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Strongly persistent centers for trigonometric Abel equations

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Abstract

In this paper we introduce the notion of strongly persistent centers, together with the condition of the annulation of some generalized moments, for Abel differential equations with trigonometric coefficients as a natural candidate to characterize the centers of composition type for these equations. We also recall several related concepts and discuss the differences between the trigonometric and the polynomial cases.

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1 Introduction

Consider planar systems of differential equations of the form

$$\begin{aligned}\frac{dx}{dt} &= -y + P(x,y),\\ \frac{dy}{dt} &= x + Q(x,y),\end{aligned} \tag{1}$$

where P and Q are analytic functions starting with second order terms. The problem of determining necessary and sufficient conditions on P and Q for system (1) to have a center at the origin is known as the *center-focus problem*. From the works of Poincaré and Lyapunov it is well-known that equation (1) has a center at the origin when an infinite sequence of polynomial conditions among the coefficients of the Taylor expansions of P and Q at the origin are satisfied. These conditions are given by the vanishing of the so called *Lyapunov constants*. Moreover, when P and Q are polynomials of a given degree this set of conditions is finite due to the Hilbert's basis Theorem. Nevertheless from these results it is not easy to obtain explicit conditions on P and Q that force the origin to be a center.

The particular case when $P(x,y) = P_n(x,y)$ and $Q(x,y) = Q_n(x,y)$ are homogeneous polynomials of degree n has received a considerable attention. The cases $n = 2$ or $n = 3$ were completely solved many time ago in the works of Dulac, Bautin, Kaptein and Sibirskii and several partial results are known for $n \geq 4$, see [14] and the references there in. In this particular case, taking polar coordinates (ρ, θ), system (1) writes as

$$\frac{d\rho}{dt} = f(\theta)\,\rho^n,$$
$$\frac{d\theta}{dt} = 1 + g(\theta)\,\rho^{n-1},$$

with

$$f(\theta) = \cos(\theta)\,P_n(\cos(\theta), \sin(\theta)) + \sin(\theta)\,Q_n(\cos(\theta), \sin(\theta)),$$
$$g(\theta) = \cos(\theta)\,Q_n(\cos(\theta), \sin(\theta)) - \sin(\theta)\,P_n(\cos(\theta), \sin(\theta)).$$

Applying the change of variables introduced by Cherkas in [11],

$$r = \frac{\rho^{n-1}}{1 + g(\theta)\,\rho^{n-1}},$$

we get

$$\frac{dr}{d\theta} = \hat{A}(\theta) r^3 + \hat{B}(\theta) r^2, \tag{2}$$

where

$$\hat{A}(\theta) = (1-n) f(\theta) g(\theta)\,, \quad \hat{B}(\theta) = g'(\theta) + (1-n) f(\theta).$$

Notice that $\hat{A}(\theta)$ and $\hat{B}(\theta)$ are trigonometric polynomials of degree $2(n+1)$ and $n+1$ respectively. Equations of the form (2) are a particular class of Abel equations.

Therefore it is natural to consider the following problem:

Center-focus problem for Abel equations. *Given the equation*

$$\dot{r} = \frac{dr}{d\theta} = A(\theta) r^3 + B(\theta) r^2, \tag{3}$$

where A and B are trigonometric polynomials, give necessary and sufficient conditions on A and B to ensure that all the solutions $r = r(\theta, r_0)$, with initial condition $r(0, r_0) = r_0$ and $|r_0|$ small enough are 2π-periodic, i.e. $r(0, r_0) = r(2\pi, r_0)$. For short, if this property holds we will say the the Abel equation has a center (at $r = 0$).

It is well known that also for the above problem the existence of a center is guaranteed if some polynomials computed from the coefficients of $A(\theta)$ and $B(\theta)$ vanish. These quantities are given by the return map between $\{\theta = 0\}$ and $\{\theta = 2\pi\}$, near $r = 0$,

$$\Pi(r_0) = r_0 + V_k r_0^k + O(r_0^{k+1})$$

and can be thought as the Lyapunov constants for the Abel equation, see for instance [3, 13]. The first two center conditions are

$$V_2 := \int_0^{2\pi} B(\theta)\, d\theta = 0 \quad \text{and} \quad V_3 := \int_0^{2\pi} A(\theta)\, d\theta = 0. \tag{4}$$

In [3] the authors introduce a simple condition called *composition condition*, for short CC, which ensures that the corresponding Abel equation has a center. Roughly speaking the composition condition, says that the primitives of the functions A and B depend functionally on a new 2π-periodic function. See Definition 1 in next section for the precise statement of the result. When an Abel equation has a center because A and B satisfy the CC we will say that the equation has a *CC-center*. In [1] it was shown that this condition is not necessary to have a center.

For planar differential equations (1) the simplest explicit conditions that imply that the origin is a center are either that the vector field is Hamiltonian, or that the vector field is reversible with respect to a straight line. The notion of CC-centers for Abel equations can also be seen as the simplest explicit condition for these equations to have a center. Moreover it is not difficult to prove that the Hamiltonian or reversible (with respect to a straight line) centers for planar differential equations (1) with homogeneous nonlinearities correspond, via the Cherkas transformation, to CC-centers for the corresponding Abel equations.

On the other hand several authors try to characterize the so-called *persistent centers*, see for instance [2] and the references therein and to relate them with the CC-centers. It is said that the Abel equation

$$\dot{r} = \epsilon\, A(\theta) r^3 + B(\theta) r^2, \tag{5}$$

has a persistent center if it has a center for all ϵ small enough. In next Section we give some equivalent formulations. It is easy to see that CC-centers are persistent centers but the converse of this implication is an open question.

Persistent centers satisfy the so-called *moments condition*, see for instance [2] or Theorem 8 for a stronger result. This last condition says that

$$\int_0^{2\pi} A(\theta) \left(\int_0^{\theta} B(\alpha)\, d\alpha \right)^k d\theta = 0,$$

for all $k \geq 0$.

Summarizing, associated to trigonometric Abel equations we have CC-centers, persistent centers and moments condition. The current interest is to relate these three concepts.

