COMPLETE ABELIAN INTEGRALS FOR POLYNOMIALS WHOSE GENERIC FIBER IS BIHOLOMORPHIC TO $\mathbb{C}^*$

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Abstract. Let H be a polynomial of degree $m+1$ on $\mathbb{C}^2$ such that its generic fiber is biholomorphic to $\mathbb{C}^*$, and let ω be an arbitrary polynomial 1-form of degree n on $\mathbb{C}^2$. We give an upper bound depending only on m and n for the number of isolated zeros of the complete Abelian integral defined by H and ω.

1. Introduction and statement of the results

Let $H : \mathbb{C}^2 \to \mathbb{C}$ be a polynomial whose generic fiber is irreducible, and let ω be a polynomial 1-form on $\mathbb{C}^2$. By the *complete Abelian integral* defined by H and ω, we mean the function

$$I(c) = \int_{[\gamma_c]} \omega,$$

where the parameter c varies over the set of generic values of H, and $[\gamma_c]$ is a *cycle* of H: $[\gamma_c]$ is the homology class of a loop $\gamma_c \subset H^{-1}(c)$, and $[\gamma_c]$ is non-trivial in the first homology group $H_1(H^{-1}(c), \mathbb{Z})$ of the generic fiber $H^{-1}(c)$ of H.

From the classical Poincaré–Pontryagin–Andronov criterion we know that the isolated zeros of $I(c)$ are related to the limit cycles of the infinitesimal *perturbed Hamiltonian system*

$$dH - \varepsilon\omega = 0 \qquad \text{with } 0 \neq \varepsilon \in (\mathbb{C}, 0) \text{ fixed},$$

that arise from the cycles of the *Hamiltonian system* $dH = 0$, which are precisely the cycles of H. In this sense, the problem of finding the upper bound $Z(m, n) \in \mathbb{N}$, depending on $m = \deg(H) - 1$ and $n = \deg(\omega)$ for the number of isolated zeros of $I(c)$, counting multiplicities, is referred to as the *weak infinitesimal Hilbert's 16th problem* (see [1]). Of course, in this problem we must consider all polynomials H of degree $m + 1$ and all the 1-forms ω of degree n.

Khovanskiĭ [10] and Varchenko [16] proved that $Z(m, n)$ is finite. Petrov and Khovanskiĭ claimed that $Z(m, n) \leq A(m)n + B(H)$, where $A(m)$ is an explicit constant depending only on m while $B(H)$ is independent of ω but depends on H. The proof of this assertion was given by Żołądek [17, Theorem 6.26]. Recently Binyamini, Novikov and Yakovenko [4] proved that $Z(n, n) \leq 2^{2^{\mathrm{Po}(n)}}$, where $\mathrm{Po}(n) = O(n^p)$ stands for an explicit polynomially growing term with the exponent p not exceeding 61.

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A difficulty in finding an explicit upper bound for $Z(m,n)$ is that even though $I(c)$ is a locally single-valued function, globally it can be multi-valued since its analytic continuation depends on the monodromy of the polynomial H (see Section 2).

If $\dim H_1(H^{-1}(c),\mathbb{Z}) = 1$ for a generic value c of H, then the generic fiber of H is irreducible and biholomorphic to $\mathbb{C}^*$; therefore, H is called a primitive polynomial of type $\mathbb{C}^*$. This is the simplest non-trivial case for studying $I(c)$ because there is a unique cycle $[\gamma_c]$ to consider, and H has trivial global monodromy (see Section 2, Remark 6). Suppose then that H is primitive of type $\mathbb{C}^*$ allows the (global) study of the complete Abelian integral $I(c)$.

In this paper we study the weak infinitesimal Hilbert's 16th problem for primitive polynomials of type $\mathbb{C}^*$. The main result of this work is the following.

Theorem 1. *Let $H : \mathbb{C}^2 \to \mathbb{C}$ be a primitive polynomial of type $\mathbb{C}^*$ of degree $m+1$, and let ω be a polynomial 1-form of degree n on $\mathbb{C}^2$.*

(a) *The complete Abelian integral $I(c)$, defined by H and ω, is a polynomial.*

(b) *$I(c)$ has at most $\left[\frac{(n+1)m}{2}\right]$ isolated zeros, where $[\cdot]$ denotes the integer part.*

Remark 1. Statement (a) of Theorem 1 provides an interesting property of $I(c)$, because a priori we only expect that $I(c)$ would be a rational function. On the other hand, at the moment we do not know if the upper bound given in statement (b) of Theorem 1 is optimal (see Remark 7 in Section 3).

We will recall a concept which will allow us to simplify the study of $I(c)$. Suppose that the polynomials H and $\widetilde{H}$ are *algebraically equivalent*, that is, there are polynomial automorphisms ψ and σ of $\mathbb{C}^2$ and $\mathbb{C}$ respectively, such that the diagram

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{\psi} & \mathbb{C}^2 \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle \widetilde{H}} \\ \mathbb{C} & \xrightarrow{\sigma} & \mathbb{C} \end{array} \tag{1}$$

commutes. The investigation of $I(c)$ is then equivalent to the study of the complete Abelian integral $\widetilde{I}(\tilde{c})$ defined by $\widetilde{H} := \sigma \circ H \circ \psi^{-1}$ and $\widetilde{\omega} := (\psi^{-1})^*(\omega)$. We will say that $\widetilde{\omega}$ is the polynomial 1-form defined by ω and the commutative diagram (1).

