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SOME NEW RESULTS ON DARBOUX INTEGRABLE DIFFERENTIAL SYSTEMS

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Abstract. We deal with complex planar differential systems having a Darboux first integral H. We present a definition of remarkable values and remarkable curves associated to H and characterize the existence of a polynomial inverse integrating factor for these systems. Furthermore, we study the relation between the characteristic polynomial $\mathcal{F}$ and the inverse integrating factors of the system and show the importance of the numerator of the exponential factor of H in the construction of $\mathcal{F}$.

1. Introduction and preliminary definitions

A *complex planar polynomial differential system* of degree d is

$$\dot{x} = P(x,y), \;\; \dot{y} = Q(x,y), \tag{1}$$

where $P, Q \in \mathbb{C}[x,y]$ are coprime and $d = \max\{\deg P, \deg Q\}$. We write $X = (P,Q)$ and $Xf = Pf_x + Qf_y$, f a $\mathcal{C}^1$-function.

Let U be an open subset of $\mathbb{C}^2$. A *first integral* of X in U is a non-constant $\mathcal{C}^1$-function $H : U \to \mathbb{C}$, possibly multi-valued, which is constant on all the solutions of X contained in U, i.e. $XH = 0$ on U. We say in this case that X is *integrable* on U.

An *inverse integrating factor* of X in U is a $\mathcal{C}^1$-function $V : U \to \mathbb{C}$ satisfying $XV = \operatorname{div}(X)V$, where $\operatorname{div}(X) = P_x + Q_y$ is the divergence function of X. The function V is a very useful tool to study integrable systems (see [5]). Indeed the set $V^{-1}(0)$ contains a lot of information about the 'skeleton' or the separatrices of the phase portrait of X in U, see [3, 16, 15, 18, 4].

We say that *the inverse integrating factor V is associated to the first integral H* of system (1) in U if $(P,Q) = (-H_y, H_x)V$ in $U \setminus \{V = 0\}$.

Let $f \in \mathbb{C}[x,y]$. We say that the algebraic curve $f = 0$ is *invariant* if there exists a polynomial $K \in \mathbb{C}[x,y]$ of degree at most $d-1$, called the *cofactor*, such that $Xf = Kf$.

Let $g, h \in \mathbb{C}[x,y]$ be coprime polynomials. The function $F = e^{g/h}$ is an *exponential factor* of system (1) if there exists a polynomial $L \in \mathbb{C}[x,y]$ of degree at most $d-1$, called the *cofactor*, such that $XF = LF$. In this case, $h = 0$ is an invariant algebraic curve. The notion of exponential factor is due to Christopher [7]. An exponential factor appears when an invariant algebraic curve has multiplicity greater than one. For more details on exponential factors see [10].

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A function of the form

$$\prod_{i=1}^{p} f_i^{\lambda_i} \prod_{j=1}^{q} \exp\left(\frac{g_j}{h_j}\right)^{\mu_j}, \tag{2}$$

where $f_i, g_j, h_j \in \mathbb{C}[x,y]$ and $\lambda_i, \mu_j \in \mathbb{C}$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, is called a *Darboux function*. If system (1) has a first integral or an inverse integrating factor of the form (2), being $f_i = 0$ invariant algebraic curves and $\exp(g_j/h_j)$ exponential factors of system (1), then system (1) is *Darboux integrable*. The Darboux Theory of Integrability (or Darboux method) relates the number of invariant algebraic curves and exponential factors with the existence of a Darboux first integral, see [17] for a summary of the main results about the Darboux method and its improvements.

It is known that a Darboux function H can be written as

$$H(x,y) = \prod_{i=1}^{p} f_i^{\lambda_i} \exp\left(\frac{g}{\prod_{i=1}^{p} f_i^{n_i}}\right), \tag{3}$$

where $f_i \in \mathbb{C}[x,y]$ is irreducible, $\lambda_i \in \mathbb{C}$, $n_i \in \mathbb{N} \cup \{0\}$ and $g \in \mathbb{C}[x,y]$ is coprime with f_i if $n_i > 0$, for $i = 1, \ldots, p$. Of course not both λ_i and n_i are zero for any i. We shall use this expression of H as Darboux function all along the paper. We note that it can be easily proved that the exponential function in (3) is an exponential factor with cofactor $L = -\sum_{i=1}^{p} \lambda_i K_i$, where K_i is the cofactor of $f_i = 0$, for all $i = 1, \ldots, p$.

If the function H in (3) is a Darboux first integral of system (1) then the system has the rational inverse integrating factor

$$V(x,y) = \frac{\prod_{i=1}^{p} f_i^{n_i+1}}{R} = \frac{1}{\gcd\left((\log H)_x, (\log H)_y\right)}, \tag{4}$$

where R is a polynomial, the so-called *remarkable factor* (see [12]). Note that V is the inverse integrating factor associated to the first integral $\log H$. If H is not rational then V is the only rational inverse integrating factor of system (1), see [6].

It is known that if system (1) has a polynomial inverse integrating factor, then in most cases it has degree $d+1$ (see [6, 23, 9, 24, 12] and see also the survey on the inverse integrating factor [14]). We define the extension of the degree function $\delta(\prod g_i^{\alpha_i}) = \sum \alpha_i \deg g_i$, where $\alpha_i \in \mathbb{C}$ and $g_i \in \mathbb{C}[x,y]$. Of course, if $\alpha_i \in \mathbb{N}$ for all i then δ is exactly the degree function. We note that $\delta(V) \in \mathbb{N} \cup \{0\}$, V defined in (4). As in the polynomial case, we shall compare $\delta(V)$ with $d+1$ and we will show in which cases they are equal.