Motivated by the above problem many authors have faced the equivalent question when the functions A and B, instead of being trigonometric polynomials are usual polynomials, see [5, 6, 7, 8, 10, 12, 17]. To fix the problem, in this situation the question is to give necessary and sufficient conditions on the real polynomials $A(t)$ and $B(t)$ to ensure that the solutions of the equation

$$\frac{dw}{dt} = A(t)w^3 + B(t)w^2$$

satisfy $w(a) = w(b)$, for some given a and b and for initial conditions close enough to $w = 0$. Notice that contrary to what happens in the trigonometric case with $a = 0$ and $b = 2\pi$, such condition does not imply the global periodicity of $w(t)$. There are many significative advances for this polynomial case. In [16] an example of a polynomial Abel equation satisfying the moments condition and not satisfying the composition condition is given. Later on, in [15] a full algebraic characterization of the moments condition in the polynomial case is done. In Section 4 we recall this result and prove that a natural trigonometric analogous to it does not hold.

In this paper we consider the following natural extension of the Abel equation (3),

$$\dot{r} = A(\theta)r^n + B(\theta)r^m, \quad \text{with} \quad n > m > 1, \tag{6}$$

where $A(\theta)$ and $B(\theta)$ are trigonometric polynomials. Notice that the case $m = 1$ can be transformed into a Riccati equation which can be explicitly solved. In this case it is easy to see that (6) has a center if and only if the two conditions given in (4) hold.

Our purpose is to point out some relations between, persistent centers, CC-centers and moments condition for equation (6). In Section 2 we give several relations between these three concepts and in Section 3 we present some examples which show that some of them are not equivalent. Also we introduce several new classes, *symmetric centers*, *degree-persistent centers* and *strongly persistent centers*. Related to this last class we also consider some generalized moments conditions, which can be useful to have a better understanding of the situation. Finally, in Section 4, we present a diagram with the known implications among the different classes of centers and conditions considered in the paper.

2 Persistent Centers, Moment Conditions and the Composition Condition

In all the paper given any function $C(\theta)$ we will denote by $\widetilde{C}(\theta) = \int_0^\theta C(t)dt$.

In [3] the authors give the following sufficient condition for equation (6) to have a center, named the *composition condition*.

Definition 1. *The functions $A(\theta)$ and $B(\theta)$ satisfy the* **composition condition (CC)** *if there exist $\mathcal{C}^1$-functions u, A_1 and B_1, with u being 2π-periodic, and such that*

$$\widetilde{A}(\theta) = A_1(u(\theta)) \quad \textit{and} \quad \widetilde{B}(\theta) = B_1(u(\theta)). \tag{7}$$

To see that the CC implies the existence of a center one can consider the differential equation

$$\frac{dR}{du} = A_1'(u)R^n + B_1'(u)R^m. \tag{8}$$

Let $R = R(u)$ one of its solutions. It is easy to see that then $r(\theta) = R(u(\theta))$ is a solution of (6). Therefore, if A and B satisfy the CC given in (7), then the corresponding differential equation (6) has center for all $n, m \in \mathbb{N}$.

As we have already explained in the introduction there are centers for (6) which are no CC-centers, see [1, 2] and also Proposition 18.

Another important notion is that of the *persistent center*. Before introducing it we prove some preliminary results.

Lemma 2. *Consider the Abel equation* (6) *and let* $r(\theta, \rho)$ *be the solution such that* $r(0, \rho) = \rho$. *Set*

$$I = \{\rho : \ r(\theta, \rho) \text{ is defined for all } \theta \in [0, 2\pi]\}.$$

Then I *is and open interval containing* 0. *If in addition equation (6) has a center then* I *coincides with the set of initial conditions for which the solution is defined for all time. Moreover, in this case, it also coincides with the set of initial conditions that correspond to periodic orbits.*

Proof. First we prove that I is and open interval containing 0. It is open because of the continuous dependence of the flow on the initial conditions. It is an interval because if $0 < \rho_1 \notin I$ then $r(\theta, \rho_1)$ goes to infinity at some point $\theta_1 \in (0, 2\pi)$ and this fact implies that for all $\rho > \rho_1$ the corresponding solution goes to infinity as well, at some point $0 < \theta_\rho \le \theta_1$. Hence $\rho \notin I$ for all $\rho > \rho_1$. A similar argument can be used when $0 > \rho_1 \notin I$. So the return map $\rho \mapsto r(2\pi, \rho)$ is well defined on I and, if the equation has a center, it is the identity for ρ small enough. Since it is analytic it follows that the return map is the identity on I. This ends the proof of the lemma. $\square$

The proof of the next lemma is straightforward.

Lemma 3. *The solution of* $dr/d\theta = C(\theta)\, r^k$ *for* $k \neq 1$, *satisfying* $r(0, \rho) = \rho$ *is*

$$r(\theta, \rho) \ = \ \rho \Big[1 + (1 - k)\, \rho^{k-1}\, \widetilde{C}(\theta) \Big]^{\frac{1}{1-k}}.$$

Moreover, this differential equation has a center if and only if $\widetilde{C}(2\pi) = \int_0^{2\pi} C(\theta)\, d\theta = 0$.

The proof of the following result follows the same ideas that the one of Proposition 2.1 of [4].

Lemma 4. *Assume that the differential equation* (6)*,*

$$\dot r = A(\theta) r^n + B(\theta) r^m, \quad \textit{with} \quad n > m > 1,$$

has a center. Then

$$V_m = \widetilde{B}(2\pi) = \int_0^{2\pi} B(\theta)\, d\theta = 0 \quad \textit{and} \quad V_n = \widetilde{A}(2\pi) = \int_0^{2\pi} A(\theta)\, d\theta = 0.$$

Proof. Let

$$r = r(\theta, \rho) = \rho + u_2(\theta)\rho^2 + u_3(\theta)\rho^3 + O(\rho^4),$$

be the solution of (6) such that $r(0,\rho) = \rho$. If the equation has a center then $u_k(2\pi) = 0$, for all $k \geq 1$. Plugging the above expression in (6) we obtain that

$$u_1(\theta) \equiv u_2(\theta) \equiv \cdots \equiv u_{m-1}(\theta) \equiv 0 \quad \text{and} \quad u_m(\theta) = \widetilde{B}(\theta).$$

Thus the condition $V_m = \widetilde{B}(2\pi) = 0$ holds. Therefore, from Lemma 3 we have that the equation $\dot r = B(\theta) r^m$ has also a center. Let $r = R(\theta, \rho)$ be its solution satisfying $R(0,\rho) = \rho$. Then, for ρ small enough, $R(2\pi, \rho) = \rho$.