We want to apply the previous argument to the study of the complete Abelian integrals for polynomials of type $\mathbb{C}^*$. Therefore, we consider the algebraic classification of primitive polynomials of type $\mathbb{C}^*$; such classification was given by Miyanishi and Sugie (see Subsection 4.1). They proved in [11] that any primitive polynomial H of type $\mathbb{C}^*$ is algebraically equivalent to a polynomial $\widetilde{H}$ of the family

$$\mathcal{S} := \left\{ x^k \left(x^l y - P(x)\right)^r \;\middle|\; \begin{array}{l} k,r \in \mathbb{N},\;\; (k,r)=1,\;\; l \in \mathbb{N}\cup\{0\}, \text{and} \deg(P(x)) < l. \\ \text{If } l>0, \text{ then } P(0)\neq 0, \text{ and if } l=0, \text{ then } P(x)\equiv 0 \end{array} \right\}.$$

In short, the examination of the complete Abelian integrals for primitive polynomials of type $\mathbb{C}^*$ essentially reduces to studying the complete Abelian integrals for the family $\mathcal{S}$ and to finding the relation between the degrees of the initial objects H and ω and the degrees of the transformed objects $\widetilde{H} \in \mathcal{S}$ and $\widetilde{\omega}$.

We note that the family $\mathcal{S}$ has an infinite number of connected components: the coefficients of $P(x)$ are continuous parameters, and the parameters k, r, and l vary over infinite discrete sets. A priori, the study of the complete Abelian integral defined by a polynomial in the family $\mathcal{S}$ and a polynomial 1-form, may depend on each connected component of $\mathcal{S}$. However, the following result shows that its behavior only depends on the degree of the polynomial and does not depend on the connected component that contains it.

Theorem 2. *Let H be a polynomial of degree $m+1$ in the family $\mathcal{S}$, and let ω be a polynomial 1-form of degree n on $\mathbb{C}^2$.*

(a) *The complete Abelian integral $I(c)$, defined by H and ω, is a polynomial.*

(b) *$I(c)$ has at most $\hat{Z}(m,n) := \left[\frac{n+1}{m+1}\right]$ isolated zeros.*

(c) *$\hat{Z}(m,n)$ is the optimal upper bound for the number of zeros of $I(c)$.*

The final ingredient in proving Theorem 1 is to determine the relationships between the degrees of H and $\widetilde{H}$ and the degrees of ω and $\widetilde{\omega}$. These relations will be studied in Proposition 3 (Section 3) where we will prove that $2 \leq \deg(\widetilde{H}) \leq \deg(H)$ and that $\deg(\widetilde{\omega}) \leq (\deg(\omega)+1)(\deg(H)-1) - 1$.

Remark 2. Theorem 1 is a generalization of the classical result: If $H = (x^2+y^2)/2$ (which is of type $\mathbb{C}^*$ whose representative in $\mathcal{S}$ is $\widetilde{H} = xy$), and ω is an arbitrary polynomial 1-form of degree n, then the complete Abelian integral defined by H and ω has at most $\left[\frac{n+1}{2}\right]$ isolated zeros (for a proof in the real case, see [8], [17]).

The paper is organized as follows. In Section 2 we recall the construction of complete Abelian integrals for primitive polynomials on $\mathbb{C}^2$. The proof of Theorem 1 will be given in Section 3. In Section 4 we recall the algebraic classification of primitive polynomials of type $\mathbb{C}^*$, and we will give the proof of Theorem 2.

2. Complete Abelian integrals

It is well-known that for each polynomial $H : \mathbb{C}^2 \to \mathbb{C}$ there is a finite set $\Sigma_H \subset \mathbb{C}$ such that

$$H : \mathbb{C}^2 - H^{-1}(\Sigma_H) \to \mathbb{C} - \Sigma_H \tag{2}$$

is a locally trivial smooth fibration (see [5] for a proof). The set Σ_H is the set of *singular values of* H and is composed of the values in $\mathbb{C}$ coming from singular points in $\mathbb{C}^2$ and "singular points at infinity" of H (see [7] for a description of these points). Any value $c \in \mathbb{C} - \Sigma_H$ is called a *generic value* of H and

$$\mathcal{L}_c := \left\{(x,y) \in \mathbb{C}^2 \,|\, H(x,y) - c = 0\right\} \subset \mathbb{C}^2$$

is called a *generic fiber* of H, which is an affine non-singular algebraic curve.

A polynomial H is called *primitive* if its generic fiber is irreducible. Thus, if a fiber $\mathcal{L}_{c_0}$ is reducible then $c_0 \in \Sigma_H$. Moreover H is called *primitive of type* (g,h) if its generic fiber is isomorphic to a compact Riemann surface of genus $g \geq 0$ punctured at $h \geq 1$ different points. H is *rational* if it is of type $(0,h)$; moreover H is of type $\mathbb{C}$ if $h=1$, and H is of type $\mathbb{C}^* := \mathbb{C} - \{0\}$ if $h=2$.

