Limit cycles appearing from the perturbation of a system with a multiple line of critical points

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Abstract

Consider the planar ordinary differential equation $\dot{x} = -y(1-y)^m$, $\dot{y} = x(1-y)^m$, where m is a positive integer number. We study the maximum number of zeroes of the Abelian integral M that controls the limit cycles that bifurcate from the period annulus of the origin when we perturb it with an arbitrary polynomial vector field. One of the key points of our approach is that we obtain a simple expression of M based on some successive reductions of the integrals appearing during the procedure.

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1   Introduction and main result

Consider the family of planar systems

$$\begin{cases} \dot{x} &= -yC(x,y) + \varepsilon P(x,y), \\ \dot{y} &= xC(x,y) + \varepsilon Q(x,y), \end{cases} \tag{1}$$

where P, Q and C are real polynomials, $C(0,0) \neq 0$, and ε is a small real parameter. It is well known that the number of zeroes of the Abelian integral

$$M(r) = \int_{\gamma_r} \frac{Q(x,y)\,dx - P(x,y)\,dy}{C(x,y)}, \qquad (2)$$

where $\gamma_r = \{(x,y) : x^2 + y^2 = r^2\}$, controls the number of limit cycles of (1) that bifurcate from the periodic orbits of the unperturbed system (1) with $\varepsilon = 0$, see [2]. Notice that since the sets γ_r are circles, the Abelian integral $M(r)$ is simpler that in the general weak Hilbert's 16th Problem, see [5].

The problem of finding upper and lower bounds for the number of zeroes of $M(r)$, when P and Q are arbitrary polynomials of a given degree, say n, and C is a particular polynomial, has been faced in several recent papers. The cases where the curve $\{C(x,y) = 0\}$ is one line; two parallel lines; two orthogonal lines; k lines, parallel to two orthogonal directions; or k isolated points are some of these situations, see [1, 3, 4, 6, 7], respectively. Notice that none of these algebraic curves has a multiple factor.

As far as we know the only studied case having a polynomial C with multiple factors is considered in [8]. In that paper the authors take $C(x,y) = (1-y)^m$ and prove that an upper bound for the number of zeroes of $M(r)$ on its domain of definition, $(0,1)$, is $n+m-1$ and that this bound is reached when $m = 1$. This bound coincides with the one obtained in [6] for $m = 1$.

Our main result improves the upper bound given in [8] and provides the optimal upper bound for the zeroes of $M(r)$. We prove

Theorem 1.1 *Consider*
$$M(r) = \int_{\gamma_r} \frac{Q(x,y)\,dx - P(x,y)\,dy}{(1-y)^m}, \qquad (3)$$

where $\gamma_r = \{(x,y) \mid x^2 + y^2 = r^2\}$, $r \in (0,1)$ and P and Q polynomials of degree n. Then the maximum number of zeroes of $M(r)$, taking into account their multiplicities, is $\left[\frac{m+n}{2}\right] - 1$ when $n < m-1$ and n when $n \geq m-1$. Moreover, in both cases the corresponding upper bounds are reached for suitable P and Q.

The approach of [8] is mainly based on the explicit computation of $M(r)$. On the other hand our point of view allows to approach to the expression of $M(r)$ keeping most of the time some simple "basic" integrals that we do not need to compute until the final step.

More concretely, first we prove that $M(r)$ can be given as

$$M(r) = \sum_{j=0}^{n+1} \alpha_{m-j} I_{m-j}(r) + U(r^2) I_1(r),$$

for some suitable constants α_i and some polynomial U, where

$$I_k(r) := \int_0^{2\pi} \frac{1}{(1 - r\sin\theta)^k}\,d\theta, \quad r \in [0,1) \text{ and } k \in \mathbb{Z}, \qquad (4)$$

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see Theorem 2.3. Afterwards we show that the integrals $I_k(r)$ can be obtained in terms of rational functions of r^2 and

$$I_1(r) = \frac{2\pi}{\sqrt{1 - r^2}},$$

see Proposition 2.4. Both results together give that

$$M(r) = \frac{V(r^2)}{(1 - r^2)^{m-1/2}} + W(r^2),$$

for some new polynomials V and W of a controlled degree. From this last expression the proof of the upper bound follows easily.

This approach for the computation of $M(r)$ is also crucial to know how to choose P and Q such that the maximum number of zeroes of M can be obtained.

