+4ab\lambda. \tag{3.8}$$

Note that $g(-\frac{2b}{3})=\frac{4b}{27}(b+3\lambda)(9a-2b^2)>0$ and $g(0)=4ab\lambda<0$, we can use Sturm Theorem to obtain that $g(u)$ has exactly one root for $u\in(-\frac{2b}{3},0)$, which implies the desired result. We give the detailed computations in Appendix 1. Note that $\sigma(0)=a\lambda<0$ and in case (C) we have $\sigma(\xi)>0$ when $\tilde{\xi}(\xi)=x_m$, hence $\tilde{x}(x_1)>x_m-\lambda$. □

Suppose that system (3.1) has a limit cycle L, then $\mathcal{F}_\lambda^+\cap\mathcal{F}_\lambda^-\setminus\{O\}\neq\emptyset$. By Corollary 2.3, it contains a unique point. Let $y=\varphi(x)$ for $y\geq E(x)$ and $y=\psi(x)$ for $y\leq E(x)$ are the expressions of L, and L intersects C_E at points P and Q, where $x_P<0<x_Q$, and intersects the y-axis at points A and B, where $y_A<0<y_B$.

We will study the sign of the following integral along L for system (3.1).

$$I_E(L):=-\oint_{L^+}E'(x)\,dt=\oint_{L^+}\frac{E'(x)}{x}dy=\oint_{L^+}\frac{E'(x)}{E(x)-y}dx, \tag{3.9}$$

where L^+ means that the integral is taken along L clockwise, given by the direction of the vector field (3.1). The different forms of $I_E(L)$, listed above, will be used in different places.

Since $-E'(x)$ is the divergence of vector field (3.1), the following result is easily obtained from Theorem 2.2 of [8] or Theorem 1.23 of [2] for instance.

Lemma 3.5. *If $I_E(L)<0$ (or >0), then the limit cycle L is stable (or unstable) and hyperbolic.*

Lemma 3.6. *Suppose that $E(x)<0<E(-x)$ for $0<x\ll 1$, and there is a $x_0>0$ such that $E(x_0)=E(-x_0)$ and $E(x)<E(-x)$ for $0<x<x_0$, then the following statements hold:*

(i) $\psi(-x)<\psi(x)<0<\varphi(-x)<\varphi(x)$ for $0<x<x_0$.

(ii) $x_P<-x_0$ and $x_Q>x_0$.

(iii) $y_P=F(x_P)<y_Q=F(x_Q)$.

(iv) In the region $x_M-\lambda\leq x<+\infty$ system (3.1) has at most one limit cycle L, and $I_E(L)<0$ if L exists.

Proof. Statement (i) follows easily from the fact that $y=\varphi(x)$ and $y=\psi(x)$ satisfy the differential equation $\frac{dy}{dx}=\frac{x}{E(x)-y}$ and by using the Differential Inequality Theorem. Statements (ii) and (iii) follow from [5], also can see the formula (4.47) of [8]. Statement (iv) follows from Lemma 3.3 and the famous Uniqueness Theorem of Zhang, see Theorem 4.6 of [8]. □

Definition 3.7. *A piece of arc $\{(x,y):y=E(x),0\leq\alpha\leq\beta\}\subset C_E$ is called a* U-arc, *if $E(\alpha)=E(\beta)$ and $E(x)<E(\beta)$ for all $x\in(\alpha,\beta)$. Similarly, an arc $\{(x,y):y=E(x),\alpha\leq\beta\leq 0\}\subset C_E$ is called a* Λ-arc, *if $E(\alpha)=E(\beta)$ and $E(x)>E(\beta)$ for all $x\in(\alpha,\beta)$.*

Note that by Lemma 2.2, if system (3.1) has a limit cycle and C_E has two local minimum points then these two minimum points are located in two sides of the origin, hence a U-arc or a Λ-arc, that we will treat, is simply a convex or concave curve with a unique local minimum or local maximum.

Lemma 3.8. *Suppose that system (3.1) has a limit cycle L, and that in the bounded region by L the curve C_E contains a U-arc or a Λ-arc, for $x \in [\alpha, \beta]$, then*

$$I_E[\alpha,\beta] := -\int_{\{x\in[\alpha,\beta]\}\cap L^+} E'(x)\,dt < 0.$$

Proof. Suppose that the orbit L meets the curve C_E at points P and Q respectively, then $y = \varphi(x)$ is monotonically increasing for $x \in (x_P, 0)$ and decreasing for $x \in (0, x_Q)$, and $y = \psi(x)$ is monotonically decreasing for $x \in (x_P, 0)$ and increasing for $x \in (0, x_Q)$.

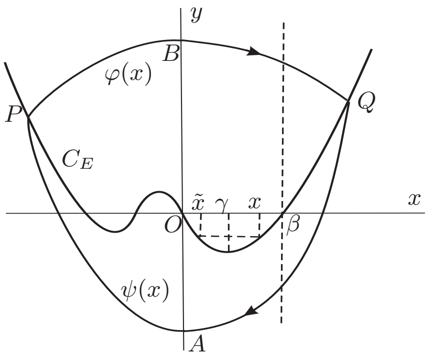

Figure 3. The curve C_E has a U-arc.

We first consider the U-arc for $x \in [\alpha, \beta]$ ($\alpha = 0$ in Figure 3). Note that $I_E[\alpha,\beta] = I_E^{\varphi}[\alpha,\beta] + I_E^{\psi}[\alpha,\beta]$, where I_E^{φ} and I_E^{ψ} are integrals taken along $y = \varphi(x)$ and along $y = \psi(x)$ respectively. By formula (3.9) we have

$$I_E^{\varphi}[\alpha,\beta] = \int_{\alpha}^{\gamma} \frac{E'(s)}{E(s)-\varphi(s)}ds + \int_{\gamma}^{\beta} \frac{E'(x)}{E(x)-\varphi(x)}dx,$$

where γ corresponds to the minimum point on the U-arc. As we did in the proof of Lemma 2.4, $\tilde{x}(x) \in (\alpha, \gamma)$ is the function defined by $E(\tilde{x}) = E(x)$ for $x \in (\gamma, \beta)$, then $\frac{d\tilde{x}}{dx} = \frac{E'(x)}{E'(\tilde{x})}$. We change variable in the first integral by $s = \tilde{x}(x)$, then we find

$$I_E^{\varphi}[\alpha,\beta] = \int_{\gamma}^{\beta} \frac{E'(x)\,(\varphi(x)-\varphi(\tilde{x}(x)))}{(E(x)-\varphi(x))(E(\tilde{x}(x))-\varphi(\tilde{x}(x)))}dx. \tag{3.10}$$

Since for $x \in (\gamma, \beta)$ we have $E'(x) > 0$, $\varphi(x) < \varphi(\tilde{x}(x))$, $E(x) < \varphi(x)$ and $E(\tilde{x}(x)) < \varphi(\tilde{x}(x))$, we obtain $I_E^{\varphi}[\alpha,\beta] < 0$. Similarly, we get $I_E^{\psi}[\alpha,\beta] < 0$. Note that $\psi(x) > \psi(\tilde{x}(x))$ for $x \in (\gamma, \beta)$, but the integral is taken from β to α for x.

The proof for a Λ-arc for $x < 0$ is completely similar. □

Proof of Theorem 3.1

(I) cases (A) and (B).

We suppose that system (3.1) has a limit cycle L, expressed by $y = \varphi(x)$ above C_E and by $y = \psi(x)$ below C_E, and L intersects C_E at P and Q respectively (we will use these notations in all cases). Since $E(x) < 0 < E(-x)$ for $0 < x < \beta$, and C_E has a unique minimum point, we have $x_0 > \beta$, where $E(x_0) = E(-x_0)$. By Lemma 3.6, $x_P < -x_0$, $x_Q > x_0 > \beta$, and $y_Q > y_P$, see Figure 4.

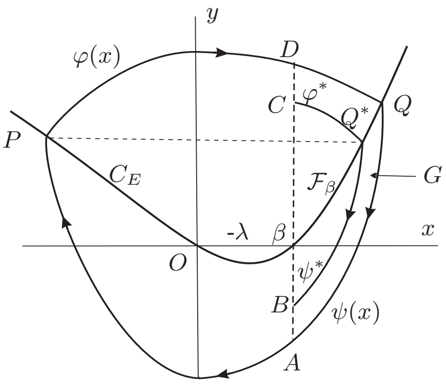

Figure 4. The cases (A) and (B).

We first prove

$$I_E[0, x_Q] < I_E[\beta, x_Q] < I_E^*[\beta, x_{Q^*}], \tag{3.11}$$

where $Q^* = (C_E \cap \{y = y_P\}) \setminus \{P\}$, and

$$I_E^*[\beta, x_{Q^*}] = \int_\beta^{x_{Q^*}} \left[\frac{E'(x)}{E(x) - \varphi^*(x)} - \frac{E'(x)}{E(x) - \psi^*(x)} \right] dx,$$

where $y = \varphi^*(x)$ and $y = \psi^*(x)$ are orbits of system (3.1) from point Q^*, above and below C_E respectively. It is obvious that $\varphi^*(x) < \varphi(x)$ and $\psi^*(x) > \psi(x)$ for $\beta \le x < x_{Q^*}$.