We recall that if H is a primitive polynomial of type (g,h), then the first homology group $H_1(\mathcal{L}_c, \mathbb{Z})$ of every generic fiber $\mathcal{L}_c$ of H is a free Abelian group finitely generated of dimension $2g+h-1$.

Let H be a primitive polynomial of type (g,h), and we consider the following:

- A generic value c_0 of H.
- A basis $\left\{[\gamma_{c_0}^\tau] \,|\, \tau = 1,2,...,2g+h-1\right\}$ of $H_1(\mathcal{L}_{c_0}, \mathbb{Z})$.
- A complex disc $\Delta(c_0, r)$ centered at c_0 of radius r such that $\Delta(c_0,r) \subset \mathbb{C} - \Sigma_H$.

- The transport γ_c^τ, induced by the fibration (2), of $\gamma_{c_0}^\tau$ into $\mathcal{L}_c$ with $c \in \Delta(c_0, r)$.
- A polynomial 1-form $\omega = Adx + Bdy$ on $\mathbb{C}^2$.

With these objects we construct, for each $\tau = 1, 2, \ldots, 2g + h - 1$, the Abelian integral

$$\begin{array}{rccl} I_\tau(c): & \Delta(c_0, r) & \to & \mathbb{C} \\ & c & \mapsto & \displaystyle\int_{[\gamma_c^\tau]} \omega, \end{array}$$

where $[\gamma_c^\tau]$ is the homology class of γ_c^τ.

Every Abelian integral $I_\tau(c)$ is well-defined and holomorphic. Indeed, we take a representative loop for our fixed cycle $[\gamma_{c_0}^\tau]$. We can transport this loop continuously into the neighboring fibers and integrate ω along the resulting loops. This transportation depends on the representative loop, but since the homology classes of the obtained loops are well-defined, the integration of ω on the resulting cycles does not depend on the mode of transportation. Therefore the Abelian integral $I_\tau(c)$ is a well-defined and holomorphic function.

The Abelian integral $I_\tau(c)$ can be analytically continued on $\mathbb{C} - \Sigma_H$ because the cycle $\gamma_{c_0}^\tau$ can be transported continuously into any generic fiber $\mathcal{L}_c$ of H. The resulting function is locally single-valued, but globally it can be multi-valued because the analytic continuation depends on the *monodromy of polynomial* H, this is, on the action of the fundamental group $\pi_1(\mathbb{C} - \Sigma_H, c_0)$ of $\mathbb{C} - \Sigma_H$ based at c_0 in $H_1(\mathcal{L}_{c_0}, \mathbb{Z})$. If this action is trivial it states that H has *trivial global monodromy* and $I_\tau(c)$ extends to a single-valued function on $\mathbb{C} - \Sigma_H$. In addition, if we consider all possible analytic continuations of all $I_\tau(c)$, $\tau = 1, 2, \ldots, 2g + h - 1$, we obtain the complete Abelian integral $I(c)$.

We note that the monodromy of H depends on the complexity of $\pi_1(\mathbb{C} - \Sigma_H, c_0)$, and this complexity increases with respect to the cardinality of Σ_H.

Remark 3. In [14, Corollary 1] Suzuki proved that if H is a primitive polynomial of type (g, h), then the cardinality of Σ_H is at most $2g + h - 1 = \dim H_1(\mathcal{L}_{c_0}, \mathbb{Z})$.

Remark 4. A primitive polynomial H is of type $\mathbb{C}$ if and only if $\Sigma_H = \emptyset$. If H is of type $\mathbb{C}$, then $I(c) \equiv 0$.

Remark 5. If H is a primitive polynomial of type $\mathbb{C}^*$, then Σ_H has exactly one point. This assertion is true because from Remark 3 the set Σ_H has at most one point and from Remark 4 the set Σ_H has at least one point.

Remark 6. If H is a primitive polynomial of type $\mathbb{C}^*$, then it has trivial global monodromy.

Proof. According to the construction of the family $\mathcal{S}$ we have the commutative diagram (1) with $\widetilde{H} \in \mathcal{S}$. From Remark 3, $\Sigma_{\widetilde{H}}$ has exactly one point, and since the fiber $\widetilde{\mathcal{L}}_0 = \{(x, y) \in \mathbb{C}^2 \mid \widetilde{H} = 0\}$ is reducible, $\Sigma_{\widetilde{H}} = \{0\}$. Hence for $c \neq 0$ the fiber $\widetilde{\mathcal{L}}_c = \{(x, y) \in \mathbb{C}^2 \mid \widetilde{H} - c = 0\}$ is irreducible, and it is easy to see that

$$(3) \qquad \begin{array}{rccl} \varphi_c: & \mathbb{C}^* & \to & \widetilde{\mathcal{L}}_c \\ & z & \mapsto & \left(z^r c^{s_2}, \dfrac{c^{s_1} + z^k P(z^r c^{s_2})}{c^{l s_2} z^{rl+k}} \right) \end{array}$$

is a parametrization, where s_1 and s_2 are integers such that $rs_1 + ks_2 = 1$ (recall that $(k, r) = 1$). In fact the map