In the same context of Darboux integrability we deal with the infinity, relating the singular points at infinity with the inverse integrating factor and with the polynomials appearing in (3). In particular we show how important is the polynomial g of the expression (3). All these new results are stated in Theorem 1, see Subsection 2.1.

Still concerning Darboux integrable systems, we define remarkable values and remarkable curves for Darboux first integrals. Remarkable values and remarkable curves were first defined by Poincaré in [19] for rational first integrals; their importance in the phase portrait of the system has been widely shown, see for example [6, 11, 12]. Here we extend these definitions to Darboux first integrals and prove a result that characterizes the existence of a polynomial inverse integrating factor by means of the number of critical remarkable values, see Theorem 4 in Subsection 2.2.

The paper is organized as follows. In Section 2 we state our new results, that we split in two subsections. Section 3 is devoted to the proof of the results of Subsection 2.1. In Subsection 4 we prove the results stated in Subsection 2.2. Finally in Section 5 we provide some examples of application and state an open question.

2. Statement of the new results

For the sake of clearness we split our results in two subsections. In the first one with deal with the infinity; in the second one we deal with remarkable values.

2.1. **The infinity and the inverse integrating factor.**

Consider the polynomial system X given by (1) and suppose that it has a Darboux first integral (3). We denote by $\tilde{f}$ the homogeneous part of highest degree of a polynomial f. Let $\mathcal{F} = x\tilde{Q} - y\tilde{P}$ be the *characteristic polynomial* associated to X. We define

$$\Pi_1 = \prod_{i=1}^{p} \tilde{f}_i^{\lambda_i}, \quad \Pi_2 = \tilde{g} / \prod_{i=1}^{p} \tilde{f}_i^{n_i}. \tag{5}$$

We note that $\delta(\Pi_1) \in \mathbb{C}$ and $\delta(\Pi_2) \in \mathbb{Z}$.

Our first main result is Theorem 1. We relate in it the characteristic polynomial $\mathcal{F}$ with some inverse integrating factor and we show the importance of the polynomial g appearing in the expression of H. Moreover we compare $\delta(V)$ with $d+1$, characterizing whether they are equal.

Theorem 1. *Consider system* (1) *and suppose that it has a Darboux first integral* (3) *which is not rational and the inverse integrating factor V given in* (4). *The following statements hold.*

(1) $\delta(V) < d+1$ if and only if $\delta(\Pi_2) > 0$. Moreover

$$\mathcal{F} = \delta(\Pi_2) \frac{\tilde{g} \prod_{i=1}^{p} \tilde{f}_i}{\tilde{R}} \not\equiv 0.$$

(2) $\delta(V) = d+1$ if and only if either $\delta(\Pi_2) < 0$ and Π_1 is not constant, or $\delta(\Pi_2) = 0$. Moreover

$$\mathcal{F} = \delta(\Pi_1)\tilde{V}.$$

(3) $\delta(V) > d+1$ if and only if $\delta(\Pi_2) < 0$ and Π_1 is constant. Moreover

$$\mathcal{F}_\infty = \left(v^{d+2} V|_C \, \frac{\partial \log(H|_C)}{\partial v} \right) \Bigg|_{v=0} \not\equiv 0,$$

where $\mathcal{F}_\infty = v^{d+1}\mathcal{F}|_C \in \mathbb{C}[u,v]$ and $C = \{(x,y) = (1,u)/v\}$.

Next corollary corrects Theorem 2 of [12] about a characterization of polynomial differential systems having a Darboux first integral and degenerate infinity. We recall that the infinity is *degenerate* if and only if $\mathcal{F} \equiv 0$.

Corollary 2. *Consider system* (1) *and suppose that it has a Darboux first integral* (3). *The infinity is degenerate if and only if $\delta(\Pi_1) = 0$ and either $\delta(\Pi_2) < 0$ and Π_1 is not constant, or $\delta(\Pi_2) = 0$.*

The rational case $H = h_1/h_2$ (that is $g \equiv 0$, $n_i = 0$ and $\lambda_i \in \mathbb{Z}$ for all $i = 1, \ldots, p$ in (3)) is a particular one of Darboux functions. We note that (4) writes then as

$$V = \frac{h_1 h_2}{R}. \tag{6}$$

The results of Theorem 1 hold, as the next corollary assures.

Corollary 3. *Suppose that system* (1) *has a rational first integral $H = h_1/h_2$, where h_1 and h_2 are coprime polynomials. Let* (6) *be an inverse integrating factor of system* (1)*. Then $\delta(V) = d + 1$ and $\mathcal{F} = \delta(H)\tilde{V}$.*

Remark 1. *If R is a constant, $g \equiv 0$ and $n_i = 0$ for all $i \in \{1, \ldots, p\}$, we have a generalization of a result due to Kooij and Christopher* [9] *and Zoladek* [24]*.*

Remark 2. *Concerning the degree of V in case that R is constant, statement (e) of the Main Theorem of* [6] *assures that $\deg V = d + 1$ under certain conditions. This statement is not always true. We provide a counterexample in Section 5 (see Example 5) with an explanation of why it fails. The correct hypotheses are the ones of Theorem 1.*

2.2. **Remarkable values of Darboux first integrals.** The remarkable values and remarkable curves were first introduced by Poincaré in [19], and afterwards studied by several authors, see [6, 12, 11]. It has been shown in the literature that the remarkable curves play an important role in the phase portrait as they are strongly related to the separatrices. It is proved in [6] that there are a finite number of them. In [6, 12] they are related with the inverse integrating factor. In [12, 11] some techniques to compute remarkable curves are provided. Moreover remarkable values and remarkable curves are used in [13, 4] to prove some results on integrability and the inverse integrating factor, and in [1] to study degenerate singular points.