Notice that $r(\theta,\rho) - R(\theta,\rho) = O(\rho^{m+1})$. Therefore, writing $r(\theta,\rho) = R(\theta,\rho) + \rho^{m+1} S(\theta,\rho)$, with

$$S(\theta,\rho) = w_0(\theta) + w_1(\theta)\rho + w_2(\theta)\rho^2 + \cdots + w_n(\theta)\rho^n + O(\rho^{n+1})$$

and plugging it again in (6) we obtain that

$$w_0(\theta) \equiv w_1(\theta) \equiv \cdots \equiv w_{n-m-2}(\theta) \equiv 0 \quad \text{and} \quad w_{n-m-1}(\theta) = \widetilde{A}(\theta).$$

Then

$$r(2\pi,\rho) = R(2\pi,\rho) + \rho^{m+1} S(2\pi,\rho) = \rho + \widetilde{A}(2\pi)\rho^n + O(\rho^{n+1}),$$

and $\widetilde{A}(2\pi) = 0$ as we wanted to prove. □

Proposition 5. *Set $n > m \geq 1$. The following three conditions are equivalent:*

(i) The differential equation

$$\dot r = A(\theta) r^n + \epsilon B(\theta) r^m, \tag{9}$$

has a center for all ϵ small enough.

(ii) The differential equation

$$\dot r = A(\theta) r^n + \epsilon\, B(\theta) r^m, \tag{10}$$

has a center for all $\epsilon \in \mathbb{R}$.

(iii) The differential equation

$$\dot{r} = \lambda\, A(\theta) r^n + \mu\, B(\theta) r^m, \tag{11}$$

has a center for all $\lambda, \mu \in \mathbb{R}$.

Moreover, if $m > 1$*, all the above conditions are also equivalent to:*

(iv) The differential equation

$$\dot{r} = \epsilon\, A(\theta) r^n + B(\theta) r^m, \tag{12}$$

has a center for all ϵ *small enough.*

Proof. i)⇒ ii). Assume that (i) holds and define

$$\mathcal{S} := \{0 < \epsilon \in \mathbb{R} : \text{ equation (10) has a center for all } \epsilon',\ 0 < \epsilon' < \epsilon\}.$$

Set $\epsilon_0 = \infty$ if $\mathcal{S}$ is unbounded and $\epsilon_0 = \sup(\mathcal{S})$ otherwise. We will denote by $r = r(\theta, \epsilon, \rho_0)$ the solution of (9) such that $r(0, \epsilon, \rho_0) = \rho_0$. We will show that $\epsilon_0 = \infty$. If not, set $\rho_0 > 0$ be such that $r(\theta, \epsilon_0, \rho_0)$ and $r(\theta, \epsilon_0, -\rho_0)$ is defined in $[0, 2\pi]$. Then, by the continuity of solutions with respect to parameters, the same holds for $r(\theta, \epsilon, \rho_0)$ and $r(\theta, \epsilon, -\rho_0)$ for $\epsilon \in (\epsilon_0 - \delta, \epsilon_0 + \delta)$ with δ small enough. Using Lemma 2 we conclude that the return map $(\epsilon, \rho) \mapsto r(2\pi, \epsilon, \rho)$ is well-defined on $(\epsilon_0 - \delta, \epsilon_0 + \delta) \times (-\rho_0, \rho_0)$. Since it is analytic and, also from Lemma 2, restricted to $(\epsilon_0 - \delta, \epsilon_0) \times (-\rho_0, \rho_0)$ is the identity we obtain that it is the identity on $(\epsilon_0 - \delta, \epsilon_0 + \delta) \times (-\rho_0, \rho_0)$. Thus equation (10) has a center for all $\epsilon \in (\epsilon_0 - \delta, \epsilon_0 + \delta)$ which gives a contradiction with the definition of ϵ_0.

ii)⇒ iii). Assume that ii) holds. By Lemma 3, $\dot{r} = \mu\, B(\theta)\, r^m$ has a center if and only if $\widetilde{B}(2\pi) = 0$. This last condition is guaranteed by Lemma (4). So (11) has a center when $\lambda = 0$. Otherwise, observe that the change $R = \alpha r$ transforms equation (11) into

$$\dot{R} = \lambda \alpha^{1-n} A(\theta) R^n + \mu \alpha^{1-m} B(\theta) R^m.$$

If n is even choosing $\alpha = \lambda^{\frac{1}{n-1}}$ we get that equation (11) is conjugated to

$$\dot{r} = A(\theta) r^n + \mu \lambda^{\frac{1-m}{n-1}} B(\theta) r^m$$

which has a center. This finish the proof in the case n even. If n is odd and $\lambda > 0$ the same argument holds. So in this case we have proved that equation (11) has a center for all $\lambda \geq 0$ and for all $\mu \in \mathbb{R}$. To obtain the desired result for all $\lambda \in \mathbb{R}$ we fix $\mu \in \mathbb{R}$ and consider

$$\lambda_0 = \inf\{\lambda \in \mathbb{R} : \text{ equation (11) has a center for all } \lambda' > \lambda\}$$

and use the same arguments than in the proof of the previous implication to show that $\lambda_0 = -\infty$.

It is obvious that iii) $\Rightarrow$ i). Lastly to see that iv)$\Leftrightarrow$ ii) we repeat the same arguments used in the proof of ii)$\Rightarrow$ iii), choosing now $\alpha = \mu^{1-m}$. Note that this last equivalence does not hold when $m = 1$. $\square$

The above result motivates the following definition:

Definition 6. *Equation (6) has a* ***persistent center*** *if*

$$\dot{r} = \lambda\, A(\theta) r^n + \mu\, B(\theta) r^m,$$

has a center for all $\lambda, \mu \in \mathbb{R}$.