Since $\{x \in [0, \beta]\} \cap C_E$ is a U-arc, by Lemma 3.8 we have $I_E[0, \beta] < 0$. On the other hand $I_E[0, x_Q] = I_E[0, \beta] + I_E[\beta, x_Q]$, hence $I_E[0, x_Q] < I_E[\beta, x_Q]$.

Denote the region for $x \ge \beta$ and bounded by the orbits L and L^* by G, and using the Green formula we obtain

$$I_E^*[\beta, x_{Q^*}] - I_E[\beta, x_Q] = \iint_G \frac{d}{dx}\left(\frac{E'(x)}{x}\right) dxdy + \frac{E'(\beta)}{\beta}(y_B - y_A + y_D - y_C). \tag{3.12}$$

Note that $E'(\beta) > 0$, and by Lemma 3.3 we have $\frac{d}{dx}\frac{E'(x)}{x} > 0$ in G, because $x \ge \beta > 0$ and $\lambda \in (-\frac{b}{3}, 0)$. Hence $I_E^*[\beta, x_{Q^*}] - I_E[\beta, x_Q] > 0$, and (3.11) is proved.

We next prove

$$I_E[x_P, 0] + I_E^*[\beta, x_{Q^*}] < 0. \tag{3.13}$$

For this purpose we use the function $\tilde{x} = \tilde{x}(x)$, defined by $E(x) = E(\tilde{x})$ for $x \in [\beta, x_{Q^*}]$ and $\tilde{x} \in [x_P, 0]$, then

$$I_E^\varphi[x_P, 0] + I_E^{\varphi^*}[\beta, x_{Q^*}] = \int_{x_P}^0 \frac{E'(s)}{E(s) - \varphi(s)} ds + \int_\beta^{x_{Q^*}} \frac{E'(x)}{E(x) - \varphi^*(x)} dx.$$

Changing variable by $s = \tilde{x}(x)$ in first integral and noting $E(x) = E(\tilde{x})$ and $\tilde{x}'(x) = E'(x)/E'(\tilde{x})$, we have

$$I_E^{\varphi}[x_P, 0] + I_E^{\varphi^*}[\beta, x_{Q^*}] = \int_{\beta}^{x_{Q^*}} \frac{E'(x)(\varphi^*(x) - \varphi(\tilde{x}(x)))}{(E(x) - \varphi^*(x))(E(\tilde{x}(x)) - \varphi(\tilde{x}(x)))} dx.$$

We prove that

$$\eta(x) := \varphi(\tilde{x}(x)) - \varphi^*(x) > 0, \quad \text{for } x \in [\beta, x_{Q^*}). \tag{3.14}$$

It is easy to see $\eta(\beta) = \varphi(0) - \varphi^*(\beta) > 0$ and $\eta(x_{Q^*}) = \varphi(x_P) - \varphi^*(x_{Q^*}) = 0$. We let $\omega(x) = \varphi(\tilde{x}(x))$, then

$$\frac{d\omega(x)}{dx} = \frac{d\varphi}{d\tilde{x}}\frac{d\tilde{x}}{dx} = \frac{\tilde{x}}{E(\tilde{x}) - \varphi(\tilde{x})}\frac{E'(x)}{E'(\tilde{x})} = \frac{x}{E(x) - \omega(x)} \frac{E'(x)}{x} \Big/ \frac{E'(\tilde{x})}{\tilde{x}}.$$

Note that $E(x) = E(\tilde{x}) < \omega(x)$, and $\frac{E'(x)}{x} \Big/ \frac{E'(\tilde{x})}{\tilde{x}} < 1$ for $0 < x - \beta \ll 1$, since $\tilde{x}(x) \to 0 - 0$ as $x \to \beta + 0$, we have

$$\frac{d\omega(x)}{dx} > \frac{x}{E(x) - \omega(x)}, \quad 0 < x - \beta \ll 1.$$

On the other hand

$$\frac{d\varphi^*(x)}{dx} = \frac{x}{E(x) - \varphi^*(x)}, \quad 0 \le \beta \le x_{Q^*}.$$

Thus $\eta(x) = \varphi(\tilde{x}(x)) - \varphi^*(x) = \omega(x) - \varphi^*(x)$ is increasing for $0 < x - \beta \ll 1$. But $\eta(x_{Q^*}) = 0$, implies that (3.14) holds, as it is shown in Figure 5 (i). The reason is as follows: if there is some $x \in (\beta, x_{x_Q^*})$ such that $\eta(x) < 0$, then $\frac{E'(x)}{x} - \frac{E'(\tilde{x})}{\tilde{x}}$ would have at least two zeros, see Figure 5(ii), this contradicts Lemma 3.4.

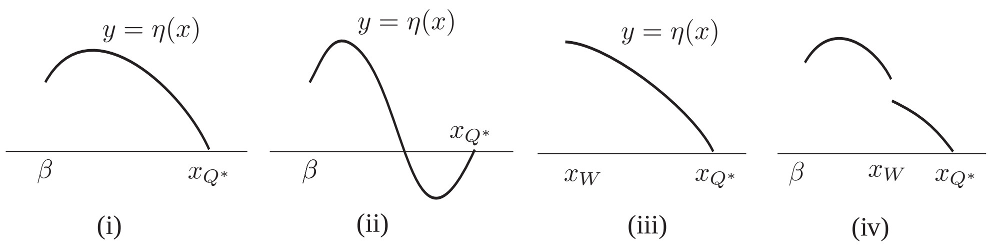

Figure 5. The different behavior of the function $\eta(x)$.

Similarly, we have $I_E^{\psi}[x_P, 0] + I_E^{\psi^*}[\beta, x_{Q^*}] < 0$, hence (3.13) follows.

From (3.11) and (3.13) we have $I_E[0, x_Q] < I_E^*[\beta, x_{Q^*}] < -I_E[x_P, 0]$, implying

$$I_E(L) = I_E[x_P, 0] + I_E[0, x_Q] < 0.$$

Therefore by Lemma 3.5 the limit cycle L, if exists, is hyperbolic and stable, and it must be unique, because two stable limit cycles surrounding a unique singularity cannot co-exist.

(II) Case (C) with $\lambda \in (-\frac{b}{3}, 0)$.

The proof is essentially the same as above with some modification. Note that $x_Z = x_M - \lambda$, where Z is the local maximum point of C_E. If a limit cycle L does not cross the line $x = x_Z$ (i. e. $x_P \geq x_Z$), then by Lemma 3.6 (iv), $I_E(L) < 0$. So we suppose $x_P < x_Z$. If $x_P \in [x_U, x_Z)$, where U is the left local minimum point of C_E, then the proof is similar and even simpler. So we suppose $x_P < x_U$, and there are three possibilities depending on $y_U > 0$, $y_U = 0$ and $y_U < 0$, shown in Figure 6 (i), (ii) and (iii) respectively.

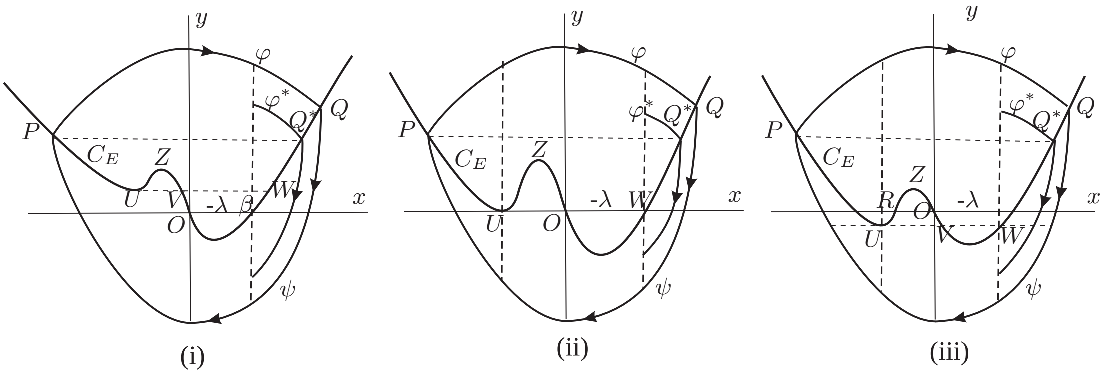

Figure 6. The case (C) with $\lambda \in (-\frac{b}{3}, 0)$.

In any case the straight line $\{y = y_U\}$ cuts C_E at points V and W for $x > 0$ ($x_W > x_V$). We take Q^* on C_E as above. Note that $x_Q > x_0 \geq x_W$, where $x_0 > 0$ with the property $E(x_0) = E(-x_0)$. In cases (i) and (ii) of Figure 6, we also have $x_Q \geq x_0 > \beta$, here β is the only positive zero of $E(x) = 0$.