Let $H = h_1/h_2$ be a rational first integral of system (1). We define the *degree* of H as $\deg H = \max\{\deg h_1, \deg h_2\}$. We say that the degree of H is *minimal* if any other rational first integral of (1) has degree at least $\deg H$.

The *remarkable values* are defined as level sets c of a minimal rational first integral $H = h_1/h_2$ for which $h_1 + c h_2$ factorizes. The factors of $h_1 + c h_2$ provide the *remarkable curves* associated to c. If in this factorization some factor has exponent greater than one, then the corresponding remarkable curve and remarkable value are said to be *critical*. We note that the critical remarkable curves are the factors of $h_2^2 \gcd(H_x, H_y)$.

An important property of the inverse integrating factor is that usually a polynomial differential system X having a first integral has an inverse integrating factor with an easier expression. In what follows we are more precise:

(i) If X has a Liouvillian first integral, then it has a Darboux inverse integrating factor.
(ii) If X has a Darboux first integral (including rational), then it has a rational inverse integrating factor.
(iii) If X has a rational first integral and has no polynomial first integrals, then it has a polynomial inverse integrating factor if and only if the first integral has at most two critical remarkable values.
(iv) If X has a polynomial first integral, then it has a polynomial inverse integrating factor.

For a definition of *Liouvillian* function see [21], roughly speaking a Liouvillian function comes from the integral of a Darboux function. Statement (i) was proved in [21] (see also [8]). Statement (ii) was proved in [20] (see also [6]). Statement (iii) was proved in [6]. Statement (iv) was proved in [13] (see also [12]). Our second main result is Theorem 4, which characterizes the polynomial differential systems (1) having a Darboux first integral and a polynomial inverse integrating factor, extending statement (iii) above to differential systems having a Darboux first integral. See Section 4 for a definition of critical remarkable values of Darboux first integrals.

Theorem 4. *Suppose that system* (1) *has a Darboux first integral* (3) *which is not rational. Then the inverse integrating factor V defined in* (4) *is a polynomial if and only if the number of critical remarkable values of H is zero.*

Now statement (iii) above can be rewritten in a similar way as Theorem 4. Without loss of generality, in the case of existence of a rational first integral, we assume that if $c \neq 0, \infty$ is a critical remarkable value, then both 0 and ∞ are also critical remarkable values.

Theorem 5 (see [6])**.** *Suppose that system* (1) *has a minimal rational first integral $H = h_1/h_2$ and has no polynomial first integrals. Then system* (1) *has a polynomial inverse integrating factor if and only if the number of critical remarkable values different from* 0 *and* ∞ *is zero. This polynomial inverse integrating factor is to be the function V defined in* (6).

As we already mentioned, Theorem 4 extends the result of [6] about the existence of a polynomial inverse integrating factor for systems having a rational first integral to systems having a Darboux first integral. In that article the authors also prove that the polynomial differential systems having a complete Darboux first integral and having no rational first integrals have a polynomial inverse integrating factor. We recall that H is *complete* if the $f_i = 0$ appearing in (3) are the unique invariant algebraic curves of system (1). In particular, it implies that there are no critical remarkable curves, and so it is clear that (4) is a polynomial. Hence our result is stronger than that one.

3. The infinity and the inverse integrating factor

The characteristic polynomial $\mathcal{F}$ is either homogeneous of degree $d+1$ or identically zero. In the former case, it provides the characteristic directions at infinity, which are in correspondance with the singular points at infinity. In the latter case the infinity is degenerate and clearly $\tilde{P} = xW$ and $\tilde{Q} = yW$, for a certain homogeneous polynomial W of degree $d-1$. This polynomial W provides the singular directions at infinity, which may correspond to singular points at infinity after removing the line of singularities. For more information see [2]. In Theorem 1 we give an expression of $\mathcal{F}$ in terms of the polynomials appearing in (3) and relate it to the inverse integrating factor. Moreover, as we mentioned before, we compare $\delta(V)$ and $d+1$.

Before proving Theorem 1 we need to do some remarks.

Remark 3. *When* $\deg g = \sum_{i=1}^{p} n_i \deg f_i \neq 0$ *and* Π_2 *is constant, we can divide* g *by* $\prod_{i=1}^{p} f_i^{n_i}$*; hence there exists a polynomial* $\bar{g}$ *such that* $g = \Pi_2 \prod_{i=1}^{p} f_i^{n_i} + \bar{g}$*, with*

$\deg \bar{g} < \deg g$. *Therefore*

$$\exp\left(\frac{g}{\prod_{i=1}^{p} f_i^{n_i}}\right) = e^{\Pi_2} \exp\left(\frac{\bar{g}}{\prod_{i=1}^{p} f_i^{n_i}}\right).$$

In this case we consider the first integral H/e^{Π_2} *instead of* H *and now we have* $\deg \bar{g} < \sum_{i=1}^{p} n_i \deg f_i$. *Hence we can assume without loss of generality that the case* $\deg g = \sum_{i=1}^{p} n_i \deg f_i$ *and* Π_2 *constant does not appear in our study.*