We want to remark that most authors use the definition of persistent center given in item (iv) of Proposition 5. Other also refer to the notion of persistent at infinity using the definition of item (i). Our result shows that all are equivalent. We have chosen the above one because it is more symmetric.

Notice that if $A(\theta)$ and $B(\theta)$ satisfy the CC then the same is true for $\lambda A(\theta)$ and $\mu B(\theta)$. So, each center which satisfies the CC is a persistent center.

Definition 7. *For each* $k \in \mathbb{N}$, *the expression* $\int_0^{2\pi} A(\theta)\, \widetilde{B}(\theta)^k\, d\theta$ *is known as the* ***moment of order*** k ***of*** A ***with respect to*** B.

Next result proves that if a center is persistent, then the moments of A with respect to B and the moments of B with respect to A must be zero. As far as we know the second fact is a new result.

Theorem 8. *If (6) has a persistent center then*

$$\int_0^{2\pi} A(\theta)\, \widetilde{B}^k(\theta)\, d\theta = 0\,,\; k \geq 0 \tag{13}$$

and

$$\int_0^{2\pi} B(\theta)\, \widetilde{A}^k(\theta)\, d\theta = 0\,,\; k \geq 0. \tag{14}$$

Proof. We prove first the new set of conditions (14). From the hypothesis and Proposition 5 we know that equation $\dot{r} = A(\theta) r^n + \epsilon\, B(\theta) r^m$ has a center for all $\epsilon \in \mathbb{R}$. From Lemma 2 we also know that there exist ρ' and ϵ' such that the above equation has the 2π-periodic solution $r(\theta, \epsilon, \rho)$ for all $|\rho| < \rho'$ and $|\epsilon| < \epsilon'$. Then, for $0 < \rho < \rho'$:

$$\int_0^{2\pi} \frac{\frac{\partial r(\theta,\epsilon,\rho)}{\partial \theta}}{(r(\theta,\epsilon,\rho))^n}\, d\theta \;=\; \int_0^{2\pi} A(\theta)\, d\theta + \epsilon \int_0^{2\pi} \frac{B(\theta)}{(r(\theta,\epsilon,\rho))^{n-m}}\, d\theta.$$

Since $r(\theta, \epsilon, \rho)$ is 2π-periodic the left hand side is 0. Moreover, since (9) has a center, from Lemma 4, $\int_0^{2\pi} A(\theta)\, d\theta = 0$. Then

$$\int_0^{2\pi} \frac{B(\theta)}{(r(\theta, \epsilon, \rho))^{n-m}}\, d\theta \;=\; 0, \quad \text{for all} \quad 0 < |\rho| < \rho',\, 0 < |\epsilon| < \epsilon'. \tag{15}$$

From the analyticity of the differential equation, we can write its solutions as $r(\theta, \epsilon, \rho) = \sum_{i \geq 0} \psi_i(\theta, \rho)\, \epsilon^i$, being $\psi_0(\theta, \rho)$ the solution when $\epsilon = 0$. From Lemma 3,

$$\psi_0(\theta, \rho) \;=\; \rho \Big[1 + (1-n)\, \rho^{n-1}\, \widetilde{A}(\theta) \Big]^{\frac{1}{1-n}}. \tag{16}$$

Fixed $\rho \neq 0$, it holds that $(r(\theta, \epsilon, \rho))^{m-n}$ tends uniformly to $(\psi_0(\theta, \rho))^{m-n}$on $[0, 2\pi]$ when $\epsilon \to 0$. Therefore, from (15),

$$\int_0^{2\pi} \frac{B(\theta)}{(\psi_0(\theta, \rho))^{n-m}}\, d\theta = \lim_{\epsilon \to 0} \int_0^{2\pi} \frac{B(\theta)}{(r(\theta, \epsilon, \rho))^{n-m}}\, d\theta = 0.$$

From (16), for $0 \neq |\rho| < \rho'$,

$$\frac{1}{\rho^{n-m}} \int_0^{2\pi} B(\theta) \Big[1 + (1-n)\, \rho^{n-1}\, \widetilde{A}(\theta) \Big]^{\frac{n-m}{n-1}} d\theta \;=\; 0.$$

Notice that for $0 \leq |\rho| < \delta$ small enough,

$$\Big[1 + (1-n)\, \rho^{n-1}\, \widetilde{A}(\theta) \Big]^{\frac{n-m}{n-1}} = \sum_{k \geq 0} b_k\, \rho^{(n-1)k}\, \widetilde{A}^k(\theta),$$

where $b_k = \binom{\frac{n-m}{n-1}}{k} (1-n)^k \neq 0$ for all $k \geq 0$. Hence

$$\sum_{k \geq 0} b_k\, \rho^{(n-1)k} \int_0^{2\pi} B(\theta)\, \widetilde{A}^k(\theta)\, d\theta = 0$$

for all ρ small enough. From this fact, equality (14) follows for all $k \geq 0$.

To prove (13) we can proceed similarly starting from (12) and the following equality:

$$\int_0^{2\pi} \frac{\frac{\partial r(\theta, \epsilon, \rho)}{\partial \theta}}{(r(\theta, \epsilon, \rho))^m}\, d\theta \;=\; \epsilon \int_0^{2\pi} A(\theta)\, (r(\theta, \epsilon, \rho))^{n-m}\, d\theta + \int_0^{2\pi} B(\theta)\, d\theta.$$

□

Remark 9. *The above result also holds for polynomial Abel equations.*

Note that from the above Theorem we know that if an equation satisfies the composition condition then all the moments vanish. From this result it is natural to formulate the following two questions.

Question 10. *Assume that the functions A and B are trigonometric polynomials such that all the moments of A with respect to B and the ones of B with respect to A vanish, see* (13) *and* (14). *Consider its associated Abel equation (6). Then:*

- *Does it have a center? If yes:*
- *Is the center persistent?*
- *Is the center a CC-center?*

In Section 3 we give examples which answer negatively all the above questions.