For case (ii) we have $x_V = 0$ and $x_W = \beta$. By using the same way as in the proof for cases (A) and (B), we can prove

$$I_E[x_W, x_Q] < I_E^*[x_W, x_{Q^*}] < -I_E[x_P, x_U], \tag{3.15}$$

i. e. $I_E[x_P, x_U] + I_E[x_W, x_Q] < 0$. In fact, by Lemma 3.4 we have $\frac{E'(x)}{x} > \frac{E'(\tilde{x})}{\tilde{x}} > 0$ for $x > x_W$ and $\tilde{x}(x) < x_U$, hence it is enough to use $\eta(x_{Q^*}) = 0$ to show that $\eta(x) = \varphi(\tilde{x}(x)) - \varphi^*(x)$ is monotonically decreasing for $x \in [x_W, x_{Q^*})$ and $\eta(x) > 0$ for $x \in (x_W, x_{Q^*})$, see Figure 5 (iii). On the other hand $I_E[x_U, x_W] < 0$ is obviously true by Lemma 3.8, because the part of C_E from point U to point W consists of a Λ-arc and a U-arc.

For case (i) of Figure 6, $\tilde{x}(x) \in (x_V, 0)$ when $x \in (\beta, x_W)$ and $\tilde{x}(x) \in (x_P, x_U)$ when $x \in (x_W, x_{Q^*})$. We first prove (3.15) by the same way as above, and find out that $\eta(x_W + 0) = \varphi(x_U) - \varphi^*(x_W) > 0$. By Lemma 3.8 we have $I_E[x_U, x_V] < 0$, and by a similar proof of (3.15) we have $I_E[x_V, 0] + I_E[\beta, x_W] < 0$, because in this case $\eta(\beta) = \varphi(0) - \varphi(\beta) > 0$, $\eta'(\beta) = 0 - \varphi'(\beta) > 0$ and $\eta(x_W - 0) = \varphi(x_V) - \varphi^*(x_W) > \varphi(x_U) - \varphi^*(x_W) = \eta(x_W + 0) > 0$, hence by using Lemma 3.4 we obtain $\eta(x) = \varphi(\tilde{x}(x)) - \varphi^*(x) > 0$ for $x \in (\beta, x_W)$, shown in Figure 5 (iv).

Finally for case (iii) of Figure 6, we have (3.15) by the same way than in case (i), and $I_E[x_R, 0] + I_E[x_V, x_W] < 0$ by Lemma 3.8, where R is the intersection point of C_E with x-axis between U and the origin. It remains to prove $I_E[x_U, x_R] + I_E[0, x_V] < 0$. We can use the same arguments than the proof of Lemma 3.8, let $\tilde{x} = \tilde{x}(x)$ for $x \in (0, x_V)$ and $\tilde{x} \in (x_U, x_R)$, defined by $E(x) = E(\tilde{x})$. Then we have a similar formula like the right hand side of (3.10), an integral from 0 to x_V. Since $E'(x) < 0$, we need to show $\varphi(x) - \varphi(\tilde{x}(x)) > 0$ for $x \in (0, x_V)$. If we reverse the part of C_E for $x < x_Z$ to the right (symmetry with respect to the line $\{x = x_Z\}$) then, by (2.4) with $\lambda = x_M$, there is no intersection with C_E for $x > x_Z$. This implies that if we reverse the part of C_E for $x < 0$ to the right (symmetry with respect to the line $\{x = 0\}$), then the image arc of C_E from point R to point U is located right to the arc of C_E from point O to point V. Thus we have $\varphi(x) > \varphi(-x) > \varphi(\tilde{x}(x))$. The first estimate is by Lemma 3.6 (i), and the second by monotonicity of the φ and $\tilde{x}(x) < -x$ for $x \in (0, x_V)$.

(III) Case (C) with $\lambda \in (x_m, x_M)$.

We will prove that if system (3.1) has a limit cycle L, then $I_E(L) > 0$. Thus by Lemma 3.5 L is hyperbolic unstable and the limit cycle is unique. For convenience we do the transformation $(x, t) \mapsto (-x, -t)$, then equation (3.1) becomes

$$\begin{aligned} \dot{x} &= y - G(x), \\ \dot{y} &= -x, \end{aligned} \tag{3.16}$$

where $G(x) = E(-x) = -a_1 x + a_2 x^2 - a_3 x^3 + \frac{1}{4}x^4$, $a_j(x)$ is the same as in (3.2) for $j = 1, 2$ or 3. For system (3.16), we prove $I_G(L) < 0$.

We still use $P = (x_P, y_P) \in C_G$ and $Q = (x_Q, y_Q) \in C_G$ to denote the most left and most right points of the limit cycle L, then by Lemma 3.6, $x_P < -x_0 < 0 < x_0 < x_Q$ and $y_P < y_Q$, where x_0 satisfies $G(x_0) = G(-x_0)$. By Lemma 3.3 and a similar result of Lemma 3.6 (iv), if $x_P \ge x_Z$, where Z is the local maximum point and $x_Z = \lambda - x_M < 0$, then $I_G(L) < 0$. Hence we only consider the case $x_P < x_Z$.

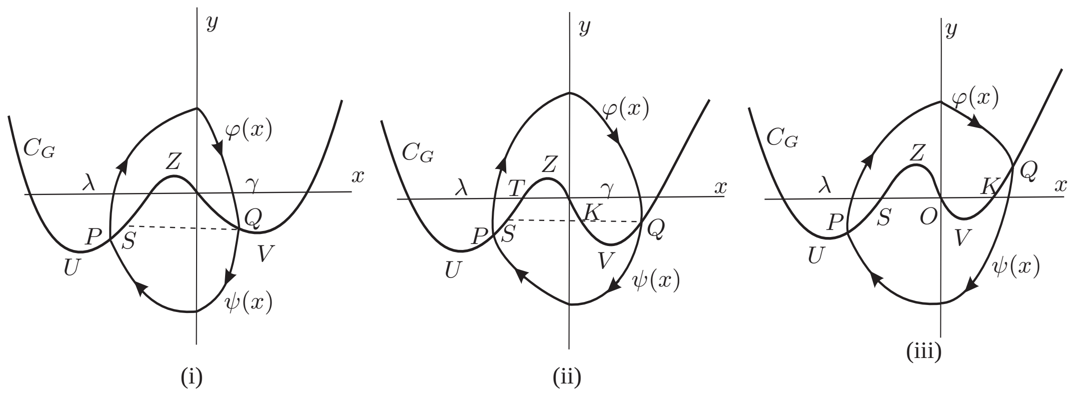

Figure 7. The three possibilities for case (C) with $\lambda \in (x_m, x_M)$ and $x_P \in (x_U, x_Z)$.

Subcase III-1: $x_P \in [x_U, x_Z)$.

Note that U is the left local minimum point of C_G and $x_U = \lambda$. It is possible $x_Q \le x_V$ (Figure 7 (i)) or $x_Q > x_V$, where V is the right local minimum point of C_G and $x_V = \gamma = \lambda - x_m > 0$. The cases (ii) and (iii) of Figure 7 correspond to $y_Q < 0$ and $y_Q \ge 0$ respectively. If $y_P \ge 0$ in case (iii), then the discussion is similar to case (ii).

In case (iii) of Figure 7, we have $I_G[x_P, x_S] < 0$, because $\frac{G'(x)}{x}dy < 0$ along this part of C_G. Similarly $I_G[x_K, x_Q] < 0$. The rest part of C_G consists of a Λ-arc and a U-arc, hence by Lemma 3.8, $I_G[x_S, x_K] < 0$. Combining them together we have $I_G(L) < 0$.

We will prove $I_G[x_P, x_Q] < 0$ in case (i) and $I_G[x_P, x_K] < 0$ in case (ii). Since the proofs are similar, we give details for the former, and only explain the difference for the later.