Notice also that Theorem 1.1 proves that if n is big enough and $m \geq 1$ the maximum number of limit cycles bifurcating from the periodic orbits γ_r, $r \in (0,1)$, for the system

$$\begin{cases} \dot{x} &= -y(1-y)^m + \varepsilon\,P(x,y), \\ \dot{y} &= x(1-y)^m + \varepsilon\,Q(x,y), \end{cases} \tag{5}$$

where $P(x,y)$, $Q(x,y)$ are polynomials of degree n is independent of m. Hence to obtain systems with many limit cycles it seems that it is not a good idea to consider polynomials C with multiple factors in (1).

2   Preliminary results

This section contains all the preliminary computations to express the Abelian integral $M(r)$ given in (3) in terms of polynomials and the family of functions $I_k(r)$, $k \in \mathbb{Z}$ introduced in (4) of the previous section.

Lemma 2.1 *Let $R_N(x)$ be a polynomial of degree N, then for $r \in [0,1)$ and $m \in \mathbb{N}$ it holds that*

$$\int_0^{2\pi} \frac{R_N(r\sin\theta)}{(1 - r\sin\theta)^m}\,d\theta = \sum_{k=0}^{N} \alpha_{m-k} I_{m-k}(r),$$

for some $\alpha_j \in \mathbb{R}$.

Proof.   Notice that for $k \geq 0$,

$$\int_0^{2\pi} \frac{r^k \sin^k\theta}{(1 - r\sin\theta)^m}\,d\theta = \int_0^{2\pi} \frac{(-(1 - r\sin\theta) + 1)^k}{(1 - r\sin\theta)^m}\,d\theta =$$

$$\sum_{j=0}^{k} \binom{k}{j}(-1)^j \int_0^{2\pi} \frac{(1 - r\sin\theta)^j}{(1 - r\sin\theta)^m}\,d\theta = \sum_{j=0}^{k} \binom{k}{j}(-1)^j I_{m-j}(r). \tag{6}$$

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The result follows by applying the above formula to each term of

$$R_N(r\sin\theta) = \sum_{k=0}^{N} r_k(r\sin\theta)^k.$$

$\square$

Lemma 2.2 *Let I_k be the functions introduced in* (4). *Then for $1 \neq k \in \mathbb{Z}$,*

$$r^2 I_k(r) = I_k(r) + \frac{3-2k}{k-1}I_{k-1}(r) + \frac{k-2}{k-1}I_{k-2}(r). \qquad (7)$$

Proof. Notice that

$$I_{k-1}(r) = \int_0^{2\pi} \frac{1 - r\sin\theta}{(1 - r\sin\theta)^k}\, d\theta = I_k(r) - r\int_0^{2\pi} \frac{\sin\theta}{(1 - r\sin\theta)^k}\, d\theta. \qquad (8)$$

Using integration by parts in the last integral we get

$$
\begin{aligned}
r\int_0^{2\pi} \frac{\sin\theta}{(1 - r\sin\theta)^k}\, d\theta &= \left[-\frac{r\cos\theta}{(1 - r\sin\theta)^k}\right]_0^{2\pi} + \int_0^{2\pi} \frac{kr^2\cos^2\theta}{(1 - r\sin\theta)^{k+1}}\, d\theta \\
&= k\int_0^{2\pi} \frac{r^2 - r^2\sin^2\theta}{(1 - r\sin\theta)^{k+1}}\, d\theta \\
&= kr^2 I_{k+1}(r) - k\int_0^{2\pi} \frac{(-1 + 1 - r\sin\theta)^2}{(1 - r\sin\theta)^{k+1}}\, d\theta \\
&= kr^2 I_{k+1}(r) - k(I_{k+1}(r) - 2I_k(r) + I_{k-1}(r)).
\end{aligned}
$$

Substituting it in (8), we find

$$r^2 I_{k+1}(r) = I_{k+1}(r) + \frac{1-2k}{k}I_k(r) + \frac{k-1}{k}I_{k-1}(r).$$

Replacing k by $k - 1$ we obtain the formula of the statement. $\square$

Theorem 2.3 *Let $M(r)$ be the Abelian integral given in* (3) *associated to system* (5). *Then, for $r \in [0, 1)$,*

$$M(r) = \sum_{j=0}^{n+1} \alpha_{m-j} I_{m-j}(r) + \begin{cases} 0, & when \quad n < m, \\ U_{\left[\frac{n-m+2}{2}\right]}(r^2)I_1(r), & when \quad n \geq m, \end{cases} \qquad (9)$$

where α_i are suitable real constants and U_k is a polynomial of degree k. Moreover $M(0) = 0$.