Remark 4. *Applying the change of variables*

$$C = \{x = 1/v, y = u/v\} \tag{7}$$

to system (1)*, we obtain the polynomial differential system*

$$\dot{u} = v^{d+1}(xQ - yP)|_C, \quad \dot{v} = -v^{d+1}P|_C. \tag{8}$$

It is clear that $v = 0$ *in this new system corresponds to the line of infinity of system* (1)*. The singular points of* (8) *on* $v = 0$ *are in correspondance with the linear factors of* $\mathcal{F}$*. We assume without loss of generality that* $x \nmid \mathcal{F}$ *if* $\mathcal{F} \not\equiv 0$*, as otherwise this linear factor* x *would not appear as a singular point of* (8)*.*

Proof of Theorem 1. We denote by X_f the Hamiltonian system $(-f_y, f_x)$, for some function f. We first compute $X_{\log H}$:

$$\begin{aligned} X_{\log H} &= \sum_{i=1}^{p} \lambda_i \frac{X_{f_i}}{f_i} + \frac{X_g \prod_{i=1}^{p} f_i - g \sum_{i=1}^{p} n_i X_{f_i} \prod_{j \neq i} f_j}{\prod_{i=1}^{p} f_i^{n_i+1}} \\ &= \frac{1}{\prod_{i=1}^{p} f_i^{n_i+1}} \left[X_g \prod_{i=1}^{p} f_i + \sum_{i=1}^{p} \left(\lambda_i \prod_{j=1}^{p} f_j^{n_j} - n_i g\right) X_{f_i} \prod_{j \neq i} f_j \right]. \end{aligned} \tag{9}$$

As P and Q are coprime and $\log H$ is a first integral of system (1), we have

$$X = (P, Q) = \frac{\prod_{i=1}^{p} f_i^{n_i+1}}{R} X_{\log H} = V X_{\log H}, \tag{10}$$

where the remarkable factor R is the greatest common divisor of the expression between brackets in (9) and V was defined in (4).

The equality (10) is also proved in [17]. We prove it here again for completeness. It is clear that $d + 1 \leq \max\{\sum_{i=1}^{p} n_i \deg f_i, \deg g\} + \sum_{i=1}^{p} \deg f_i - \deg R$, the inequality holding only in the case that the highest order terms of the two expressions between brackets in (9) are both identically zero.

As all the possibilities for $\delta(\Pi_1), \delta(\Pi_2)$ and $\delta(V)$ are considered in the three statements of Theorem 1 (taking into account Remark 3), we only need to prove one direction of the statements to prove the whole Theorem. Hence we distinguish some different cases depending on the sign of $\delta(\Pi_2)$.

If $\delta(\Pi_2) > 0$ then

$$\tilde{R}(\tilde{P}, \tilde{Q}) = \prod_{i=1}^{p} \tilde{f}_i \Big(X_{\tilde{g}} - \tilde{g} \sum_{i=1}^{p} n_i \frac{X_{\tilde{f}_i}}{\tilde{f}_i} \Big) = \tilde{g} \prod_{i=1}^{p} \tilde{f}_i X_{\log \Pi_2} \not\equiv 0, \tag{11}$$

as Π_2 is not constant because $\delta(\Pi_2) > 0$. Then

$$\mathcal{F} = \tilde{g} \frac{\prod_{i=1}^{p} \tilde{f}_i}{\tilde{R}} \frac{x \frac{\partial \Pi_2}{\partial x} + y \frac{\partial \Pi_2}{\partial y}}{\Pi_2} = \delta(\Pi_2) \frac{\tilde{g} \prod_{i=1}^{p} \tilde{f}_i}{\tilde{R}} \not\equiv 0,$$

where we have applied the Euler Theorem for homogeneous functions. Moreover, as $V = \prod_{i=1}^p f_i^{n_i+1}/R$ and $\delta(\Pi_2) > 0$, we obtain $\delta(V) < \deg \mathcal{F} = d+1$. The 'only if' part of statement (1) follows.

If $\delta(\Pi_2) = 0$ then

$$\begin{aligned} \tilde{R}(\tilde{P},\tilde{Q}) &= \prod_{i=1}^p \tilde{f}_i \left(X_{\tilde{g}} + \sum_{i=1}^p (\lambda_i \prod_{j=1}^p \tilde{f}_j^{n_j} - n_i \tilde{g}) \frac{X_{\tilde{f}_i}}{\tilde{f}_i} \right) \\ &= \prod_{i=1}^p \tilde{f}_i^{n_i+1} \left(X_{\log \prod_{i=1}^p \tilde{f}_i^{\lambda_i}} + X_{\tilde{g}/\prod_{i=1}^p \tilde{f}_i^{n_i}} \right) \qquad (12) \\ &= \prod_{i=1}^p \tilde{f}_i^{n_i+1} X_{\log \Pi_1 + \Pi_2}, \end{aligned}$$

provided that $X_{\log \Pi_1+\Pi_2} \not\equiv 0$. Indeed $\log \Pi_1 + \Pi_2$ is constant if and only if Π_1 and Π_2 are constant, but we are assuming that Π_2 is not constant in this case, see Remark 3. Hence (12) always holds. Therefore $\delta(V) = d+1$ and

$$\mathcal{F} = \Big(\delta(\Pi_1) + \overbrace{\delta(\Pi_2)}^{0} \Pi_2 \Big) \tilde{V} = \delta(\Pi_1)\tilde{V}.$$

If $\delta(\Pi_2) < 0$ then

$$\tilde{R}(\tilde{P},\tilde{Q}) = \prod_{i=1}^p \tilde{f}_i^{n_i+1} \sum_{i=1}^p \lambda_i \frac{X_{\tilde{f}_i}}{\tilde{f}_i} = \prod_{i=1}^p \tilde{f}_i^{n_i+1} X_{\log \Pi_1}, \qquad (13)$$

provided that Π_1 is not constant. If this is the case, we have $\delta(V) = d+1$ and

$$\mathcal{F} = \left(\sum_{i=1}^p \lambda_i \deg f_i \right) \tilde{V} = \delta(\Pi_1)\tilde{V}.$$

We have proved the 'only if' part of statement (2). If Π_1 is constant then $\delta(V) > d+1$.