Observing case (i) of Figure 7, $I_G[x_P, x_S] < 0$, we need to prove $I_G[x_S, x_Q] < 0$ for the Λ-like-arc SZQ. We use the same method than for the Λ-like-arc in Figure 6 (iii). We need to prove $I_G[x_S, x_Z] + I_G[x_Z, x_Q] < 0$. For $x \in (x_Z, x_Q)$ we define $\tilde{x} = \tilde{x}(x) \in (x_S, x_Z)$ by $G(x) = G(\tilde{x})$. Since $G'(x) < 0$ for $x \in (x_Z, x_Q)$, we need to prove $\eta(x) = \varphi(x) - \varphi(\tilde{x}(x)) > 0$ in $x \in (x_Z, x_Q)$. It is obvious that $\varphi(x) > 0$ for $x \in (x_Z, 0]$. A sufficient condition of $\varphi(x) > 0$ for $x \in (0, x_Q)$ is $x_0 \geq \gamma$, see Figure 8 (i), because in this case we have $x < -\tilde{x}(x)$ for $x \in (0, x_Q)$, hence $\varphi(x) > \varphi(-x) > \varphi(\tilde{x}(x))$. The first estimate is by Lemma 3.6 (i), and the second estimate is by the monotonicity of $\varphi(x)$ for $x < 0$. We next prove that if $\lambda \in (x_m, -\frac{b}{2}]$, then $x_0 > \gamma$. In fact from $G(-x) = G(x)$ we find a unique positive solution

$$x_0 = \sqrt{\frac{3\lambda(\lambda^2 + b\lambda + a)}{-(b + 3\lambda)}}, \tag{3.17}$$

where $b + 3\lambda < 0$ and $\lambda(\lambda^2 + b\lambda + a) > 0$ for $\lambda \in (x_m, x_M)$, and $x_0^2 - \gamma^2 = x_0^2 - (\lambda - x_m)^2 = (-2(b + 3\lambda))^{-1}\chi(\lambda, a, b)$, where

$$\chi(\lambda, a, b) = (b + 3\lambda)(b + 2\lambda)\sqrt{b^2 - 4a} + 12\lambda^3 + 14b\lambda^2 + 5b^2\lambda + b(b^2 - 2a).$$

By using the Fourier-Budan Criterion (explained in Appendix 1), it is not hard to find $\chi(\lambda, a, b) > 0$ for $\lambda \in (x_m, -\frac{b}{2}]$, where $a > 0, b > 0$ and $\frac{2}{9}b^2 < a < \frac{1}{4}b^2$.

For case (ii) of Figure 7, we need to prove $I_G[x_S, x_K] < 0$ for the Λ-arc SZK. If $\lambda \in (x_m, -\frac{b}{2}]$, the proof is exactly the same as above.

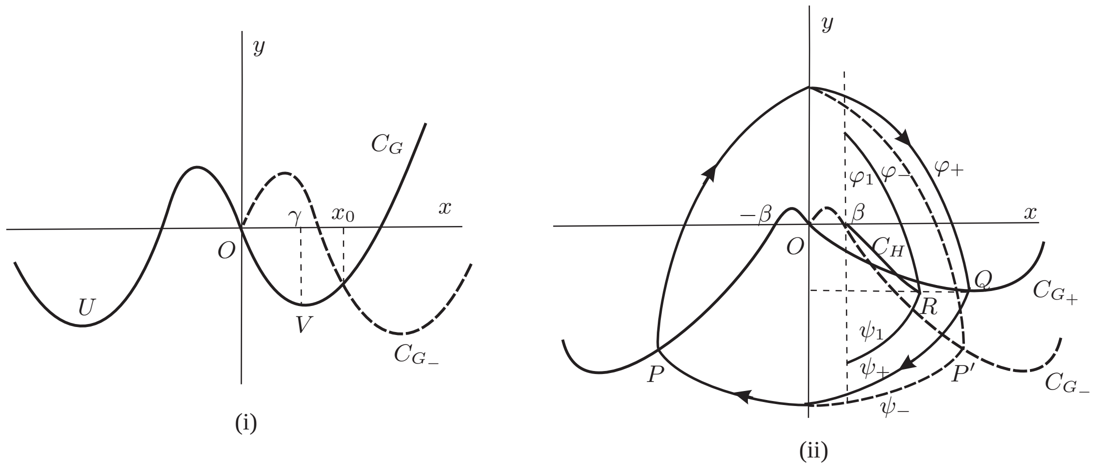

Figure 8. The study for $x_0 \geq \gamma$ and $x_0 < \gamma$ respectively.

It remains to consider the case $\lambda \in (-\frac{b}{2}, x_M)$ and $x_0 < \gamma$. In this case we will prove $I_G[x_P, x_Q] < 0$ directly by using some suitable transformations, borrowed from [6] due to Xianwu Zeng.

We denote $C_{G_+} = \{(x,y)|\, y = G(x), x > 0\}$ and $C_{G_-} = \{(x,y)|\, y = G(-x), x > 0\}$, $\varphi_+(x) = \varphi(x)$ and $\psi_+(x) = \psi(x)$ for $x \in (0, x_Q)$, and $\varphi_-(x) = \varphi(-x)$ and $\psi_-(x) = \psi(-x)$ for $x \in (0, x_{P'})$, where P' is the symmetry point of P. $(\beta, 0)$ is the only intersection point of C_{G_-} with the x-axis for $x \in (0, |\lambda|)$ and $(x_0, G(x_0))$ is the only intersection point of C_{G_+} and C_{G_-}. We certainly have $\beta < x_0$, and by assumption $x_0 < \gamma$, see Figure 8 (ii).

Let $k = \frac{y_{P'}}{y_Q} > 1$ and let $\bar{x}(x) = k^2(x-\beta)+\beta$ for $x \in (\beta, x_R)$, where $x_R = \frac{1}{k^2}(x_{P'} - \beta) + \beta$. Then $\bar{x}(x) \in (\beta, x_{P'})$ for $x \in (\beta, x_R)$. Let $H(x) = \frac{1}{k}G_-(\bar{x}(x))$, where $G_-(x) = G(-x)$ for $x > 0$, and let $\varphi_1(x) = \frac{1}{k}\varphi_-(\bar{x}(x))$ and $\psi_1(x) = \frac{1}{k}\psi_-(\bar{x}(x))$. Note that $G'_-(x) = -G'(-x)$ and by change of variable $t = -s$ we have

$$\begin{aligned} I_G[x_P, -\beta] &= \int_{x_P}^{-\beta} \left[\frac{G'(t)}{G(t) - \varphi(t)} - \frac{G'(t)}{G(t) - \psi(t)} \right] dt \\ &= \int_{\beta}^{x_{P'}} \left[-\frac{G'_-(s)}{G_-(s) - \varphi_-(s)} + \frac{G'_-(s)}{G_-(s) - \psi_-(s)} \right] ds. \end{aligned} \tag{3.18}$$

Then changing variable by $s = \bar{x}(x)$, we obtain

$$I_G[x_P, -\beta] = \int_{\beta}^{x_R} \left[-\frac{H'(x)}{H(x) - \varphi_1(x)} + \frac{H'(x)}{H(x) - \psi_1(x)} \right] dx. \tag{3.19}$$

By definition we have $y_R = H(x_R) = \frac{1}{k}G_-(\bar{x}(x_R)) = \frac{1}{k}y_{P'} = y_Q$. It is easy to find that

$$G'_+(x) - G'_-(x) = G'(x) + G'(-x) = 2[-(b+3\lambda)x^2 - \lambda(\lambda^2 + b\lambda + a)] > 0, \text{ if } x > x_1, \tag{3.20}$$

where $x_1 = \sqrt{-\frac{\lambda(\lambda^2+b\lambda+a)}{b+3\lambda}} < x_0$, see (3.17). Hence $G'_-(x) < G'_+(x) < 0$ for $x \in (x_0, x_2)$, where $x_2 =\max(x_{P'}, x_Q)$. This implies $H'(x) < 0$ for $x \in (\beta, x_R)$.

We let $C_H = \{(x,y)|\, y = H(x), \beta \le x \le x_R\}$, and prove that $C_H \cap C_{G_+}$ consists of a unique point, hence $x_R < x_Q$, since $y_R = y_Q$. In fact, if $C_H \cap C_{G_+} = \emptyset$ for $x \in (\beta, x_R)$, then $x_R > x_Q$ and $H(x) > G_+(x)$ for $x \in (\beta, x_R)$. Note that the function $y = \varphi_1(x) = \frac{1}{k}\varphi_-(\bar{x}(x))$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{\bar{x}(x)}{H(x) - y} =: f_1(x,y), \quad y > H(x), \tag{3.21}$$

and the function $y = \varphi_+(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{x}{G_+(x) - y} =: f_+(x,y), \quad y > G_+(x). \tag{3.22}$$

Hence for any $x \in (\beta, x_Q)$ and $y > H(x) > G_+(x)$ we have

$$f_1(x,y) < \frac{x}{H(x) - y} < f_+(x,y) < 0,$$

because $\bar{x}(x) > x > 0$. By the Differential Inequality Theorem we find $\varphi_1(x) \ge \varphi_+(x)$ for $x \in (\beta, x_Q)$. This implies $\varphi_1(\beta) \ge \varphi_+(\beta)$, contradicting the fact $0 < \varphi_1(\beta) = \frac{1}{k}\varphi_-(\beta) < \varphi_-(\beta) < \varphi_+(\beta)$ by Lemma 3.6 (i).

Therefore $C_H \cap C_{G_+} \neq \emptyset$. Since $0 > H(x) = \frac{1}{k}G_-(\bar{x}(x)) > G_-(\bar{x}(x))$, the intersection $C_H \cap C_{G_+}$ happens for $x > x_0$. In Appendix 2 we will prove the following property:

$$G'_+(x) - H'(x) > 0 \text{ if } G_+(x) = H(x) \text{ and } x \in (x_0, x_R). \tag{3.23}$$

Hence $C_H \cap C_{G_+}$ consists of a unique point, and $x_R < x_Q$.