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Proof. Parameterizing the curve γ_r in polar coordinates, $(x,y) = (r\cos\theta, y = r\sin\theta)$, the integral $M(r)$ writes as

$$M(r) = -\int_0^{2\pi} \frac{Q(r\cos\theta, r\sin\theta)\, r\sin\theta + P(r\cos\theta, r\sin\theta)\, r\cos\theta}{(1 - r\sin\theta)^m}\, d\theta =$$

$$\sum_{j=1}^{n+1} \int_0^{2\pi} \frac{R_j(\cos\theta, \sin\theta)\, r^j}{(1 - r\sin\theta)^m}\, d\theta = \sum_{j=1}^{n+1} \int_0^{2\pi} \frac{S_j(\sin\theta)\, r^j}{(1 - r\sin\theta)^m}\, d\theta, \tag{10}$$

where $R_j(x,y)$ denotes a homogeneous polynomial of degree j, $S_j(x)$ denotes a polynomial of degree j and in the last equality we have used that

$$\int_0^{2\pi} \frac{\sin^p\theta \cos^{2q+1}\theta}{(1 - r\sin\theta)^m}\, d\theta = \int_0^{2\pi} \frac{\sin^p\theta(1 - \sin^2\theta)^q \cos\theta}{(1 - r\sin\theta)^m}\, d\theta = 0$$

and

$$\int_0^{2\pi} \frac{\sin^p\theta \cos^{2q}\theta}{(1 - r\sin\theta)^m}\, d\theta = \int_0^{2\pi} \frac{\sin^p\theta(1 - \sin^2\theta)^q}{(1 - r\sin\theta)^m}\, d\theta,$$

for any nonnegative integer numbers p and q. Moreover using the homogeneity of the polynomials R_j we also know that the polynomials S_j in (10) are odd polynomials when j is odd and even polynomials when j is even. That is, to fix the ideas,

$$\sum_{j=1}^{n+1} S_j(\sin\theta)r^j = (s_{1,1}\sin\theta)\, r + \left(s_{2,0} + s_{2,2}\sin^2\theta\right) r^2 + \left(s_{3,1}\sin\theta + s_{3,3}\sin^3\theta\right) r^3$$

$$+ \left(s_{4,0} + s_{4,2}\sin^2\theta + s_{4,4}\sin^4\theta\right) r^4 + \cdots + S_{n+1}(\sin\theta)r^{n+1}.$$

Notice that collecting the higher powers in $\sin\theta$ for each term r^j we define

$$(s_{1,1}\sin\theta)r + (s_{2,2}\sin^2\theta)r^2 + \cdots + (s_{i,i}\sin^i\theta)r^i =: T_i(r\sin\theta),$$

where T_i, for $i = 1, \ldots, n+1$, is a polynomial of degree i vanishing at the origin. Continuing this process we obtain that

$$\sum_{j=1}^{n+1} S_j(\sin\theta)r^j = T_{n+1}(r\sin\theta) + r^2 T_{n-1}(r\sin\theta) + \cdots + r^{2[(n+1)/2]} T_{n-2[(n+1)/2]+1}(r\sin\theta),$$

for some polynomials T_j. Hence

$$M(r) = \sum_{j=1}^{n+1} \int_0^{2\pi} \frac{S_j(\sin\theta)\, r^j}{(1 - r\sin\theta)^m}\, d\theta = \sum_{j=0}^{[(n+1)/2]} r^{2j} \int_0^{2\pi} \frac{T_{n+1-2j}(r\sin\theta)}{(1 - r\sin\theta)^m}\, d\theta$$

$$= \sum_{j=0}^{[(n+1)/2]} r^{2j} \left[\alpha_m^j I_m(r) + \alpha_{m-1}^j I_{m-1}(r) + \cdots + \alpha_{m+2j-n-1}^j I_{m+2j-n-1}(r)\right],$$

where we have used Lemma 2.1 and α_k^j are real constants.