It is worth to comment that the question of whether one of the moments condition implies or not the composition conjecture, for the case of polynomials (not trigonometric polynomials) has attracted during the last years a wide interest. This question has been known as the *Composition Conjecture*. Finally it was proved to be false and in [16] the author gave a pair of polynomials such that the moments of one of them respect to other vanish but they do not satisfy the composition condition. For the sake of completeness we reproduce here this example:

Example 11. ([16]) *Take $A(t) = T_2'(t) + T_3'(t)$ and $B(t) = T_6'(t)$ where T_i denotes the i-th Chebyshev polynomial and T_i' its derivative. Thus $T_2(t) = 2t^2 - 1$ and $T_3(t) = 4t^3 - 3t$. It is well known that $T_6(t) = (T_3 \circ T_2)(t) = (T_2 \circ T_3)(t) = 32t^6 - 48t^4 + 18t^2 - 1$. Moreover*

$$T_2(\sqrt{3}/2) - T_2(-\sqrt{3}/2) = T_3(\sqrt{3}/2) - T_3(-\sqrt{3}/2) = 0.$$

Thus we get

$$\begin{aligned}
\int_{-\sqrt{3}/2}^{\sqrt{3}/2} A(t)\, \widetilde{B}^k(t)\, dt &= \int_{-\sqrt{3}/2}^{\sqrt{3}/2} (T_2'(t) + T_3'(t))\, T_6^k(t)\, dt \\
&= \int_{-\sqrt{3}/2}^{\sqrt{3}/2} T_2'(t)(T_3(T_2(t)))^k dt + \int_{-\sqrt{3}/2}^{\sqrt{3}/2} T_3'(t)(T_2(T_3(t)))^k dt \\
&= \int_{-\sqrt{3}/2}^{\sqrt{3}/2} T_2'(t) P(T_2(t))\, dt + \int_{-\sqrt{3}/2}^{\sqrt{3}/2} T_3'(t) Q(T_3(t))\, dt \\
&= \left(\widetilde{P}(T_2(t)) + \widetilde{Q}(T_3(t))\right)\Big|_{-\sqrt{3}/2}^{\sqrt{3}/2} = 0,
\end{aligned}$$

where P and Q are some suitable polynomials.

However A and B do not satisfy the composition condition. An easy way to see this is to show that some moment of B respect to A does not vanish. This is the case because

$$\int_{-\sqrt{3}/2}^{\sqrt{3}/2} B(t)\, \widetilde{A}^2(t)\, dt = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} T_6'(t)\, (T_2(t) + T_3(t))^2\, dt = \frac{864\sqrt{3}}{385}.$$

Since for the above example not all the the moments of B respect to A vanish, for the polynomial case, remains the following question, which seems to us much more natural that the so called composition conjecture:

Question 12. *Given an interval $[a,b]$, let A and B be polynomials such that all the moments of A with respect to B and the ones of B with respect to A vanish. Is it true that A and B satisfy the composition condition?*

The tools developed in [15] will surely be very useful to answer the above question. As we have said, in next section we will see that the answer to this question is "no" in the case of trigonometric polynomials.

All the above results and comments lead us to believe that the composition condition for trigonometric polynomials must be related with a stronger condition than the moments conditions (13) and (14). We introduce now the following definition.

Definition 13. *We say that equation (6) has a* ***strongly persistent center*** *if*

$$\frac{dr}{d\theta} = (\alpha\, A(\theta) + \beta\, B(\theta))r^n + (\gamma\, A(\theta) + \delta\, B(\theta))r^m,$$

has a center for all $\alpha, \beta, \gamma, \delta \in \mathbb{R}$.

Clearly CC-centers are strongly persistent. Moreover strongly persistent centers satisfy that some "generalized moments" have to be equal to zero, as we will see in the next lemma. Brudnyi in ([8, 9, 10]) also introduced iterated integrals and some generalized moments to express the Taylor expansion of the return map of generalized Abel equations.

Lemma 14. *If (6) has a strongly persistent center then the following generalized moments vanish:*

$$\int_0^{2\pi} A(\theta)\, \widetilde{A}^p(\theta)\, \widetilde{B}^q(\theta)\, d\theta = 0 \quad \text{and} \quad \int_0^{2\pi} B(\theta)\, \widetilde{A}^p(\theta)\, \widetilde{B}^q(\theta)\, d\theta = 0 \tag{17}$$

for all $p, q \in \mathbb{N}$.

Proof. From Theorem 8 we know that

$$\int_0^{2\pi} (\alpha\, A(\theta) + \beta\, B(\theta))\, (\gamma\, \widetilde{A}(\theta) + \delta\, \widetilde{B}(\theta))^k\, d\theta = 0\,,\ k \geq 0. \tag{18}$$

Taking $\beta = 0$ and $\alpha = 1$:

$$\int_0^{2\pi} A(\theta)\, (\gamma\, \widetilde{A}(\theta) + \delta\, \widetilde{B}(\theta))^k\, A(\theta)\, d\theta = \sum_{i=0}^{k} a_i \int_0^{2\pi} \gamma^i\, \delta^{k-i} \widetilde{A}^i(\theta)\, \widetilde{B}^{k-i}(\theta)\, A(\theta)\, d\theta = 0,$$

for some $a_i \neq 0$. Notice that for $i = 0$ and $i = k$ the corresponding integrals are equal to zero. Hence:

$$\sum_{i=1}^{k-1} a_i\, \gamma\, \delta \int_0^{2\pi} \gamma^{i-1}\, \delta^{k-i-1}\, A(\theta)\, \widetilde{A}^i(\theta) \widetilde{B}^{k-i}(\theta)\, d\theta = 0,$$

which implies that

$$\sum_{i=1}^{i=k-1} a_i \int_0^{2\pi} \gamma^{i-1}\,\delta^{k-i-1}\,A(\theta)\,\widetilde{A}^i(\theta)\widetilde{B}^{k-i}(\theta)\,d\theta = 0$$

for all $\gamma \neq 0$ and $\delta \neq 0$. Taking the limits when γ and δ tend to zero, we get, respectively

$$\int_0^{2\pi} A(\theta)\,\widetilde{A}(\theta)\,\widetilde{B}^{k-1}(\theta)\,d\theta = 0, \quad \int_0^{2\pi} A(\theta)\,\widetilde{A}^{k-1}(\theta)\,\widetilde{B}(\theta)\,d\theta = 0.$$

Reasoning inductively we see that for all $i \in \mathbb{N}, 0 \leq i \leq k$,

$$\int_0^{2\pi} A(\theta)\,\widetilde{A}^i(\theta)\,\widetilde{B}^{k-i}(\theta)\,d\theta = 0 \quad \text{and} \quad \int_0^{2\pi} A(\theta)\,\widetilde{A}^{k-i}(\theta)\,\widetilde{B}^i(\theta)\,d\theta = 0,$$

and hence that $\int_0^{2\pi} A(\theta)\,\widetilde{A}^p(\theta)\,\widetilde{B}^q(\theta)\,d\theta = 0$ for all $p, q \in \mathbb{N}$.