Thus we can define $\hat{x} = \hat{x}(x) \in (0, x_Q)$ by $H(x) = G_+(\hat{x}(x))$ for $x \in (\beta, x_R)$. Let $\varphi_2(x) = \varphi_+(\hat{x}(x))$ and $\psi_2(x) = \psi_+(\hat{x}(x))$, then by a change of variable $s = \hat{x}(x)$, noting $\frac{d\hat{x}}{dx} = \frac{H'(x)}{G'(\hat{x}(x))}$, we have

$$\begin{aligned} I_G[0, x_Q] &= \int_0^{x_Q} \left[\frac{G'_+(s)}{G_+(s) - \varphi_+(s)} - \frac{G'_+(s)}{G_+(s) - \psi_+(s)} \right] ds \\ &= \int_\beta^{x_R} \left[\frac{H'(x)}{H(x) - \varphi_2(x)} - \frac{H'(x)}{H(x) - \psi_2(x)} \right] dx. \end{aligned} \tag{3.24}$$

From (3.19) and (3.24) we have that $I_G[X_P, -\beta] + I_G[0, x_Q]$ equals to

$$\int_\beta^{x_R} \left[\frac{H'(x)(\varphi_2(x) - \varphi_1(x))}{(H(x) - \varphi_1(x))(H(x) - \varphi_2(x))} + \frac{H'(x)(\psi_1(x) - \psi_2(x))}{(H(x) - \psi_1(x))(H(x) - \psi_2(x))} \right] dx. \tag{3.25}$$

We prove that for $x \in (\beta, x_R)$

$$\xi(x) := \varphi_2(x) - \varphi_1(x) > 0, \quad \eta(x) := \psi_2(x) - \psi_1(x) < 0. \tag{3.26}$$

Therefore, by (3.25) and (3.26), combining with the fact that $I_G[-\beta, 0] < 0$ (by Lemma 3.8), we have $I_G[x_P, x_Q] < 0$.

To prove (3.26), we first study the behavior of $\xi(x) = \varphi_+(\hat{x}(x)) - \frac{1}{k}\varphi_-(\bar{x}(x))$ and $\eta(x) = \psi_+(\hat{x}(x)) - \frac{1}{k}\psi_-(\bar{x}(x))$ at the endpoints $x = \beta$ and $x = x_R$. It is obvious that for $k = \frac{y_{P'}}{y_Q} > 1$ we have

$$\varphi_2(\beta) = \varphi_+(0) > \varphi_-(\beta) > \frac{1}{k}\varphi_-(\beta) = \varphi_1(\beta) > 0,$$

$$\psi_2(\beta) = \psi_+(0) < \psi_-(\beta) < \frac{1}{k}\psi_-(\beta) = \psi_1(\beta) < 0,$$

and

$$\xi'(\beta) = \varphi'_+(0)\frac{k\,G'_-(\beta)}{G'_+(0)} - k\,\varphi'_-(\beta) = 0 - k\,\varphi'_-(\beta) > 0,$$

$$\eta'(\beta) = \psi'_+(0)\frac{k\,G'_-(\beta)}{G'_+(0)} - k\,\psi'_-(\beta) = 0 - k\,\psi'_-(\beta) < 0.$$

Hence we obtain

$$\xi(\beta) > 0,\ \xi'(\beta) > 0; \quad \eta(\beta) < 0,\ \eta'(\beta) < 0. \tag{3.27}$$

On the other hand we have

$$\xi(x_R) = \varphi_+(x_Q) - \frac{1}{k}\varphi_-(x_{P'}) = y_Q - \frac{1}{k}y_{P'} = 0,$$

and similarly $\eta(x_R) = 0$. So we have

$$\xi(x_R) = 0,\ \eta(x_R) = 0. \tag{3.28}$$

We have shown that the functions $y = \varphi_1(x)$ and $y = \psi_1(x)$ satisfy the differential equation (3.21); it is easy to find that the functions $y = \varphi_2(x) > H(x)$ and $y = \psi_2(x) < H(x)$ satisfy

$$\frac{dy}{dx} = \frac{\hat{x}(x)}{G_+(\hat{x}(x)) - y} \frac{k\, G'_-(\bar{x}(x))}{G'_+(\hat{x}(x))} = \frac{\bar{x}(x)}{H(x) - y} \frac{k\, G'_-(\bar{x}(x))}{\bar{x}(x)} \left(\frac{G'_+(\hat{x}(x))}{\hat{x}(x)} \right)^{-1}. \tag{3.29}$$

In Appendix 2 we will prove that for $x \in (\beta, x_R)$

$$\frac{G'_+(\hat{x}(x))}{\hat{x}(x)} - \frac{k\, G'_-(\bar{x}(x))}{\bar{x}(x)} = 0 \text{ has at most one zero,} \tag{3.30}$$

under condition $G_+(\hat{x}(x)) = \frac{G_-(\bar{x}(x))}{k}$ (i. e. the definition $H(x) = G_+(\hat{x}(x))$). Comparing two differential equations (3.21) and (3.29) and using the fact (3.30), we obtain that $\xi'(x)$ and $\eta'(x)$ have at most one zero in $x \in (\beta, x_R)$. Thus, by using the facts (3.27) and (3.28) we obtain (3.26), the behavior of $\xi(x)$ looks like Figure 5 (i), and the behavior of $\eta(x)$ looks like the symmetry of $\xi(x)$ with respect to y-axis.

Finally, for case (ii) of Figure 7, the proof of $I_G[x_P, x_K] < 0$ is the same. Instead of (3.28) we use $\xi(x_R) > 0$ and $\eta(x_R) < 0$.

Subcase III-2: $x_P < x_U = \lambda$.

In this case we need to prove three facts: (a) $x_Q > x_V = \gamma$, (b) $I_G[\lambda, \gamma] < 0$, and (c) $I_G[x_P, \lambda] + I_G[\gamma, x_Q] < 0$.

We first reverse the left part of C_G, symmetric with respect to the line $\{x = x_Z\}$, and denote the image of C_G by $C_{\bar{G}}$, then $C_{\bar{G}}$ is entirely below the right part of C_G for $x > x_Z$, see Figure 9 (i). To check this, we take $\lambda = x_M$ in (2.4), then $z = 0$ (equivalent to $x = x_Z$ here) is the only zero. We also use $\bar{\varphi}$ and $\bar{\psi}$ as the images of φ and ψ for $x < x_Z$.

Note that $\ell_l := x_Z - x_U = |x_M| = -x_M$ and $\ell_r := x_V - x_Z = x_M - x_m$, hence $\ell_l - \ell_r = x_m - 2x_M = \frac{1}{2}(b - 3\sqrt{b^2 - 4a}) > 0$, since $\frac{2}{9}b^2 < a < \frac{1}{4}b^2$. This implies $\bar{\gamma} = x_{\bar{U}} > \gamma$, where $\bar{U}$ is the image of U.

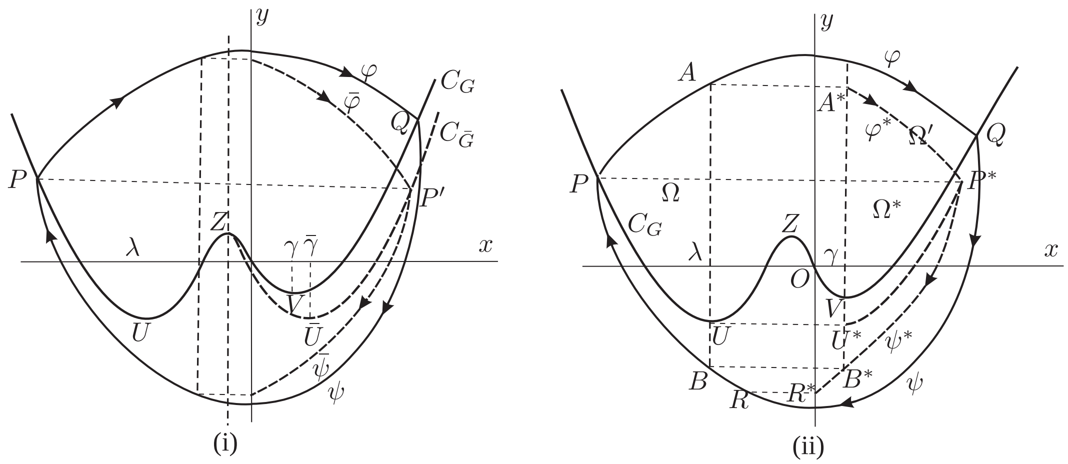

Figure 9. The case (C) with $\lambda \in (x_m, x_M)$ and $x_P > \lambda$.