Finally, if $\delta(\Pi_2) < 0$ and Π_1 is constant, then $v^{-\delta(\Pi_2)}$ divides $(g/\prod_{i=1}^p f_i^{n_i})|_C$. Moreover $(\prod_{i=1}^p f_i^{\lambda_i})|_C|_{v=0} = \Pi_1$. Hence $H|_C|_{v=0} = \Pi_1$ is constant, and thus $v=0$ is invariant under the flow of system (8). Hence the infinity of system (1) is not degenerate.

To compute $\mathcal{F}_\infty$ we note that $H|_C$ and $v^{d+2}V|_C$ are respectively a first integral and an inverse integrating factor of system (8). Moreover $\dot{u} = v^{d+2}V|_C(\log(H|_C))_v$. The infinite singular points of system (1) are in correspondance with the singular points of system (8) on $v=0$, as $\dot{u} = v^{d+1}(xQ-yP)|_C$. Hence we obtain the expression of $\mathcal{F}_\infty$ of statement (3), as we wanted. Therefore the 'only if' part of statement (3) follows and the Theorem is proved. □

Remark 5. *We note that $V \log H$ is another inverse integrating factor of system* (1). *Moreover, we have*

$$\tilde{V} \log \tilde{H} = \frac{\tilde{g} \prod_{i=1}^p \tilde{f}_i}{\tilde{R}} + \frac{\prod_{i=1}^p \tilde{f}_i^{n_i}}{\tilde{R}} \sum_{i=1}^p \lambda_i \log \tilde{f}_i.$$

Observe that in statement (1) of Theorem 1 $\mathcal{F}$ is, up to a constant, the rational summand of this expression. In particular this summand is to be a homogeneous polynomial of degree $d+1$.

Remark 6. *In statement (2) it is clear that if $\mathcal{F} \not\equiv 0$ then $\tilde{V}$ is a polynomial. But if $\mathcal{F} \equiv 0$ then $\tilde{V}$ is not to be a polynomial in general, as we will see in Proposition 9 of Section 5.*

Proof of Corollary 2. The only case in Theorem 1 for which $\mathcal{F} \equiv 0$ is statement (2) with $\delta(\Pi_1) = 0$. □

Proof of Corollary 3. Without loss of generality we make the assumption that either $\deg h_1 \neq \deg h_2$, or $\deg h_2 = \deg(h_1 + ch_2)$ for all $c \in \mathbb{C}$. We claim that $\Pi_1 = \tilde{h}_1/\tilde{h}_2$ is not constant. To prove the claim we distinguish two cases. If $\deg h_1 \neq \deg h_2$ then $\delta(\Pi_1) \neq 0$ and thus Π_1 is not constant. If $\deg h_2 = \deg(h_1 + ch_2)$ for all $c \in \mathbb{C}$ and Π_1 is constant, then $\deg(h_1 - \Pi_1 h_2) < \deg h_2$, a contradiction.

Now we apply similar arguments to those of statement (2) of Theorem 1 to get $\delta(V) = d + 1$ and $\mathcal{F} = \delta(H)\tilde{V}$. □

We provide next a result similar to Theorem 1 (the proof is similar) but concerning the origin instead of infinity. We state it as a corollary of Theorem 1. First we need some definitions.

We denote by $\hat{f}$ the homogeneous part of lowest degree of a polynomial f, and by $\operatorname{sdeg} f$ the degree of $\hat{f}$. Let $m = \min\{\operatorname{sdeg} P, \operatorname{sdeg} Q\}$. We suppose that the origin of system (1) is a singular point at the origin and we call $\mathcal{F}_O = x\hat{Q} - y\hat{P}$ the *characteristic polynomial* associated to this singular point. Let

$$\Sigma_1 = \prod_{i=1}^{p} \hat{f}_i^{\lambda_i}, \quad \Sigma_2 = \hat{g} / \prod_{i=1}^{p} \hat{f}_i^{n_i}.$$

Finally we define $\mu(\prod g_i^{\alpha_i}) = \sum \alpha_i \operatorname{sdeg} g_i$, where $\alpha_i \in \mathbb{C}$ and $g_i \in \mathbb{C}[x, y]$. We note that $\mu(\Sigma_1) \in \mathbb{C}$ and $\mu(\Sigma_2) \in \mathbb{Z}$.

Corollary 6. *Consider system* (1) *and suppose that it has a Darboux first integral* (3) *which is not rational and a inverse integrating factor* (4). *The following statements hold.*

(1) $\mu(V) > m + 1$ *if and only if* $\mu(\Sigma_2) < 0$. *Moreover*
$$\mathcal{F}_O = \mu(\Sigma_2) \frac{\hat{g} \prod_{i=1}^{p} \hat{f}_i}{\hat{R}} \not\equiv 0.$$
(2) $\mu(V) = m + 1$ *if and only if either* $\mu(\Sigma_2) > 0$ *and* Σ_1 *is not constant, or* $\mu(\Sigma_2) = 0$. *Moreover in this case*
$$\mathcal{F}_O = \mu(\Sigma_1)\hat{V}.$$
(3) $\mu(V) < m + 1$ *if and only if* $\mu(\Sigma_2) > 0$ *and* Σ_1 *is constant. Moreover* $\mathcal{F}_O \not\equiv 0$ *and we can compute the characteristic directions by means of a blow-up.*

Remark 7. *After Corollary 6 we remark two particular cases.*

(1) *The origin is dicritical (that is,* $\mathcal{F}_O \equiv 0$*) if and only if* $\mu(\Sigma_1) = 0$ *and either* $\mu(\Sigma_2) > 0$ *and* Σ_1 *is not constant, or* $\mu(\Sigma_2) = 0$.
(2) *Suppose that system* (1) *has a rational first integral* $H = h_1/h_2$ *and the inverse integrating factor* (6). *Then* $\mu(V) = m + 1$ *and* $\mathcal{F} = \mu(H)\tilde{V}$.