Starting with $\beta = 1$ and $\alpha = 0$ we obtain the other set of conditions. □

Hence it is natural to introduce the following open questions:

Question 15. *Assume that the functions A and B are trigonometric polynomials such that all the generalized moments given in* (17) *vanish. Consider its associated Abel equation (6). Then:*

- *Does it have a center? If yes:*
- *Is the center strongly persistent or persistent?*
- *Is the center a CC-center?*

We want to stress that the example that we will construct in the next section, which answers negatively the items stated in Question 10, gives no information about Question 15 because as we will see it has at least one non-zero generalized moment.

A smaller (or equal) class that strongly persistent centers is introduced in the following definition:

Definition 16. *We say that equation (6) has a* ***symmetric center*** *if it has a center and*

$$\frac{dr}{d\theta} = B(\theta)r^n + A(\theta)r^m,$$

has also a center.

Clearly CC-centers are symmetric centers. As far as we know, no example is known of symmetric center not being a CC-center.

Finally we introduce the class of *degree-persistent centers*, as follows:

Definition 17. *We say that equation (6) has a* ***degree-persistent center*** *if*

$$\frac{dr}{d\theta} = A(\theta)r^p + B(\theta)r^q,$$

has a center for all p and q in $\mathbb{N}$.

As in the previous case, CC-centers are degree-persistent centers and it would be nice to study the converse implication. Notice that degree-persistent centers form, in principle, a smaller class that the one of the symmetric centers.

3 The example

Systems (1), with P and Q of degree 2, can be written as $\dot{z} = iz + Az^2 + Bz\bar{z} + C\bar{z}^2$, where $z = x + iy$ and A, B, C are complex numbers and are usually called *quadratic systems.* Quadratic systems having a center are classified in four families. The family given by the single condition $B = 0$ is known as the Lotka-Volterra class. When $C = 1/4$, the corresponding Abel equation obtained using the Cherkas transformation is (6) with $A(\theta) = a\cos(2\theta) + b\sin(2\theta) + (1/32)\sin(6\theta)$ and $B(\theta) = \cos(3\theta)$, where a and b are arbitrary real parameters. Motivated for this equation we consider the trigonometric polynomials A and B given in next proposition.

Proposition 18. *Consider the trigonometric polynomials*

$$A(\theta) = a\cos(2\theta) + b\sin(2\theta) + c\sin(6\theta) \quad \text{and} \quad B(\theta) = \cos(3\theta).$$

Then:

(i) For all $k \geq 0$,

$$I_k := \int_0^{2\pi} A(\theta)\widetilde{B}^k(\theta)\, d\theta = 0, \qquad J_k := \int_0^{2\pi} B(\theta)\widetilde{A}^k(\theta)\, d\theta = 0$$

and

$$\int_0^{2\pi} B(\theta)\, \widetilde{A}^3(\theta)\, \widetilde{B}(\theta)\, d\theta = \frac{a(3b^2 - a^2)}{192}\pi. \tag{19}$$

(ii) For $a = 0$ or $a^2 = 3b^2$, the equation $dr/d\theta = A(\theta)r^n + B(\theta)r^m$, with $n > m$ has a CC-center.

(iii) For all a, b with $a \neq 0$, $a^2 \neq 3b^2$ and $c = 1/32$, the equation $dr/d\theta = A(\theta)r^3 + B(\theta)r^2$ has a non-persistent center.

(iv) For all a, b with $a \neq 0$, $a^2 \neq 3b^2$ and $c \neq 1/32$, the equation $dr/d\theta = A(\theta)r^3 + B(\theta)r^2$ has not a center.

(v) For all a, b with $a \neq 0$, $a^2 \neq 3b^2$ and $c = 1/32$, the equation $dr/d\theta = B(\theta)r^3 + A(\theta)r^2$ has not a center.

Proof. (i) To see that for all a, b, c it holds that $I_k = 0$ we will prove that each one of the three following integrals

$$\int_0^{2\pi} \cos(2\,\theta)\sin^k(3\,\theta)\,d\theta\,, \int_0^{2\pi} \sin(2\,\theta)\sin^k(3\,\theta)\,d\theta \quad \text{and} \quad \int_0^{2\pi} \sin(6\,\theta)\sin^k(3\,\theta)\,d\theta$$

is zero. Calling M_k the first one of them and applying the integration by parts method it can be seen that M_k satisfies a linear recurrence of order two, concretely $M_k = \frac{9k(k-1)}{9k^2-4}\,M_{k-2}$. Since $M_0 = 0, M_1 = 0$, we deduce that $M_k = 0$ for all $k \in \mathbb{N}$.

With respect to the second one:

$$\int_0^{2\pi} \sin(2\,\theta)\sin^k(3\,\theta)\,d\theta = 2\int_0^{2\pi} \sin(\theta)\cos(\theta)\left(3\sin(\theta) - 4\sin^3(\theta)\right)^k\,d\theta$$

and hence the integral is $2\int_0^{2\pi}\cos(\theta)\,P(\sin(\theta))\,d\theta$ for a certain real polynomial $P(x) \in \mathbb{R}[x]$. So the result follows.