Now we compare the relative positions of φ with $\bar{\varphi}$, and that of ψ with $\bar{\psi}$, for $x \geq 0$. Since $\bar{\psi}|_{x=0} > \psi|_{x=0}$, and the function $y = \bar{\psi}(x)$ satisfies the differential equation

$$\frac{dy}{dx} = \frac{x+c}{\bar{G}(x) - y}, \quad y < \bar{G}(x),$$

where $\bar{G}(x) = G(-(x+c))$, $c = 2\,|x_Z| > 0$; and $y = \psi(x) < G(x)$ satisfies

$$\frac{dy}{dx} = \frac{x}{G(x) - y} < \frac{x+c}{G(-(x+c)) - y},$$

because $G(-(x+c)) < G(x)$ as we mentioned above. Hence $\bar{\psi}$ is entirely stay above ψ. On the other hand $\bar{\varphi}|_{x=0} < \varphi|_{x=0}$. If $\bar{\varphi}$ crosses φ at some point $(x', \varphi(x'))$, then it would stay above φ for all $x > x'$, because $y = \bar{\varphi}(x) > \bar{G}(x)$ and $y = \varphi(x) > G(x)$ satisfy the above equation respectively, but

$$0 > \frac{x}{G(x) - y} > \frac{x+c}{G(-(x+c)) - y}.$$

In this case, $\bar{\varphi}$ would not meet $\bar{\psi}$ at P', leading to a contradiction.

Thus we obtain that the region bounded by $\{x = 0\}$, $\bar{\varphi}$ and $\bar{\psi}$ is entirely contained in the region bounded by $\{x = 0\}$, φ and ψ, hence fact (a) is true, i. e. $x_Q > \gamma$, because $x_Q > \bar{\gamma}$ and $\bar{\gamma} > \gamma$.

To prove fact (b), i. e. $I_G[\lambda, \gamma] = I_G[x_U, x_V] < 0$, we can use exactly the same way as we prove $I_G[x_P, x_Q] < 0$ for the case (i) of Figure 7, using points U and V instead of the points P and Q respectively. Note that $y_U < y_V < 0$ and the discussion of Appendix 2 is made for $\bar{x}(x) \in (\beta, |\lambda|)$ and $\hat{x}(x) \in (0, \gamma)$.

Finally we prove fact (c), i. e.

$$I_G[x_P, \lambda] + I_G[\gamma, x_Q] < 0. \tag{3.31}$$

Along the straight lines $\{x = \lambda\}$ and $\{x = \gamma\}$ we have $G'(x) = 0$. If we shift the region Ω, bounded by $y = \varphi(x)$, $y = \psi(x)$ and $x \leq \lambda$, to right for a distance $c = |\lambda| - \gamma = x_m - 2\lambda > 0$, and reverse it to the right hand side, symmetric with respect to the y-axis, then Ω maps to Ω^*, and the line $\{x = \lambda\}$ maps to the line $\{x = \gamma\}$. By the same discussion as above we obtain that Ω^* is entirely contained in the region Ω', bounded by $y = \varphi(x)$, $y = \psi(x)$ and $x \geq \gamma$, see Figure 9 (ii). We remark here that it is not necessary that the image of C_G for $x \in (x_P, \lambda)$ is entirely below C_G for $x \in (\gamma, x_Q)$, but it is enough for our purpose that this is true at least for some interval of $x \geq \gamma$.

We denote $\{(x, y) : y = \varphi(x) \cup \psi(x), x_P \leq x \leq \lambda\}$ by $\{(x, y) : x = x(y), y_B \leq y \leq y_A\}$ and $\{(x, y) : y = \varphi^*(x) \cup \psi^*(x), \gamma \leq x \leq x_{P^*}\}$ by $\{(x, y) : x = x^*(y), y_{B^*} \leq y \leq y_{A^*}\}$, where $y_A = y_{A^*}$ and $y_B = y_{B^*}$, see Figure 9 (ii). Then $x(y) = -(x^*(y) + c)$, and by using (3.9)

$$I_G[x_P, \lambda] = \int_{y_B}^{y_A} \frac{G'(x(y))}{x(y)} dy = \int_{y_{B^*}}^{y_{A^*}} \frac{G'(-(x^*(y) + c))}{-(x^*(y) + c)} dy.$$

Let $I^*_G[\gamma, x_{Q^*}] = -I_G[x_P, \lambda] = \int_{y_{A^*}}^{y_{B^*}} \frac{G'(-(x^*(y)+c))}{-(x^*(y)+c)} dy$, then to verify (3.31) is equivalent to show

$$-(I_G[\gamma, x_Q] - I^*_G[\gamma, x_{Q^*}]) > 0. \tag{3.32}$$

Note that $G'(\gamma) = 0$, and $G'(-(\gamma + c)) = G'(\lambda) = 0$, by using Green formula we express the left side of (3.32) as

$$\iint_{\Omega^*} \left[\frac{d}{dx} \left(\frac{G'(x)}{x} \right) - \frac{d}{dx} \left(\frac{G'(-(x+c))}{-(x+c)} \right) \right] dxdy + \iint_{\Omega' \setminus \Omega^*} \frac{d}{dx} \left(\frac{G'(x)}{x} \right) dxdy.$$

Computations show that the first integrand is equal to

$$-2(b + 3\lambda + c) + \lambda(\lambda^2 + b\lambda + a) \left(\frac{1}{x^2} + \frac{1}{(x+c)^2} \right) > 0,$$

because $x \geq \gamma > 0$, $b + 3\lambda + c = b + x_m + \lambda < b + x_m + x_M = 0$ and $\lambda(\lambda^2 + b\lambda + a) > 0$ for $\lambda \in (x_m, x_M)$. And the second integrand is given in (3.6), which is also positive because $b + 3\lambda < 0$. Therefore (3.32), hence (3.31) is proved.

(IV) Case (D).

By Lemma 2.1 $F(x) = \frac{1}{4}x^2(x + \frac{2b}{3})^2$. We do the change of variables and parameter $(x, t, \lambda) \mapsto (-(x + \frac{2b}{3}), -t, -(\lambda + \frac{2b}{3}))$, then the case (D) with $\lambda \in (-\frac{2b}{3}, -\frac{b}{3})$ becomes case (D) with $\lambda \in (-\frac{b}{3}, 0)$, and the proof of later case is similar to case (C) with $\lambda \in (-\frac{b}{3}, 0)$, see Figure 10. Similarly to case (iii) in Figure 6 we can obtain $I_E[x_V, x_Q] < I_E^*[x_V, x_{Q^*}]$ and $I_E[x_U, x_V] < 0$.

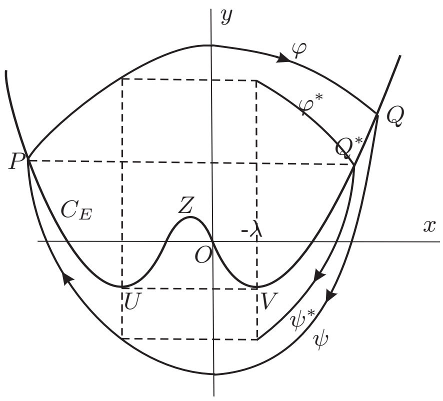

Figure 10. The case (D) with $\lambda \in (-\frac{b}{3}, 0)$.

In this case by symmetry we have $I_E^*[x_V, x_{Q^*}] + I_E[x_p, x_U] = 0$. Therefore $I_E(L) = I_E[x_p, x_U] + I_E[x_U, x_V] + I_E[x_V, x_Q] < 0$.

The proof of Theorem 3.1 is finished. □

Remark 3.9. *By using his results in [6], Xianwu Zeng proved in [7] that if system*

$$\begin{aligned} \dot{x} &= y - (b_1 x + b_2 x^2 + b_3 x^3 + x^4), \\ \dot{y} &= -x, \end{aligned} \tag{3.33}$$

satisfies $b_1 < 0 < b_3$ and $b_3^3 - 4b_2 b_3 + 8b_1 \leq 0$, then it has at most one limit cycle.

Doing scaling $(x,y) \mapsto (4^{\frac{1}{3}}x, 4^{\frac{1}{3}}y)$ *we can change system* (3.1) *to the form* (3.33) *with* $b_1 = a_1, b_2 = 4^{\frac{1}{3}}a_2$ *and* $b_3 = 4^{\frac{2}{3}}a_3$*, hence the first condition is equivalent to* $a_1 < 0 < a_3$ *and the second condition becomes* $8(2a_3^3 - 2a_2a_3 + a_1) = \frac{8}{27}b(2b^2 - 9a) \le 0$*. It is clear that the uniqueness of limit cycle for cases (A), (B), and cases (C) and (D) with* $\lambda \in (-\frac{b}{3}, 0)$ *can be directly obtained by Zheng's result. But the case (C) with* $\lambda \in (x_m, x_M)$ *does not satisfy the conditions, see Lemma* 3.2.