$$\begin{array}{rcl}\Phi : (\mathbb{C} - \Sigma_{\widetilde{H}}) \times \mathbb{C}^* & \to & \mathbb{C}^2 - \widetilde{H}^{-1}(\Sigma_{\widetilde{H}}) \\ (c, z) & \mapsto & \Phi(c, z) = \varphi_c(z)\end{array}$$

is a biholomorphism such that $p_1 = \widetilde{H} \circ \Phi$, where $p_1 : (\mathbb{C} - \Sigma_{\widetilde{H}}) \times \mathbb{C}^* \to \mathbb{C} - \Sigma_{\widetilde{H}}$ is the projection on the first factor. Thus $\widetilde{H} : \mathbb{C}^2 - \widetilde{H}^{-1}(\Sigma_{\widetilde{H}}) \to \mathbb{C} - \Sigma_{\widetilde{H}}$ is a globally trivial smooth fibration, so the action of $\pi_1(\mathbb{C} - \Sigma_{\widetilde{H}}, c_0)$ on $H_1(\widetilde{\mathcal{L}}_c, \mathbb{Z})$ is trivial. Therefore $\widetilde{H}$ has trivial global monodromy.

Analogously, since the map $\psi^{-1} \circ \Phi : (\mathbb{C} - \Sigma_{\widetilde{H}}) \times \mathbb{C}^* \to \mathbb{C}^2 - H^{-1}(\Sigma_H)$ is a biholomorphism such that $p_1 = \sigma \circ H \circ \psi^{-1} \circ \Phi$, H has trivial global monodromy. □

An alternative proof of Remark 6 can be deduced from [2, Corollary 2] and [11, Section 1.8].

In short, the simplest non-trivial case for the study of complete Abelian integrals is when H is of type $\mathbb{C}^*$. Indeed, in such a case $H_1(\mathcal{L}_{c_0}, \mathbb{Z})$ and $\pi_1(\mathbb{C} - \Sigma_H, c_0)$ are the simplest non-trivial; moreover the monodromy of H is trivial. Therefore we have a unique Abelian integral $I(c) := I_1(c)$, which extends to a single-valued function on $\mathbb{C} - \Sigma_H$. In addition, Theorem 1 claims that $I(c)$ is a polynomial, so $I(c)$ extends to the whole $\mathbb{C}$.

3. Proof of Theorem 1

To prove Theorem 1 we will use the following technical result which will be proved later on.

Proposition 3. *Let $H : \mathbb{C}^2 \to \mathbb{C}$ be a primitive polynomial of type $\mathbb{C}^*$ of degree $m + 1$, and let ω be a polynomial 1-form of degree n on $\mathbb{C}^2$. Let $\widetilde{H}$ and $\widetilde{\omega}$ be the representative of H in $\mathcal{S}$ and the 1-form defined by ω and* (1)*, respectively.*

(*a*) *The degree of $\widetilde{H}$ is at least* 2 *and at most $m + 1$.*
(*b*) *The degree of $\widetilde{\omega}$ is at most $(n + 1)m - 1$.*

We will give the proof of Theorem 1 by assuming Theorem 2 and Proposition 3.

Proof of Theorem 1. Proof of statement (a). From the Miyanishi–Sugie classification we have the commutative diagram (1), with $\widetilde{H} \in \mathcal{S}$, and we get the polynomial 1-form $\widetilde{\omega}$ defined by ω and (1). By statement (a) of Theorem 2, the Abelian integral $\widetilde{I}(\tilde{c})$ defined by $\widetilde{H}$ and $\widetilde{\omega}$ is a polynomial, and since $I(c) = \widetilde{I}(\sigma(c))$ and σ is linear, $I(c)$ is a polynomial.

Proof of statement (b). If $\widetilde{m} + 1$ is the degree of $\widetilde{H}$, then from statement (a) of Proposition 3 it follows that $2 \leq \widetilde{m}+1 \leq m+1$. By statement (b) of Proposition 3, the degree $\widetilde{n}$ of $\widetilde{\omega}$ satisfies $1 \leq \widetilde{n} \leq (n + 1)m - 1$. Therefore, by statement (b) of Theorem 2, the Abelian integral $\widetilde{I}(\tilde{c})$ has at most $[(\widetilde{n} + 1) / (\widetilde{m} + 1)]$ zeros. As $\widetilde{m}+1 \geq 2$ and $\widetilde{n} \leq (n+1)m-1$, then $\widetilde{I}(\tilde{c})$ has at most $[(n + 1)m/2]$ zeros. Finally, since $I(c) = \widetilde{I}(\sigma(c))$ and σ is linear, $I(c)$ has at most $[(n + 1)m/2]$ zeros in $\mathbb{C}$, counting multiplicities. □

Remark 7. For proving that the upper bound $[(n + 1)m/2]$, given in the above proof, is optimal, we must demonstrate the existence of a polynomial H of degree $m+1$ and a polynomial 1-form ω of degree n such that H is algebraically equivalent to $\widetilde{H} \in \mathcal{S}$ of degree 2, the 1-form $\widetilde{\omega}$ defined by ω and (1) is of degree $(n + 1)m - 1$, and $\widetilde{I}(\tilde{c})$ has exactly $[(n+1)m/2]$ zeros in $\mathbb{C}$, counting multiplicities. However, we do not have the proof of this fact.