Similarly,

$$\int_0^{2\pi} \sin(6\,\theta)\sin^k(3\,\theta)\,d\theta \;=\; 2\int_0^{2\pi} \sin^{k+1}(3\,\theta)\cos(3\,\theta)\,d\theta \;=\; 0.$$

To see that $J_k = \int_0^{2\pi} B(\theta)\,\widetilde{A}^k(\theta)\,d\theta = 0$, we will prove that

$$\int_0^{2\pi} \cos(3\,\theta)\;\sin^i(2\,\theta)\;\cos^j(2\,\theta)\;\cos^k(6\,\theta)\,d\theta = 0$$

for all natural numbers i, j, k. This assertion is trivially true when i is an odd number. So, assume that that $i = 2\,\ell$. Using the equalities:

$$\sin^i(2\,\theta) = [\sin^2(2\,\theta)]^\ell = [1 - \cos^2(2\,\theta)]^\ell = \sum_{i=0}^{\ell} a_i\,\cos^{2\,i}(2\,\theta)$$

and

$$\cos^k(6\,\theta) = [\cos^2(3\,\theta) - \sin^2(3\,\theta)]^k = [2\,\cos^2(3\,\theta) - 1]^k = \sum_{j=0}^{k} b_j\,\cos^{2\,j}(3\,\theta)$$

the problem reduces to prove that the integrals of type

$$K := \int_0^{2\pi} \cos^{2s+1}(3\,\theta)\;\cos^r(2\,\theta)\,d\theta$$

are always zero. Observing that

$$\cos^{2s+1}(3\,\theta) = \cos^{2s+1}(\theta)\;\left(1 - 4\,\sin^2(\theta)\right)^{2s+1} \;=\; \cos(\theta)\,(1 - \sin^2(\theta))^s\,(1 - 4\,\sin^2(\theta))^{2s+1}$$

and

$$\cos^r(2\,\theta) \;=\; \big(1-2\,\sin^2(\theta)\big)^r$$

we get that there exists a polynomial $Q(x)\in\mathbb{R}[x]$ such that $K=\int_0^{2\pi}\cos(\theta)\,Q(\sin^2(\theta))\,d\theta$ and hence K is zero. Lastly, equality (19) is obtained by direct computations.

In order to prove assertion (ii) assume that $a=0$. Since

$$\widetilde{B}(\theta)=\frac{1}{3}\sin(3\,\theta)=\frac{1}{3}\sin\theta\,(3-4\sin^2\theta)$$

we see that $\widetilde{B}(\theta)$ is a function of $u(\theta)=\sin\theta$. Using that $\cos(2\,\theta)$ and $\cos(6\,\theta)$ are also polynomials in $\sin\theta$ we deduce that $\widetilde{A}(\theta)$ also depends on $u(\theta)=\sin\theta$. So, the CC is satisfied and equation $dr/d\theta=A(\theta)r^n+B(\theta)r^m$ has a center for all n,m. If $a^2=3\,b^2$ then it can be seen that $\widetilde{A}(\theta)$ and $\widetilde{B}(\theta)$ are polynomials in $\cos(\theta+\pi/6)$ and the CC is also satisfied.

In order to prove (iii) and (iv) we compute, following [3] or [13, Prop. 3.1], the first coefficients V_i of the return map of the Abel equation. We obtain that

$$V_i=0,\ \text{for}\quad i=2,\ldots,10,\quad\text{and}\quad V_{11}=\frac{a(3b^2-a^2)(1-32c)}{4320}\pi.$$

Thus a necessary condition to have a center is that either $a=0$ or $a^2=3b^2$ or $c=1/32$. The fact that these conditions are sufficient comes from (ii) and from the fact that when $c=1/32$, these cases corresponds to a quadratic center in the plane with a center at the origin. So if $a\neq 0$ and $a^2\neq 3b^2$ a center exists if and only if $c=1/32$. This implies that such a center is not persistent. Note also that from (i) these centers do not satisfy that all generalized moments associated to A and B vanish.

(v) Computing the coefficients of the return map, as in (iii-iv), we get that $V_2=V_3=\cdots=V_9=0$ and

$$V_{10}=\frac{a\left(3b^2-a^2\right)\left(7285+1292032b+8999424a^2+902845440b^2\right)}{495452160}\pi\neq 0,$$

so the equation has not a center, as we wanted to prove. □

Remark 19. *(i) Observe that the family presented in the above proposition answers negatively the questions stated in Question 10. Notice also that from this proposition conditions* $I_k=0\,,\ J_k=0\,,\ k\geq 0$ *does not imply the existence of center.*

(ii) Note that item (v) of the above proposition shows that the Abel equation considered has neither a degree-persistent center nor a symmetric center. Moreover, computing also several Lyapunov constants, it can also be seen that for all a,b *with* $a\neq 0$, $a^2\neq 3b^2$, *the equation* $dr/d\theta=A(\theta)r^4+B(\theta)r^2$ *has neither a center.*

In [15] the following characterization of the moments condition in the polynomial case was proved.

Theorem 20. ([15]) *Set $A(t), B(t) \in \mathbb{R}[t]$ and $a < b \in \mathbb{R}$. Then*

$$\int_a^b A(t)\, \widetilde{B}^k(t)\, dt = 0$$

for all $k \geq 0$ if and only if there exists $w_1(t), \ldots, w_m(t) \in \mathbb{R}[t]$ with $w_i(a) = w_i(b)$ such that

$$\widetilde{B}(t) = B_1(w_1(t)) = \ldots = B_k(w_m(t)) \quad \text{and } \widetilde{A}(t) = \sum_{i=1}^{m} A_i(w_i(t))$$

where $B_i(t), A_i(t) \in \mathbb{R}[t]$.

Notice that the above result gives the reason for which a couple of polynomials A and B satisfying the moment condition does not necessarily satisfy the CC. The point is the existence of at least two essentially different functions w_1 and w_2 in the above decompositions. This is the case for the example obtained in [16], recalled in Example 11.

The following result shows that the natural translation of the above result to the trigonometric case does not hold.