Appendix 1

We prove that the polynomial $g(u)$ in (3.8) has exactly one real root for $u \in (u_1, 0)$, where $u_1 = -\frac{2b}{3}$ and $b > 0$. For a real series $\{c_0, c_1, ..., c_n\}$ we denote by $N\{c_0, c_1, ..., c_n\}$ the number of change of signs in this series (skip zero(s) if it appears in this series). To find the number of real roots of $f(x)$ for $x \in (a, b)$, the following two criteria are well known.

Criterion A (Fourier-Budan) *If*

$$N\{f(a), f'(a), f''(a), ..., f^{(n)}(a)\} = p,$$

$$N\{f(b), f'(b), f''(b), ..., f^{(n)}(b)\} = q,$$

then $p \ge q$*, and the number of real roots (counting the multiplicity) of* $f(x)$ *for* $x \in (a,b)$ *is equal to either* $p - q$ *or* $p - q - r$*, where* r *is a positive even integer. In particular, if* $p = q$ *(resp.* $p = q + 1$*), then* $f(x)$ *has no (resp. has unique) real root in* (a, b).

Criterion B (Sturm) *Assume that* $f(x)$ *has no multiple root in* (a,b)*, and we construct the series* $\{f_0(x), f_1(x), f_2(x), ..., f_s(x)\}$ *as follows:* $f_0(x) = f(x)$*,* $f_1(x) = f'(x)$*. Divide* $f_0(x)$ *by* $f_1(x)$ *and take the remainder with negative sign as* $f_2(x)$*; then divide* $f_1(x)$ *by* $f_2(x)$ *and take the remainder with negative sign as* $f_3(x)$*,..., the last remainder with negative sign (a non-zero number) is* $f_s(x)$*. If*

$$N\{f_0(a), f_1(a), f_2(a), ..., f_s(a)\} = p,$$

$$N\{f_0(b), f_1(b), f_2(b), ..., f_s(b)\} = q,$$

then $p \ge q$ *and the number of real roots of* $f(x)$ *for* $x \in (a,b)$ *is qual to* $p - q$.

We first use Criterion B to prove that $g(u)$ has no multiple root for $u \in (u_1, 0)$. Elimination a from $g(u) = 0$ and $g'(u) = 0$ we find $(3u + 2b)h(u, \lambda, b) = 0$, where

$$h(u, \lambda, b) = 2b(3u + 2b)\lambda^2 - u(3u^2 + 12bu + 8b^2)\lambda + 6u^2(u + b)^2.$$

By Lemma 2.4 we have $3u + 2b > 0$. Let us show $h(u, \lambda, b) \ne 0$ for $u \in (u_1, 0)$, $\lambda \in (-\frac{b}{3}, 0)$ and $b > 0$. This contradiction gives the desired result. In fact, $h(u, 0, b) = 6u^2(u+b)^2 > 0$, $h(u, -\frac{b}{3}, b) = \frac{1}{9}(3u+2b)(18u^3 + 27bu^2 + 12b^2u + 2b^3) > 0$. Here $18u^3 + 27bu^2 + 12b^2u + 2b^3 > 0$ for $u \in (u_1, 0)$ and $b > 0$ can be checked easily by using Criterion B. Then eliminating u from $h(u, \lambda, b) = 0$ and $\frac{\partial}{\partial \lambda}h(u, \lambda, b) = 0$ we obtain $-9\lambda^4 + 306\,b\lambda^3 + 264\,b^2\lambda^2 + 72\,b^3\lambda + 8\,b^4 = 0$, which is impossible for $b > 0$ and $\lambda \in (-\frac{b}{3}, 0)$ by using Criterion B again. Hence we may use special values of (u, b) to check the sign of $h(u, \lambda, b)$, and it is easy to find that $h(-2, \lambda, 6) > 0$ for $\lambda \in (-2, 0)$.

Now we have proved that $g(u)$ has no multiple root for $u \in (u_1, 0)$. Since $g(u_1) = \frac{4b}{27}(b + 3\lambda)(9a - 2b^2) > 0$ and $g(0) = 4ab\lambda < 0$, it is enough to find the number of real roots for $g(u)$ by choosing special values of $b > 0$, $\frac{2}{9}b^2 < a < \frac{1}{4}b^2$ and $\lambda \in (-\frac{b}{3}, 0)$. We can verify by Criterion B that if we choose $(a, b, \lambda) = (8.6, 6, -1)$ then $g(u)$ has exactly one zero $u \approx -0.638407$ in $(-4, 0)$. Therefore $g(u)$ has exactly one root for $u \in (u_1, 0)$.

Appendix 2

We first prove (3.23), i. e.

$$G'_+(x) - H'(x) = G'(x) + k\,G'(-\bar{x}(x)) > 0, \text{ for } x \in (x_0, x_R).$$

Note that $k > 1$ and

$$k\,G'(-\bar{x}(x)) > G'(-\bar{x}(x)) > 0, \; G'(x) < 0, \text{ for } x \in (x_0, x_R),$$

hence it is enough to prove

$$G'(x) + G'(-\bar{x}(x)) \geq 0, \text{ for } x \in (x_0, x_R). \tag{3.34}$$

Since $\bar{x}(x) = k^2(x - \beta) + \beta$ we write $\bar{x}(x) = x + c$, where $c = c(x)$ increases with $x > x_0 > \beta$. Since $x_R \leq x_Q \leq \gamma$ and $\bar{x}(x_R) = x_{P'} \leq |\lambda|$, we have $c \in (0, c_m)$, where $c_m = |\lambda| - \gamma = x_m - 2\lambda > 0$. It is easy to find that $G'(x) + G'(-(x + c))$ has the expression

$$f(x, c) = -[3c + 2(b + 3\lambda)](x^2 + cx) - (c^2 + 2c\lambda + 2\lambda^2)b - (c + 2\lambda)(c^2 + c\lambda + a + \lambda^2).$$

We will prove that $f(x, c) \geq 0$ for $x \in [x_0, \gamma]$ and $c \in [0, c_m]$ with $\frac{2}{9}b^2 < a < \frac{1}{4}b^2$ and $\lambda \in (-\frac{b}{2}, x_M)$.

If $c = c_2 := -\frac{2}{3}(b + 3\lambda)$, then $f(x, c_2) = \frac{2}{27}b(9a - 2b^2) > 0$. If $c \neq c_2$, then we can rewrite $f(x, c)$ as

$$f(x, c) = -3(c - c_2)\left((x + \frac{c}{2})^2 - g(c)\right), \tag{3.35}$$

where

$$g(c) = -\frac{(c + 2\lambda)(c^2 + 2bc + 4c\lambda + 4a + 4b\lambda + 4\lambda^2)}{12(c - c_2)}.$$

Since $c + 2\lambda < 0$ for $c \in (0, c_m)$ and the second factor in the above numerator, as a quadratic polynomial in c, has a negative root and a positive root $c_1 = \sqrt{b^2 - 4a} - (b + 2\lambda) < c_2 < c_m$, it is clear that

$$f(x, c) > 0, \text{ for all } x > 0, \text{ if } c \in [c_1, c_2].$$

We next prove that

$$f(x, c) > 0, \text{ for } x \in (x_0, \gamma), \text{ if } c \in (0, c_1) \cup (c_2, c_m).$$

When $c \in (0, c_1) \cup (c_2, c_m)$, we have $g(c) > 0$ and

$$f(x, c) = -3(c - c_1)\left(x + \frac{c}{2} + \sqrt{g(c)}\right)(x - x(c)),$$

where $x(c)=\sqrt{g(c)}-\frac{c}{2}$ is the only possible positive root of f. We will prove that

$$x'(c)<0, \quad \text{for } c\in(0,c_1)\cup(c_2,c_m). \tag{3.36}$$

This implies the desired result. In fact, if $c\in(0,c_1)$, then $x(c)<x(0)=\sqrt{\frac{\lambda(\lambda^2+b\lambda+a)}{-(b+3\lambda)}}<x_0$ (see (3.17)), hence $f(x,c)>0$ for $x\in(x_0,+\infty)$; if c increases from c_2 then $x(c)$ decreases from $+\infty$, hence $f(x,c)>0$ for $x\in(0,x(c))$. By (3.36) we have $x(c)>x(c_m)$ for $c\in(c_2,c_m)$, and a computation gives $x(c_m)=\gamma$, this would finish the proof.