Observe that from Lemma 2.2 we can transform each term

$$r^{2j}\left[\alpha_m^j I_m(r) + \alpha_{m-1}^j I_{m-1}(r) + \cdots + \alpha_{m+2j-n-1}^j I_{m+2j-n-1}(r)\right]$$

with $j > 0$ to another one of the form

$$r^{2j-2}\left[\beta_m^{j-1} I_m(r) + \cdots + \beta_{m+2j-n+1}^{j-1} I_{m+2j-n+1}(r)\right] + \delta r^{2j}\alpha_1^j I_1(r),$$

for some real constants β_ℓ^{j-1} and $\delta = 1$ if $1 \in \{m, m-1, \ldots, m+2j-n-1\}$ and $\delta = 0$ otherwise. Then by using reiteratively the above procedure we get that

$$M(r) = \alpha_m I_m(r) + \alpha_{m-1} I_{m-1}(r) + \cdots + \alpha_{m-n-1} I_{m-n-1}(r) + U_{\left[\frac{n-m+2}{2}\right]}(r^2) I_1(r),$$

where α_j are real constants and $U_{\left[\frac{n-m+2}{2}\right]}$ is a polynomial of degree $\left[\frac{n-m+2}{2}\right]$ when $n - m + 2 > 0$ or vanishes identically otherwise.

Notice that since the numerator of (10) is a polynomial that vanishes at $r = 0$ then $M(0) = 0$.
□

Finally, next results gives the structure of the functions $I_k(r)$.

Proposition 2.4 *The functions* (4) *satisfy*

$$I_k(r) = \frac{1}{(1 - r^2)^{k-\frac{1}{2}}} I_{1-k}(r). \tag{11}$$

Moreover,

$$I_k(r) = \begin{cases} R_{\left[\frac{-k}{2}\right]}^k(r^2) & k \leq 0, \\ \dfrac{R_{\left[\frac{k-1}{2}\right]}^{1-k}(r^2)}{(1-r^2)^{k-1/2}}, & k \geq 1, \end{cases} \tag{12}$$

where R_ℓ^j are polynomials of (exact) degree ℓ.

Proof. It is easy to check that equality (11) is true for $k = 0, 1$. First we prove that it is true for any $k \geq 2$ by induction. Suppose that it is true for k and $k + 1$, that is

$$\begin{aligned} (1 - r^2)^{k-\frac{1}{2}} I_k(r) &= I_{1-k}(r), \\ (1 - r^2)^{k+\frac{1}{2}} I_{k+1}(r) &= I_{-k}(r). \end{aligned} \tag{13}$$

We need to prove that

$$(1 - r^2)^{k+\frac{3}{2}} I_{k+2}(r) = I_{-(k+1)}(r). \tag{14}$$

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Taking $k + 2$ instead of k in Lemma 2.2 we have that

$$(1 - r^2)I_{k+2}(r) = \frac{2k + 1}{k + 1}I_{k+1}(r) - \frac{k}{k + 1}I_k(r).$$

Then (14) writes as

$$(1 - r^2)^{k+\frac{3}{2}}I_{k+2}(r) = \frac{2k + 1}{k + 1}(1 - r^2)^{k+\frac{1}{2}}I_{k+1}(r) - \frac{k}{k + 1}(1 - r^2)^{k+\frac{1}{2}}I_k(r).$$

By using the assumption (13) we find

$$(1 - r^2)^{k+\frac{3}{2}}I_{k+2}(r) = \frac{2k + 1}{k + 1}I_{-k}(r) - \frac{k}{k + 1}(1 - r^2)I_{1-k}(r). \qquad (15)$$

Then, taking $1 - k$ instead of k in Lemma 2.2 we get

$$(1 - r^2)I_{1-k}(r) = \frac{2k + 1}{k}I_{-k}(r) - \frac{k + 1}{k}I_{-(k+1)}(r).$$

Substituting this last relation in (15) we obtain

$$(1 - r^2)^{k+\frac{3}{2}}I_{k+2}(r) = \frac{2k + 1}{k + 1}I_{-k}(r) - \frac{k}{k + 1}\left[\frac{2k + 1}{k}I_{-k}(r) - \frac{k + 1}{k}I_{-(k+1)}(r)\right],$$

which gives (14) and so equality (11) for $k \geq 0$ holds. Notice that, indeed, the proof also gives the equality (11) for $k < 0$.