4. Critical remarkable values of Darboux first integrals

The purpose of this section is to provide a definition of critical remarkable values and curves of Darboux first integrals, and afterwards to prove Theorem 4, which characterizes polynomial differential systems having a Darboux first integral and a polynomial inverse integrating factor by means of the number of critical remarkable values of the first integral. We begin the section with a lemma that we will use later on.

Lemma 7. *Suppose that system* (1) *has a Darboux first integral* (3). *Then* $f_i \nmid R$ *for all* $i \in \{1, \ldots, p\}$.

Proof. Suppose that $f_i | R$ for a certain $i \in \{1, \ldots, p\}$. If $n_i \neq 0$ then from (9) we have that $f_i | n_i g X_{f_i} \prod_{j \neq i} f_j$, and this cannot happen because $(f_i, f_j) = 1$, because $(f_i, g) = 1$ as $n_i > 0$ and because a polynomial cannot divide its derivatives.

If $n_i = 0$ then from (9) again we have $f_i | \lambda_i \prod_{j=1}^{p} f_j^{n_j} X_{f_i} \prod_{j \neq i} f_j$. We note that f_i does not appear in the products as $n_i = 0$. Hence $\lambda_i = 0$, a contradiction again. $\square$

Suppose that system (1) has a Darboux first integral (3) which is not rational. Let $f = 0$ be an irreducible invariant algebraic curve of system (1). We say that $f = 0$ is a *critical remarkable curve* of H if either $f = f_i$ and $n_i > 0$, for some $i \in \{1, \ldots, p\}$, or $f | R$. In the second case we say that $c = H|_{f=0} \in \mathbb{C} \setminus \{0\}$ is a *critical remarkable value* and we define the *exponent* of $f = 0$ as its exponent in the factorization of R plus one.

We note that the function (6) has in its numerator all the factors of $h_1 h_2$ powered to one because of the remarkable factor. It coincides with the function V of Theorem 4 with $n_i = 0$ for all $i \in \{1, \ldots, p\}$, as in this case $f_i \nmid R$ by Lemma 7. We do not associate critical remarkable values to the f_i, but according to the above observation we define the exponent of $f_i = 0$ as a critical remarkable curve as $n_i + 1$, for $i = 1, \ldots, p$.

The remarkable factor R is formed by invariant algebraic curves different from the f_i. But the curves f_i and the ones appearing in the factorization of R are not the unique invariant algebraic curves that system (1) can have, as we will show in some of the examples of Section 5. We call these curves *non-critical remarkable curves* and the corresponding level sets of H *non-critical remarkable values*. We define their exponent as 1. The polynomials f_i such that $n_i = 0$ are also considered non-critical remarkable curves, again without an associated remarkable value.

When dealing with critical remarkable values of a rational first integral $H = h_1/h_2$ we assume that the degree of H is minimal, see Subsection 2.2. This important assumption in the rational case is not a matter in the Darboux case, as the next lemma shows.

Lemma 8. *Suppose that system* (1) *has two Darboux (non-rational) first integrals* H *and* $\bar{H}$. *Then* $\bar{H} = H^{\alpha}$, *where* $\alpha \in \mathbb{C} \setminus \{0\}$.

Proof. Let V and $\bar{V}$ be the rational inverse integrating factors associated to H and $\bar{H}$, respectively. Then either $V/\bar{V}$ is a rational first integral of system (1), or there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $V = \alpha \bar{V}$. As the system has a Darboux first integral it cannot have rational first integrals, hence $V = \alpha \bar{V}$. This implies that the first integral associated to $\bar{V}$ is $\bar{H} = H^{\alpha}$, and therefore the proof is finished. $\square$

Remark 8. *It makes no difference to use H or H^α, with $\alpha \in \mathbb{C}\backslash\{0\}$, in the definition of critical remarkable values and curves. The associated inverse integrating factors are V and V/α, respectively.*

Proof of Theorem 4. It is clear that V is a polynomial if and only if either R is constant, or $R|\prod_{i=1}^{p} f_i^{n_i+1}$. The former case holds if and only if the number of critical remarkable values is zero. The latter case is equivalent to say that $R = \prod_{j\in J} f_j^{l_j}$, where $J \subseteq \{1,\ldots,p\}$, $l_j \in \mathbb{N}$ and $l_j \le n_j + 1$ for all $j \in J$, as the f_i are irreducible. But this is a contradiction with Lemma 7. Hence the Theorem follows. □

5. Examples

As Lemma 7 shows, the factor R is formed by invariant algebraic curves different from the f_i. Next example shows that the curves f_i and the ones appearing in the factorization of R are not the unique invariant algebraic curves that system (1) can have. Moreover the characteristic polynomial $\mathcal{F}$ of the differential system is computed using Theorem 1.