It remains to prove (3.36). It is easy to see that

$$x'(c)=\frac{g'(c)}{2\sqrt{g(c)}}-\frac{1}{2}. \tag{3.37}$$

We will prove that for $\mu=(a,b,\lambda)\in K=\left\{\frac{2}{9}b^2<a<\frac{1}{4}b^2, b>0, -\frac{b}{2}<\lambda<x_M\right\}$ we have

$$x'(0)<0,\ x'(c_1)<0,\ x'(c_2)<0,\ x'(c_m)<0, \tag{3.38}$$

and it is not hard to check that if we take $\mu_0=(a_0,b_0,\lambda_0)=(8.6,6,-2.4)\in K$ then $x'(c)<0$ for $c\in[0,c_1]\cup[c_2,c_m]$. Thus, if there are $\mu_1\in K$ and $\bar{c}\in(0,c_1)\cup(c_2,c_m)$, such that $x'(\bar{c})\geq 0$ for $\mu=\mu_1$, then by continuity we would find a $\mu_2\in K$ and $\hat{c}\in(0,c_1)\cup(c_2,c_m)$, satisfying $x'(\hat{c})=x''(\hat{c})=0$ for $\mu=\mu_2$. Eliminating c from $x'(c)=0$ and $x''(c)=0$ we obtain

$$a^6b^{10}(b^2-4a)^3(b^2-3a)^3(9a-2b^2)^{10}=0,$$

which is impossible for $\mu\in K$. This contradiction proves (3.36).

At last we need to show (3.38). Since

$$x'(0)(2\sqrt{g(0)}):=r(\lambda)=-\frac{s(\lambda)}{2(b+2\lambda)^2}-\sqrt{\frac{\lambda(\lambda^2+b\lambda+a)}{-(b+3\lambda)}},$$

where $s(\lambda)=6\lambda^3+6b\lambda^2+2b^2\lambda+ab$. If $s(\lambda)\geq 0$, we immediately have $x'(0)<0$. So we use the same continuity argument to show $x'(0)<0$ in case $s(\lambda)<0$. Computation gives

(i) $r(-\frac{b}{2})=-\frac{\sqrt{b^2-4a}(b-\sqrt{b^2-4a})}{2b}<0$.

(ii) $r(x_M)=-\frac{\sqrt{b^2-4a}(b^2-3a-b\sqrt{b^2-4a})}{2(b+3x_M)^2}<0$, since $(b^2-3a)^2-b^2(b^2-4a)=a(9a-2b^2)>0$.

(iii) $r(\lambda)<0$ for $\lambda\in[-\frac{b}{2},x_M]$ when $(a,b)=(8.6,6)$.

(iv) Eliminating $\sqrt{\frac{\lambda(\lambda^2+b\lambda+a)}{-(b+3\lambda)}}$ from $r(\lambda)=0$ and $r'(\lambda)=0$ we have

$$s(\lambda)(36\lambda^3+36b\lambda^2+12b^2\lambda+2b^3-3ab)=0,$$

which is impossible, because $s(\lambda)<0$ as we supposed and $36\lambda^3+36b\lambda^2+12b^2\lambda+2b^3-3ab=6s(\lambda)+b(2b^2-9a)<0$. Thus we have $x'(0)<0$.

To prove $x'(c_j)<0$ for $j=1,2,m$ it is enough to show $g'(c_j)<0$, see (3.37), and we have

$$g'(c_1)=-2\sqrt{b^2-4a}\,\frac{(b^2-3a)-b\sqrt{b^2-4a}}{(b-3\sqrt{b^2-4a})^2}<0.$$

The numerator of $g'(c_2)$ is $-\frac{4}{9}b(9a-2b^2)<0$, hence $\lim_{c\to c_2+0} g'(c)=-\infty$. And

$$g'(c_m)=-\frac{b(5a-b^2)-(b^2-3a)\sqrt{b^2-4a}}{(b-3\sqrt{b^2-4a})^2}<0,$$

because $(b(5a-b^2))^2-(b^2-3a)^2(b^2-4a)=4a^2(9a-2b^2)>0$. The proof of (3.23) is finished.

We next prove (3.30), i. e.

$$\frac{G'(\hat{x}(x))}{\hat{x}(x)}+\frac{k\,G'(-\bar{x}(x))}{\bar{x}(x)} \tag{3.39}$$

has at most one zero in $x\in(\beta,x_R)$ for $k>1$. If $\bar{x}(x)\le\hat{x}(x)$, then by Lemma 3.3 and (3.20) we have

$$\frac{G'(\hat{x}(x))}{\hat{x}(x)}+\frac{k\,G'(-\bar{x}(x))}{\bar{x}(x)}\ge\frac{1}{\bar{x}(x)}(G'(\bar{x}(x))+G'(-\bar{x}(x))>0,$$

where $-\bar{x}(x)\in(\lambda,-\beta)$. Hence we suppose $\bar{x}(x)>\hat{x}(x)$ and let $\bar{x}(x)=\hat{x}(x)+c$. Note that $\hat{x}(x_R)=x_Q$ and $\bar{x}(x_R)=x_{P'}$, hence $c\in(0,c_m)$, and (3.39) becomes

$$A(\xi,c):=\frac{G'(\xi)}{\xi}+\frac{k\,G'(-(\xi+c))}{\xi+c}=(1-k)\xi^4+\alpha_3\xi^3+\alpha_2\xi^2+\alpha_1\xi+\alpha_0, \tag{3.40}$$

where

$$\begin{aligned}
\alpha_3&=-(3k-1)c-(1+k)(b+3\lambda),\\
\alpha_2&=-3kc^2-(2k+1)(b+3\lambda)c-(k-1)(a+2b\lambda+3\lambda^2),\\
\alpha_1&=-kc^3-k(b+3\lambda)c^2-(k-1)(a+2b\lambda+3\lambda^2)c-(1+k)\lambda(\lambda^2+b\lambda+a),\\
\alpha_0&=-c\lambda(\lambda^2+b\lambda+a).
\end{aligned}$$

We need to prove that $A(\xi,c)=0$ has at most one zero in $\xi\in(0,\gamma)$ for $c\in(0,c_m)$, $k>1$ and $(a,b,\lambda)\in K$, where c_m and K are the same as above. Note that $A(0,c)<0$ and $A(+\infty,c)<0$, if $\alpha_3<0$ then by the Fourier-Budan Criterion (see Appendix 1) $A(\xi,c)$ has at most two zeros for $\xi\in(0,+\infty)$; if $\alpha_3\ge0$, i. e. $0<c\le-\frac{(k+1)(b+3\lambda)}{3k-1}$, then

$$\alpha_2\ge-\frac{(3k+1)(k-1)}{3k-1}c(b+3\lambda)-(k-1)(a+2b\lambda+3\lambda^2)>0,$$

because $k>1$, $b+3\lambda<0$ and $a+2b\lambda+3\lambda^2=(\lambda^2+b\lambda+a)+\lambda(b+2\lambda)<0$. Thus by the Fourier-Budan Criterion $A(\xi,c)$ still has at most two zeros for $\xi\in(0,+\infty)$. We will prove that $A(\gamma,c)>0$ for $c\in(0,c_m)$ and $(a,b,\lambda)\in K$, this immediately implies that $A(\xi,c)=0$ has exactly one simple zero in $\xi\in(0,\gamma)$ for $c\in(0,c_m)$. From (3.40) we have

$$A(\gamma,c)=\frac{k}{2}\,(b+2\lambda+c)\,(\beta_2c^2+\beta_1c+\beta_0), \tag{3.41}$$

where

$$\begin{aligned}
\beta_2&=-(b+2\lambda+\sqrt{b^2-4a}),\\
\beta_1&=-(3b+7\lambda)\sqrt{b^2-4a}-3b^2-7b\lambda-8\lambda^2+6a,\\
\beta_0&=-(2b^2+8b\lambda+10\lambda^2-2a)\sqrt{b^2-4a}-2b^3+8(2a-b^2)\lambda-10b\lambda^2-8\lambda^3+6ab.
\end{aligned}$$

Since $\beta_2 < 0$ and we will prove $\beta_0 > 0$, hence $\beta_2 c^2 + \beta_1 c + \beta_0 = 0$ has a unique positive zero point, and direct computation shows this zero point is exactly c_m, hence $A(\gamma, c) > 0$ for $c \in (0, c_m)$ and $(a, b, \lambda) \in K$.

To check $\beta_0 > 0$, we note that

$$\begin{aligned}\beta_0|_{\lambda=-\frac{b}{2}} &= \tfrac{1}{2}(b^2-4a)(b-\sqrt{b^2-4a}) > 0,\\ \beta_0|_{\lambda=x_M} &= 2(b^2-4a)(b-3\sqrt{b^2-4a}) > 0,\\ \beta_0'(\lambda)|_{\lambda=-\frac{b}{2}} &= 2b\sqrt{b^2-4a} - 4(b^2-4a).\end{aligned}$$

Since $4b^2(b^2-4a) - 16(b^2-4a)^2 = 4(b^2-4a)(16a-3b^2) > 0$, we have $\beta_0'(\lambda)|_{\lambda=-\frac{b}{2}} > 0$. It is obvious $\beta_0'''(\lambda) = -48 < 0$, hence the number of change of signs of β_0 at $\lambda = -\frac{b}{2}$ is 1, no matter the sign of $\beta_0''(-\frac{b}{2})$. On the other hand, β_0 has even number of zeros for $\lambda \in (-\frac{b}{2}, x_M)$, implying $\beta_0 > 0$ for $\lambda \in (-\frac{b}{2}, x_M)$. The proof of (3.30) is finished.

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