The second statement when $k \leq 0$ is a direct consequence of the fact that we integrate functions of θ that are polynomials in r. Moreover the coefficients with odd powers vanish by symmetry. Finally the case $k \geq 1$ follows from equality (4). $\qquad \square$

Remark 2.5 *Some explicit expressions of $I_k(r)$ are:*

$$I_1(r) = \frac{2\pi}{(1 - r^2)^{1/2}}, \qquad I_2(r) = \frac{2\pi}{(1 - r^2)^{3/2}}, \qquad I_3(r) = \frac{(r^2 + 2)\pi}{(1 - r^2)^{5/2}},$$

$$I_4(r) = \frac{(3r^2 + 2)\pi}{(1 - r^2)^{7/2}}, \qquad I_5(r) = \frac{(3r^4 + 24r^2 + 8)\pi}{4(1 - r^2)^{9/2}}, \qquad I_6(r) = \frac{(15r^4 + 40r^2 + 8)\pi}{4(1 - r^2)^{11/2}}$$

and

$$I_0(r) = 2\pi, \qquad I_{-1}(r) = 2\pi, \qquad I_{-2}(r) = (r^2 + 2)\pi,$$

$$I_{-3}(r) = (3r^2 + 2)\pi, \qquad I_{-4}(r) = \frac{(3r^4 + 24r^2 + 8)\pi}{4}, \qquad I_{-5}(r) = \frac{(15r^4 + 40r^2 + 8)\pi}{4}.$$

Lemma 2.6 *Consider the family of functions*

$$F(x) = A_i(x) + B_j(x)(a - x)^\alpha,$$

defined on $(-\infty, a)$, where A_i and B_j are polynomials of degree i and j respectively and $\alpha \notin \mathbb{Z}$. Then each non trivial function has at most $i + j + 1$ real zeroes, taking into account their multiplicities. Moreover there exist polynomials A_i and B_j such that the corresponding function has exactly this number of zeroes.

Proof. First we will prove that the maximum number of zeroes is $i + j + 1$. Note that

$$\frac{d}{dx}\left(B_j(x)(a - x)^\alpha\right) = \widetilde{B}_j(x)(a - x)^{\alpha - 1}$$

for some polynomial $\widetilde{B}$ also of degree j. Hence $\frac{d^{i+1}}{dx^{i+1}}F(x) = \widehat{B}_j(x)(a - x)^{\alpha - (i+1)}$, for some polynomial $\widehat{B}_j$ of degree j. This function has at most j zeroes in $(-\infty, a)$. Hence the result follows by Rolle's Theorem.

The lower bound is a consequence of the fact that the functions

$$1, x, x^2, \ldots, x^i, (a - x)^\alpha, \ldots, x^j(a - x)^\alpha$$

are linearly independent because $\alpha \notin \mathbb{Z}$. □

3 Proof of Theorem 1.1

We consider the expression of $M(r)$ given in Theorem 2.3. Before studying its number of zeroes we will transform M into a simpler form. We start first with the case $n < m - 1$. For these values of n and m we know that

$$M(r) = \sum_{j=0}^{n+1} \alpha_{m-j} I_{m-j}(r)$$

and $m - j \geq 1$ for $j = 0, \ldots, n + 1$. Then, from Proposition 2.4, we can write

$$M(r) = \sum_{j=0}^{n+1} \alpha_{m-j} \frac{R_{\left[\frac{m-j-1}{2}\right]}^{1-(m-j)}(r^2)}{(1 - r^2)^{m-j-1/2}} = \frac{\sum_{j=0}^{n+1} \alpha_{m-j}(1 - r^2)^j R_{\left[\frac{m-j-1}{2}\right]}^{1-(m-j)}(r^2)}{(1 - r^2)^{m-1/2}}.$$

As the numerator of the previous expression is a sum of polynomials in r^2 of degree $j + \left[\frac{m-j-1}{2}\right] = \left[\frac{m+j-1}{2}\right]$, $j = 0, \ldots, n + 1$, the maximum number of zeroes of $M(r)$ in $[0, 1)$ is the total degree of the numerator, that is $\left[\frac{m+n}{2}\right]$.