Proof of Proposition 3. Proof of statement (a). We have $\widetilde{H} = \sigma \circ H \circ \psi^{-1}$ or equivalently $H = \sigma^{-1} \circ \widetilde{H} \circ \psi$. Let $\widetilde{m}+1 = k + r(l+1)$ be the degree of $\widetilde{H}$. Let ψ_1 and ψ_2 be the two polynomial components of ψ of degrees n_1 and n_2, respectively. For $i \in \{1,2\}$ we write $\psi_i = \bar{\psi}_i + \psi_{in_i}$, where $\bar{\psi}_i = \psi_{i0} + \cdots + \psi_{i(n_i-1)}$ and ψ_{ij} is the homogeneous part of degree j of ψ_i, with $j = 0, 1, \ldots, n_i$. Thus

$$\begin{aligned}
\widetilde{H} \circ \psi &= \left(\bar{\psi}_1 + \psi_{1n_1}\right)^k \left(\left(\bar{\psi}_1 + \psi_{1n_1}\right)^l \left(\bar{\psi}_2 + \psi_{2n_2}\right) - P\left(\bar{\psi}_1 + \psi_{1n_1}\right)\right)^r \\
&= \sum_{\mu=0}^{k} \binom{k}{\mu} \left(\bar{\psi}_1\right)^{k-\mu} \psi_{1n_1}^{\mu} \left(\left(\bar{\psi}_1 + \psi_{1n_1}\right)^l \left(\bar{\psi}_2 + \psi_{2n_2}\right) - P\left(\bar{\psi}_1 + \psi_{1n_1}\right)\right)^r \\
&= A_1 + \psi_{1n_1}^k \left(\sum_{\nu=0}^{l} \binom{l}{\nu} \left(\bar{\psi}_1\right)^{l-\nu} \psi_{1n_1}^{\nu} \left(\bar{\psi}_2 + \psi_{2n_2}\right) - P\left(\bar{\psi}_1 + \psi_{1n_1}\right)\right)^r \\
&= A_1 + \psi_{1n_1}^k \left(A_2 + \psi_{1n_1}^l \psi_{2n_2}\right)^r,
\end{aligned}$$

where

$$A_1 := \sum_{\mu=0}^{k-1} \binom{k}{\mu} \left(\bar{\psi}_1\right)^{k-\mu} \psi_{1n_1}^{\mu} \left(\left(\bar{\psi}_1 + \psi_{1n_1}\right)^l \left(\bar{\psi}_2 + \psi_{2n_2}\right) - P\left(\bar{\psi}_1 + \psi_{1n_1}\right)\right)^r$$

and

$$A_2 := \sum_{\nu=0}^{l-1} \binom{l}{\nu} \left(\bar{\psi}_1\right)^{l-\nu} \psi_{1n_1}^{\nu} \left(\bar{\psi}_2 + \psi_{2n_2}\right) + \psi_{1n_1}^l \bar{\psi}_2 - P\left(\bar{\psi}_1 + \psi_{1n_1}\right).$$

In addition,

$$\begin{aligned}
A_1 + \psi_{1n_1}^k \left(A_2 + \psi_{1n_1}^l \psi_{2n_2}\right)^r &= A_1 + \psi_{1n_1}^k \left(\sum_{\tau=0}^{r} \binom{r}{\tau} \left(A_2\right)^{r-\tau} \left(\psi_{1n_1}^l \psi_{2n_2}\right)^\tau\right) \\
&= A_1 + A_3 + \psi_{1n_1}^k \left(\psi_{1n_1}^l \psi_{2n_2}\right)^r,
\end{aligned}$$

where

$$A_3 := \psi_{1n_1}^k \left(\sum_{\tau=0}^{r-1} \binom{r}{\tau} \left(A_2\right)^{r-\tau} \left(\psi_{1n_1}^l \psi_{2n_2}\right)^\tau\right).$$

Hence

$$\widetilde{H} \circ \psi = A_1 + A_3 + \psi_{1n_1}^k \left(\psi_{1n_1}^l \psi_{2n_2}\right)^r.$$

It is easy to see that $\deg(A_1) \leq n_1(k+rl) + rn_2 - 1$, $\deg(A_2) \leq ln_1 + n_2 - 1$, $\deg(A_3) \leq n_1(k+rl) + rn_2 - 1$ and $\deg(\psi_{1n_1}^k \left(\psi_{1n_1}^l \psi_{2n_2}\right)^r) = n_1(rl+k) + n_2 r$. Therefore as σ^{-1} is a linear polynomial we have

$$m + 1 = \deg(H) = \deg(\sigma^{-1} \circ \widetilde{H} \circ \psi) = n_1(rl+k) + n_2 r. \tag{4}$$

Now, as n_1, n_2, r, and k are positive integers and $l \geq 0$, then

$$m + 1 = n_1(rl+k) + n_2 r \geq (rl+k) + r = k + r(l+1) = \widetilde{m} + 1 = \deg \widetilde{H} \geq 2.$$