Example 1. Consider the Darboux function

$$H = \frac{(-1+x-2y+2x^2-y^2)^2}{(-1+x-y^2)(1-x-3y+3x^2+y^2)} \exp\left(\frac{x^3(y-x^2)}{2y^4}\right).$$

Its associated polynomial differential system has degree $d = 11$. We note that $y-x^2 = 0$ is an invariant algebraic curve of the system; indeed, $H|_{y=x^2} = -1$. The system has the polynomial inverse integrating factor $V = 2y^5(-1+x-2y+2x^2-y^2)(-1+x-y^2)(1-x-3y+3x^2+y^2)$, associated to $\log H$. Hence $y-x^2 = 0$ is different from the invariant algebraic curves $f_i = 0$ appearing in the expression of H and is not a factor of R.

The characteristic polynomial can be computed from H: as $\deg g = 5 > 4 = \sum_{i=1}^{3} n_i \deg f_i$ we apply statement (1) of Theorem 1 to obtain $\mathcal{F} = x^5y^3(2x^2-y^2)(3x^2+y^2) \not\equiv 0$. □

In the following two examples we apply the results of Theorem 1 and Theorem 4.

Example 2. Let $f_1 = 1+x+2y-x^2-2xy-y^2$ and $f_2 = 1+4x+2y+2x^2+4xy+2y^2$. Consider the polynomial differential system which has the Darboux first integral $H = f_1/f_2^2 \exp(x^3/(f_1^2 f_2))$ and the (polynomial) inverse integrating factor $V = f_1^3 f_2^2$ of degree 10. As V is a polynomial, by Theorem 4 the system has no critical remarkable values. Moreover, $\deg g < 2\deg f_1 + \deg f_2$ and Π_1 is not constant, hence by statement (2) of Theorem 1 we have $\delta(V) = d+1$, which means that the system has degree 9. Moreover we can compute the singular points at infinity from $\tilde{V}$: $\mathcal{F} = (\deg f_1 - 2\deg f_2)\tilde{V} = 8(x+y)^{10}$. □

Example 3. Consider the quadratic differential system $(-1-3x-2y+x^2-2xy-3y^2, 1+3x+3y+4xy+4y^2)$. This system has the Darboux first integral $H = f_1/f_2 \exp(3(1-x-y)/f_1)$ and the polynomial inverse integrating factor $V = f_1^2 f_2/9$, where $f_1 = 1+x+2y-x^2-2xy-y^2$ and $f_2 = 1+4x+2y+2x^2+4xy+2y^2$. As V is a polynomial, by Theorem 4 the system has no critical remarkable values. Moreover, as $\deg g < \deg f_1$ and $\Pi_1 = -1/2$, by statement (3) of Theorem 1 we have $\delta(V) > d+1$. Indeed $\delta(V) = 6 > 3 = d+1$. We can compute the infinite singular points from Theorem 1: $\mathcal{F}_\infty = 3u(1+u)^2$. Hence $\mathcal{F} = 3y(x+y)^2 \not\equiv 0$. □

We wonder whether we can find a relation betweeen V and $\mathcal{F}$ in case V is a Darboux inverse integrating factor. We check a simple example.

Example 4. Consider the quadratic differential system $(1, 2xy + y^2)$. This system has the Darboux inverse integrating factor $V = y^2 \mathrm{e}^{-x^2}$ and the first integral $H = -\mathrm{e}^{-x^2}/y - \sqrt{\pi}/2\,\mathrm{erfi}(x)$, where $\mathrm{erfi}(x)$ is the imaginary error function. The characteristic polynomial is $\mathcal{F} = xy(2x + y)$. □

We state the following open question.

Open Question. How can we relate the characteristic polynomial with the inverse integrating factor for polynomial differential systems having a Darboux inverse integrating factor, if such a relation exists?

There are many examples in the literature showing the existence of a polynomial inverse integrating factor when system (1) has a Darboux first integral (3) and the infinity is degenerate (see for example [22]). Next proposition shows that this is not always the case.

Proposition 9. *The cubic differential system* $(2x(4 + a + 6(4 + a)x + 8(4 + a)x^2 + 8ay^2), y(4 + a + 4(8 + 3a)x + 16(4 + a)x^2 + 16ay^2))$*, with* $a \in \mathbb{R}$*, appearing in* [6] *has a Darboux first integral and degenerate infinity, but it has no polynomial inverse integrating factors.*

Proof. The system has the Darboux first integrals $H = y^{-2(4+a)}(x+2x^2+2y^2)^{4-a}((a-4)x(1 + 4x) + 4ay^2)^{2a}$ if $a \neq 4$ and $H = y^2/(x + 2x^2 + 2y^2)\exp\big(x(1 + 4x)/(2y^2)\big)$ if $a = 4$. Moreover it has the rational inverse integrating factor $V = y(x + 2x^2 + 2y^2)((a - 4)x(1 + 4x) + 4ay^2)/x$.

Applying statement (2) of Theorem 1 we have that $\delta(V) = d + 1 = 4$ and that $\mathcal{F} \equiv 0$.

If H is not rational then V is the unique rational (including polynomial) inverse integrating factor of the system, and hence there are no polynomial inverse integrating factors. If H is rational then it has three critical remarkable values (that is, 0, ∞ and the one corresponding to the critical remarkable curve $x = 0$) and therefore there are no polynomial inverse integrating factors by Theorem 5 or [6]. □

We finally provide an example showing that statement (e) of the Main Theorem of [6] is not correct (see Remark 2 in Subsection 2.1).