Proposition 21. *Set $A(\theta) = 3\cos(3\,\theta)$ and $B(\theta) = 2\cos(2\,\theta) + 6\sin(6\,\theta)$. Then $A(\theta)$ and $B(\theta)$ satisfy the moments condition* (13). *However if $\widetilde{B}(\theta) = B_1(w(\theta))$ with $B_1 \in \mathbb{R}[x]$ and $w(\theta)$ trigonometric polynomial then B_1 must be linear and $w(\theta) = k\widetilde{B}(\theta)$ for some non-zero constant k. Moreover there is no $A_1 \in \mathbb{R}[x]$ such that $\widetilde{A}(\theta) = A_1(\widetilde{B}(\theta))$.*

Proof. We know from Proposition 18 that

$$\int_0^{2\pi} A(\theta)\, \widetilde{B}^k(\theta) = 0$$

for all $k \geq 0$.

Now we look for the decompositions of $\widetilde{B}(\theta) = B_1(w(\theta))$. There are only three possibilities: either B_1 has degree 6 and w is linear or B_1 has degree 3 and w is quadratic or B_1 has degree 2 and and w is cubic. We examine with detail the case when B_1 has degree 6 and w is linear. In this case we would get:

$$\widetilde{B}(\theta) = \sin(2\,\theta) - \cos(6\,\theta) = \sum_{i=0}^{6} p_i(s\cos(\theta) + r\sin(\theta))^i$$

with $p_6 = \pm 1$. Assume for instance that $p_6 = -1$. Doing the Fourier series of the right term of this equality, looking at its coefficients of $\cos(6\,\theta)$ and $\sin(6\,\theta)$ and equaling them to the corresponding ones of $\widetilde{B}(\theta)$ we obtain:

$$(r^2 - s^2)((s^2 + r^2)^2 - 16r^2 s^2) = 0 \quad \text{and} \quad rs(-3s^2 + r^2)(-s^2 + 3r^2) = 16,$$

which has a finite number of solutions all of them satisfying that $rs \neq 0$.

Looking now to the coefficients of $\cos(5\,\theta)$ and $\sin(5\,\theta)$ we obtain that

$$p_5 s(-10r^2s^2 + s^4 + 5r^4) = 0 \quad \text{and} \quad p_5 r(5s^4 + r^4 - 10r^2s^2) = 0.$$

Since $rs \neq 0$ we have that $p_5 = 0$.

Computing the coefficients of $\cos(4\,\theta)$ and $\sin(4\,\theta)$ we get

$$(-s^2 - 2rs + r^2)(-s^2 + 2rs + r^2)(-3r^2 - 3s^2 + 2p_4) = 0,$$
$$rs(-s + r)(s + r)(-3r^2 - 3s^2 + 2p_4) = 0.$$

Using again that $rs \neq 0$ we will have $p_4 = 3(r^2 + s^2)/2$. Now looking at the coefficients of $\cos(3\,\theta)$ and $\sin(3\,\theta)$ we obtain $p_3 s(-s^2 + 3r^2) = 0$ and $p_3 r(-3s^2 + r^2) = 0$. These equations force $p_3 = 0$.

Lastly looking at the coefficients of $\cos(2\,\theta)$ and $\sin(2\,\theta)$ we have

$$rs(9r^4 + 18r^2s^2 + 9s^4 + 16p_2) = 0 \quad \text{and} \quad (-s + r)(s + r)(9r^4 + 18r^2s^2 + 9s^4 + 16p_2) = -32.$$

This system has no solution with $rs \neq 0$. This ends the proof of the non-existence of a decomposition of this type in this case. The proof in the other cases follows by similar computations.

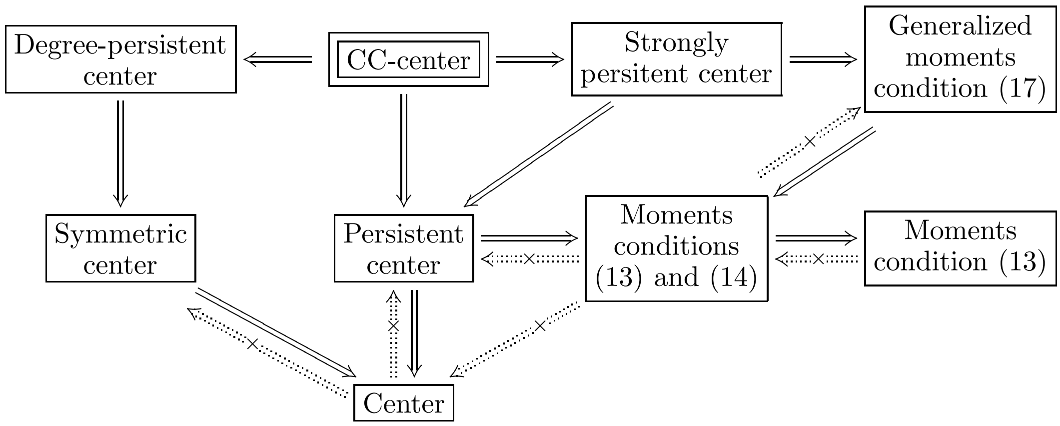
On the other hand it is clear that $\widetilde{A}(\theta) = \sin(3\,\theta)$ is not a polynomial function of $\widetilde{B}(\theta) = \sin(2\,\theta) - \cos(6\,\theta)$.

□

4 Conclusions

We have seen that the study of the CC-centers for trigonometric Abel equations and other problems surrounding them, like the characterization of the persistent centers or the relation with the moments condition, is quite different to the same questions for polynomial Abel equations. We have also introduced in Definition 13 the notion of strongly persistent centers and related with them the generalized moments condition, see (17) in Lemma 14. These generalized moments seem a natural candidate to characterize persistent and CC-centers. Other possible characterization of CC-centers are given by the classes of symmetric centers or the one degree-persistent centers.

Next diagram shows the known relations among the concepts appearing in this paper. A crossed dotted implication means that the implication does not hold. We believe that the implications not given in the diagram suggest interesting problems to be studied. Perhaps the more important ones are: Are all the persistents centers, CC-centers? Do the generalized moments conditions imply the existence of strongly persistent center?



Finally notice that many of the above concepts, problems and results can be extended to general equations of the form

$$\dot{r} = \sum_{k\geq 2} A_k(\theta) r^k,$$

having either a finite or an infinite sum, see [8, 9, 10].

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