Proof of statement (b). As $l \geq 0$, $k \geq 1$ and $r \geq 1$ it follows from (4) that

$$n_1 + n_2 \leq n_1(rl+k) + n_2 r = m + 1.$$

In addition, $n_1 + 1 \le n_1 + n_2 \le m + 1$ and $n_2 + 1 \le n_1 + n_2 \le m + 1$, whence $n_1 \le m$ and $n_2 \le m$. Hence, the degree of ψ is at most m. This implies that the degree of the polynomial automorphism ψ^{-1} is at most m [3, 6]. Therefore

$$\deg(\widetilde{\omega}) = \deg((\psi^{-1})^*(\omega)) \le nm + (m-1) = (n+1)m - 1. \qquad \square$$

4. Complete Abelian integrals for the family $\mathcal{S}$.

4.1. **The algebraic classification of primitive polynomials of type $\mathbb{C}^*$.** In [11] Miyanishi and Sugie consider an algebraically closed field $\mathbb{K}$ of characteristic zero. An irreducible polynomial $f \in \mathbb{K}[x, y]$ is *generically rational* if its generic fiber is an irreducible rational curve. They assign to f a nonnegative integer ν, where $\nu + 1$ is the number of places at infinity of the generic fiber of f.

In our context $\mathbb{K} = \mathbb{C}$ and a generically rational irreducible polynomial with $\nu = 1$ is precisely a primitive polynomial of type $\mathbb{C}^*$.

Miyanishi and Sugie gave the algebraic classification of generically rational irreducible polynomials in $\mathbb{K}[x, y]$ with $\nu = 1$ as follows.

Theorem [11, Theorem 2.3]. Let f be a generically rational, irreducible polynomial in $\mathbb{K}[x, y]$ with $\nu = 1$. Then, after a suitable change of coordinates, f is reduced to either one of the following two forms:

- $f \sim x^\alpha y^\beta + 1$, where $\alpha > 0, \beta > 0$ and $(\alpha, \beta) = 1$.
- $f \sim x^\alpha (x^l y + P(x))^\beta + 1$, where $\alpha, \beta, l > 0, (\alpha, \beta) = 1$ and $P(x) \in \mathbb{K}[x]$, with $\deg(P(x)) < l$ and $P(0) \ne 0$.

In this result a suitable change of coordinates means a change of coordinates of $\mathbb{K}[x, y]$ (see [11, Lemma 2.2]) and hence a polynomial automorphism of $\mathbb{K}^2$.

Saito obtained essentially the same result by considering the analytic classification of primitive holomorphic functions in two complex variables of type $\mathbb{C}^*$ [13, p. 332]. As far as we know, the previous theorem can be deduced from the proof of a result of Suzuki ([15, pp. 527-529]), where he considered the analytic classification of primitive meromorphic functions in two complex variables of type $\mathbb{C}^*$.

By using the polynomial automorphism $\sigma = z - 1$ of $\mathbb{C}$ and changing $\alpha = k$ and $\beta = r$ we obtain the algebraic classification of primitive polynomials of type $\mathbb{C}^*$ as the family $\mathcal{S}$ given in the introduction because the case $l = 0$ in the family $\mathcal{S}$ corresponds to the first form in the previous theorem.

4.2. **Proof of Theorem 2.** By following Ilyashenko's ideas [9], we consider the set of polynomial 1-forms

$$\{\omega_{ij} := x^i y^{j-i} dx \,|\, 1 \le j \le n,\, 0 \le i \le j-1\},$$

which is a basis for the quotient vector space of polynomial 1-forms $\omega = Adx + Bdy$ of degree $\le n$ modulo exact polynomial 1-forms dQ of degree $\le n$. Hence each polynomial 1-form ω can be written as $\omega = dQ + \sum_{j=1}^{n} \sum_{i=0}^{j-1} a_{ij}\omega_{ij}$ with $a_{ij} \in \mathbb{C}$. Thus, we need only prove Theorem 2 for the polynomial 1-forms ω_{ij}.

Let $H = x^k \left(x^l y - P(x)\right)^r$ be a polynomial of degree $m + 1 = k + r(l + 1) \ge 2$ in the family $\mathcal{S}$. We split the proof of Theorem 2 into two cases, $l = 0$ and $l > 0$. In each case we develop the following steps:

1. Each Abelian integral $P_{ij}(c) := \int_{[\gamma_c]} \omega_{ij}$, defined by H and ω_{ij}, is a polynomial.
2. We compute the upper bound for the degree of $P_{ij}(c)$, whence we attain the upper bound $\hat{Z}(m, n)$ for the number of zeros of $I(c) = \int_{[\gamma_c]} \omega = \sum_{j=1}^{n} \sum_{i=0}^{j-1} a_{ij} P_{ij}(c)$.

3. We show that the upper bound $\hat{Z}(m,n)$ is optimal; this means that there are a polynomial $H \in \mathcal{S}$ of degree $m+1$ and a polynomial 1-form ω of degree n such that $I(c) = \int_{[\gamma_c]} \omega$ has exactly $\hat{Z}(m,n)$ isolated zeros.