Example 5. The polynomial differential system $X = (P, Q)$ having the Darboux first integral

$$H = (x + iy)^{3+2i}(x - iy)^{3-2i}(x + y)^2 \exp\left(\frac{x^3}{(2x + iy)^2}\right)$$

and having the (polynomial) inverse integrating factor $V = (2x + iy)^3(x + y)(x^2 + y^2)$ has degree $d = 6$. It is easy to check that the system satisfies the hypotheses of statement (e) of the Main Theorem of [6], but $\deg V = 6 \neq 7 = d + 1$.

The theorem fails because in its proof it considers the homogeneous polynomial system $(\tilde{P}, \tilde{Q})$ of degree $d = 6$ and the polynomial inverse integrating factor $\bar{V} = x\tilde{Q} - y\tilde{P} = x^3(x^2 + y^2)(2x + iy)(x + y)$ of degree $d + 1 = 7$ to arrive to contradiction. In this example $\tilde{P}$ and $\tilde{Q}$ are not coprime. Indeed the homogeneous system is linear after removing the common factors. Moreover this linear system has the polynomial inverse integrating factor $x(2x+iy)$ and the rational first integral $x^3/(2x+iy)^2$, which is the exponent of the exponential factor of H.

We finally note that the inverse integrating factor V of degree 6 is the one of (4) and that $\bar{V}$ is just the non-logarithmic part of $V \log H$ appearing in statement (1) of Theorem 1, for we are under its hypotheses. □

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References

[1] M.J. Álvarez and A. Ferragut, *Local behavior of planar analytic vector fields via integrability*, J. Math. Anal. Appl. **385** (2012), 264–277.

[2] A. A. Andronov, E.A. Leontovich, I.I. Gordon and A.G. Maier, *Qualitative theory of second order differential equations*, John Wiley and Sons, New York, 1973.

[3] L.R. Berrone and H. Giacomini, *On the vanishing set of inverse integrating factors*, Qual. Theory Dyn. Syst. **1** (2000), 211–230.

[4] B. Coll, A. Ferragut and J. Llibre, *Polynomial inverse integrating factors for quadratic differential systems*, Nonl. Anal. **73** (2010), 881–914.

[5] J. Chavarriga, H. Giacomini, J. Giné and J. Llibre, *On the integrability of two-dimensional flows*, J. Diff. Eq. **157** (1999), 163–182.

[6] J. Chavarriga, H. Giacomini, J. Giné and J. Llibre, *Darboux integrability and the inverse integrating factor*, J. Diff. Eq. **194** (2003), 116–139.

[7] C.J. Christopher, *Invariant algebraic curves and conditions for a center*, Proc. Roy. Soc. Edinburgh **124A** (1994), 1209–1229.

[8] C.J. Christopher, *Liouvillian first integrals of second order polynomial differential equations*, Electronic J. Diff. Eq. **49** (1999), 1–7.

[9] C.J. Christopher and R.E. Kooij, *Algebraic invariant curves and the integrability of polynomial systems*, Appl. Math. Lett. **6** (1993) 51–53.

[10] C. Christopher, J. Llibre and J.V. Pereira, *Multiplicity of invariant algebraic curves and Darboux integrability*, Pacific J. of Math. **229** (2007), 63–118.

[11] A. Ferragut and H. Giacomini, *A new algorithm for finding rational first integrals of polynomial vector fields*, Qual. Theory Dyn. Syst. **9** (2010), 89–99.

[12] A. Ferragut and J. Llibre, *On the remarkable values of the rational first integrals of polynomial vector fields*, J. Diff. Eq. **241** (2007), 399–417.

[13] A. Ferragut, J. Llibre and A. Mahdi, *Polynomial inverse integrating factors for polynomial vector fields*, Disc. Cont. Dyn. Syst. **17** (2007), 387–395.

[14] I.A. García and M. Grau, *A Survey on the Inverse Integrating Factor*, Qual. Theory Dyn. Syst. **9** (2010), 115–166.

[15] I.A. García and D.S. Shafer, *Integral invariants and limit sets of planar vector fields*, J. Diff. Eq. **217** (2005), 363–376.

[16] H. Giacomini, J. Llibre and M. Viano, *On the nonexistence, existence, and uniqueness of limit cycles*, Nonlinearity **9** (1996), 501–516.

[17] J. Llibre and Ch. Pantazi, *Polynomial differential systems having a given Darbouxian first integral*, Bull. Sci. math. **128**, (2004), 775–788.

[18] J. Llibre and G. Rodríguez, *Configurations of limit cycles and planar polynomial vector fields*, J. Diff. Eq. **198** (2004), 374–380.

[19] H. Poincaré, *Sur l'intégration des équations différentielles du premier ordre et du premier degré I and II*, Rendiconti del Circolo Matematico di Palermo **5** (1891), 161–191; **11** (1897), 193–239.

[20] M.J. Prelle and M.F. Singer, *Elementary first integrals of differential equations*, Trans. Amer. Math. Soc. **279** (1983), 215–229.

[21] M.F. Singer, *Liouvillian first integrals of differential equations*, Trans. Amer. Math. Soc. **333** (1992), 673–688.

[22] D. Schlomiuk and N. Vulpe, *The full study of planar quadratic differential systems possessing a line of singularities at infinity*, J. Dyn. Diff. Eq. **20** (2008), 737–775.

[23] S. Walcher, *On the Poincaré problem*, J. Diff. Eq. **166** (2000), 51–78.

[24] H. Żołądek, *On algebraic solutions of algebraic Pfaff equations*, Stud. Math. **114** (1995) 117–126.

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