Now we deal with the case $n \geq m - 1$. From Theorem 2.3 and Proposition 2.4 we can write

$$
M(r) = \sum_{j=0}^{m-1} \alpha_{m-j} I_{m-j}(r) + \sum_{j=m}^{n+1} \alpha_{m-j} I_{m-j}(r) + U_{\left[\frac{n-m+2}{2}\right]}(r^2) I_1(r)
$$

$$
= \sum_{j=0}^{m-1} \alpha_{m-j} \frac{R^{1-(m-j)}_{\left[\frac{m-j-1}{2}\right]}(r^2)}{(1-r^2)^{m-j-1/2}} + \sum_{j=m}^{n+1} \alpha_{m-j} R^{m-j}_{\left[\frac{j-m}{2}\right]}(r^2) + U_{\left[\frac{n-m+2}{2}\right]}(r^2) I_1(r)
$$

$$
= \frac{\sum_{j=0}^{m-1} \alpha_{m-j}(1-r^2)^j R^{1-(m-j)}_{\left[\frac{m-j-1}{2}\right]}(r^2) + 2\pi(1-r^2)^{m-1} U_{\left[\frac{n-m+2}{2}\right]}(r^2)}{(1-r^2)^{m-1/2}}
$$

$$
+ \sum_{j=m}^{n+1} \alpha_{m-j} R^{m-j}_{\left[\frac{j-m}{2}\right]}(r^2) = \frac{S_{\left[\frac{n+m}{2}\right]}(r^2)}{(1-r^2)^{m-1/2}} + T_{\left[\frac{n-m+1}{2}\right]}(r^2),
$$

where S_j and T_j are given polynomials of degree j. Then, using Lemma 2.6, $M(r)$ has at most $\left[\frac{n+m}{2}\right] + \left[\frac{n-m+1}{2}\right] + 1 = n + 1$ zeroes in $[0, 1)$.

Finally, by Theorem 2.3, for all n and m we know that $M(0) = 0$. Then $M(r)$ has at most $\left[\frac{m+n}{2}\right] - 1$ or n zeroes in $(0, 1)$, taking into account their multiplicities, when $n < m - 1$ or $n \geq m - 1$, respectively.

Notice that, from the above proof, to show that the obtained upper bounds are realizable it suffices to find polynomials P and Q, in (3), such that the coefficients of (9) are arbitrary with the only restriction $M(0) = 0$. We will again split the study in the same two cases: $n < m - 1$ and $n \geq m - 1$.

In the first case, consider $P(x, y) = 0$ and $Q(x, y) = -\sum_{k=0}^{n} a_k y^k$. Then

$$
M(r) = -\int_0^{2\pi} \frac{Q(r\cos\theta, r\sin\theta) r \sin\theta}{(1 - r\sin\theta)^m} d\theta = \sum_{k=1}^{n+1} \int_0^{2\pi} \frac{a_{k-1} r^k \sin^k \theta}{(1 - r\sin\theta)^m} d\theta
$$

$$
= \sum_{k=1}^{n+1} a_{k-1} \int_0^{2\pi} \frac{r^k \sin^k \theta}{(1 - r\sin\theta)^m} d\theta = \sum_{k=1}^{n+1} a_{k-1} \sum_{j=0}^{k} \binom{k}{j} (-1)^j I_{m-j}(r) = \sum_{j=0}^{n+1} \widetilde{a}_{m-j} I_{m-j}(r)
$$

where we have used (6). Note that, choosing suitably constants a_k, the values $\widetilde{a}_k$ are arbitrary with the only restriction $\sum_{j=0}^{n+1} \widetilde{a}_{m-j} I_{m-j}(0) = 0$, for compatibility with the condition $M(0) = 0$. Hence the lower bound follows in this situation.

In the second case, we choose the polynomials of degree at most n,

$$P(x,y) = -(1-y)^{m-1} \sum_{j=0}^{\left[\frac{n-m}{2}\right]} b_j x(x^2 + y^2)^j,$$

$$Q(x,y) = -\sum_{k=0}^{n} a_k y^k - (1-y)^{m-1} \sum_{j=0}^{\left[\frac{n-m}{2}\right]} b_j y(x^2 + y^2)^j.$$

Then

$$M(r) = \sum_{j=0}^{n+1} \widetilde{a}_{m-j} I_{m-j}(r) + \sum_{j=1}^{\left[\frac{n-m+2}{2}\right]} b_{j-1} r^{2j} \int_0^{2\pi} \frac{d\theta}{1 - r\sin\theta}$$

$$= \sum_{j=0}^{n+1} \widetilde{a}_{m-j} I_{m-j}(r) + \sum_{j=1}^{\left[\frac{n-m+2}{2}\right]} b_{j-1} r^{2j} I_1(r)$$

and so the result holds.

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