Proof of Theorem 2. **Case $l = 0$.** From the definition of the family $\mathcal{S}$ it follows that $P(x) \equiv 0$; hence $H = x^k y^r$. In this case (3) takes the form

$$\begin{array}{rccl} \varphi_c : & \mathbb{C}^* & \to & \mathcal{L}_c \\ & z & \mapsto & \left(z^r c^{s_2}, \dfrac{c^{s_1}}{z^k} \right). \end{array} \tag{5}$$

Step 1. Let $\alpha := \left\{ e^{2\pi\sqrt{-1}t} \,|\, t \in [0,1] \right\} \subset \mathbb{C}^*$ be the unit circle in the domain of φ_c. Thus $[\gamma_c] := [\varphi_c(\alpha)]$ is the generator cycle of $H_1(\mathcal{L}_c, \mathbb{Z})$. In addition, as the family $\{\gamma_c\}$ is given by $\Phi((\mathbb{C} - \Sigma_H) \times \alpha)$ then $\{\gamma_c\}$ depends continuously on c. Thus

$$\begin{aligned} \int_{[\gamma_c]} \omega_{ij} = \int_\alpha \varphi_c^*(\omega_{ij}) &= \int_\alpha (z^r c^{s_2})^i \frac{(c^{s_1}))^{j-i}}{z^{(j-i)(k)}} (r z^{r-1} c^{s_2}) dz \\ &= r \left(\int_\alpha \frac{1}{z^{-r(i+1)+k(j-i)+1}} dz \right) c^{s_2(i+1)+s_1(j-i)}. \end{aligned}$$

Hence, if $\int_{[\gamma_c]} \omega_{ij} \not\equiv 0$ then $-r(i+1) + k(j-i) = 0$, that is, $k(j-i) = r(i+1)$. Since $(k,r) = 1$, there exists a positive integer q such that

$$kq = i + 1 \tag{6}$$

and

$$rq = j - i. \tag{7}$$

Therefore the power of c is $s_2(i+1) + s_1(j-i) = s_2(kq) + s_1(rq) = q$, whence we obtain that $\int_{[\gamma_c]} \omega_{ij}$ is the polynomial $P_{ij}(c) = r(2\pi\sqrt{-1})c^q$.

Step 2. Next we will compute the upper bound for the degree of the polynomials $P_{ij}(c)$ by finding an upper bound for the positive integer q.

The addition of (6) and (7) gives $q(k+r) = j+1$, whence

$$q \leq \left[\frac{j+1}{k+r} \right] = \left[\frac{j+1}{m+1} \right],$$

because $k + r = \deg(H) = m + 1$. Since $j \leq n$, $P_{ij}(c)$ is a polynomial of degree at most $\left[\frac{n+1}{m+1} \right]$. Therefore, the Abelian integral $I(c) = \int_{[\gamma_c]} \omega = \sum_{j=1}^{n} \sum_{i=0}^{j-1} a_{ij} P_{ij}(c)$ is a polynomial of degree at most $\hat{Z}(m,n) := \left[\frac{n+1}{m+1} \right]$, which is an upper bound for the number of isolated zeros of $I(c)$.

Step 3. Now, we will show that the upper bound $\hat{Z}(m,n)$ is optimal. We consider the polynomial $H = x^m y \in \mathcal{S}$ of degree $m + 1 \geq 2$. The generic fiber $\mathcal{L}_c$ of H is parameterized by

$$\begin{array}{rcl} \varphi_c : \mathbb{C}^* & \to & \mathcal{L}_c \\ z & \mapsto & \left(z, \dfrac{c}{z^m} \right). \end{array} \tag{8}$$

For each positive integer n we define the polynomial 1-form

$$\Omega_n^m := \left(y^n + \hat{Z}(m,n) \left(2x^{m\hat{Z}(m,n)-1} y^{\hat{Z}(m,n)} - x^{m-1} y \right) \right) dx.$$

It is clear that Ω_n^m is of degree n. To study the complete Abelian integral defined by H and Ω_n^m we will consider two possibilities $n = 1$ and $n \geq 2$.

i) If $n = 1$, then $\Omega_1^1 = 2ydx$ and $\Omega_1^m = ydx$ for $m > 1$. Hence, by using the parametrization (8), we obtain

$$\int_{[\gamma_c]} ydx = \int_\alpha \frac{c}{z^m} dz = \begin{cases} (2\pi\sqrt{-1})c & \text{if } m = 1, \\ 0 & \text{if } m > 1. \end{cases}$$

ii) If $n \geq 2$ then, by using the parametrization (8), we get

$$\int_{[\gamma_c]} y^n dx = \int_\alpha \frac{c^n}{z^{nm}} dz = 0 \quad (\text{since } nm \geq 2) \tag{9}$$

and

$$\int_{[\gamma_c]} x^{m\hat{Z}(m,n)-1} y^{\hat{Z}(m,n)} dx = \int_\alpha \frac{c^{\hat{Z}(m,n)}}{z} dz = (2\pi\sqrt{-1})c^{\hat{Z}(m,n)}. \tag{10}$$

Moreover we have

$$\int_{[\gamma_c]} x^{m-1} ydx = \int_\alpha \frac{c}{z} dz = (2\pi\sqrt{-1})c. \tag{11}$$

It follows from (9), (10) and (11) that

$$\int_{[\gamma_c]} \Omega_n^m = (2\pi\sqrt{-1})\hat{Z}(m,n)c\left(2c^{\hat{Z}(m,n)-1} - 1\right).$$

From *i)* and *ii)* we conclude that $\int_{[\gamma_c]} \Omega_n^m$ is a polynomial of degree $\hat{Z}(m,n)$. In addition, the zeros of $\int_{[\gamma_c]} \Omega_n^m$ are all different. Iliev in [8] proved that the upper bound $\hat{Z}(m,n)$ is optimal for the case $m = 1$.

Case $l > 0$. Suppose that $H = x^k\left(x^l y - P(x)\right)^r$, where $P(x) = p_0 + p_1 x + \cdots + p_s x^s$, with $0 \leq s \leq l-1$ and $p_0 \neq 0$. Thus $\deg(H) = m + 1 = k + r(l+1) \geq 2$.

Step 1. Analogously as in the case $l = 0$ we consider the parametrization φ_c of the generic fiber $\mathcal{L}_c$ of H given by (3). Let $\alpha := \left\{e^{2\pi\sqrt{-1}t} \,|\, t \in [0,1]\right\} \subset \mathbb{C}^*$ be the unit circle in the domain of φ_c and the cycle $[\gamma_c] := [\varphi_c(\alpha)]$, which is a generator of $H_1(\mathcal{L}_c, \mathbb{Z})$. Then

$$\int_{[\gamma_c]} \omega_{ij} = \int_\alpha \varphi_c^*(\omega_{ij}) = \int_\alpha (z^r c^{s_2})^i \frac{\left(c^{s_1} + z^k P(z^r c^{s_2})\right)^{j-i}}{c^{ls_2(j-i)} z^{(j-i)(rl+k)}} (rz^{r-1}c^{s_2})dz,$$

and by developing $\left(c^{s_1} + z^k P(z^r c^{s_2})\right)^{j-i}$ we obtain

$$\int_{[\gamma_c]} \omega_{ij} = \sum_{\mu=0}^{j-i} r\binom{j-i}{\mu}\left(\int_\alpha \frac{(P(z^r c^{s_2}))^{j-i-\mu}}{z^{r((j-i)l-i-1)+k\mu+1}} dz\right) c^{-s_2((j-i)l-i-1)+s_1\mu}. \tag{12}$$

As $P(x) = p_0 + p_1 x + \cdots + p_s x^s$, with $0 \leq s \leq l-1$ and $p_0 \neq 0$, then

$$(P(z^r c^{s_2}))^{j-i-\mu} = \sum_{n_0+\cdots+n_s=j-i-\mu} \frac{(j-i-\mu)!}{n_0! \cdots n_s!} p_0^{n_0} \cdots p_s^{n_s} z^{rN_s} c^{s_2 N_s}, \tag{13}$$

where $n_0 \geq 0, \ldots, n_s \geq 0$ and $N_s := n_1 + 2n_2 + \cdots + sn_s$. Hence if in (12) we replace the expression $(P(z^r c^{s_2}))^{j-i-\mu}$ with the right-hand side of (13), then we get

$$\int_{[\gamma_c]} \omega_{ij} = \sum_{\mu=0}^{j-i}\left(\sum_{n_0+\cdots+n_s=j-i-\mu} A^\mu_{n_0\ldots n_s}\left(\int_\alpha z^{r\widetilde{N}_s-k\mu-1}dz\right)c^{s_2\widetilde{N}_s+s_1\mu}\right), \tag{14}$$

where

$$\widetilde{N}_s := N_s - ((j-i)l - i - 1) \quad \text{and} \quad A^{\mu}_{n_0 \ldots n_s} := r\binom{j-i}{\mu}\left(\frac{(j-i-\mu)!}{n_0! \cdots n_s!}\right)p_0^{n_0}\cdots p_s^{n_s}.$$

Above we defined the integer N_s as $N_s = n_1 + 2n_2 + \cdots + sn_s$ and since $n_i \geq 0$ for $i = 0, \ldots, s$, we have the inequality $N_s \leq s(n_0 + \cdots + n_s)$. In addition, with $n_0 + \cdots + n_s = j - i - \mu$ and $s \leq l - 1$ we obtain $N_s \leq (l-1)(j-i-\mu)$. The last inequality implies that $\widetilde{N}_s \leq (l-1)(j-i-\mu) - ((j-i)l - i - 1)$, whence

$$\widetilde{N}_s \leq -j + 2i - (l-1)\mu + 1. \tag{15}$$

This inequality will be useful in step 2 of the proof.

Now, we will demonstrate that $\int_{[\gamma_c]} \omega_{ij}$ is a polynomial. We must assume that the integral $\int_\alpha z^{r\widetilde{N}_s - k\mu - 1}dz$ in (14) is different from zero. Then $r\widetilde{N}_s - k\mu = 0$: $r\widetilde{N}_s = k\mu$. Since $(k, r) = 1$, there exists a positive integer $q_{s\mu}$ such that

$$kq_{s\mu} = \widetilde{N}_s \tag{16}$$

and

$$rq_{s\mu} = \mu. \tag{17}$$

From (14) the integral $\int_\alpha z^{r\widetilde{N}_s - k\mu - 1}dz$ multiplies the variable c whose power is $s_2\widetilde{N}_s + \mu s_1$. From (16) and (17) we obtain $s_2\widetilde{N}_s + \mu s_1 = q_{s\mu}(s_2k + rs_1) = q_{s\mu}$. Hence the Abelian integral $\int_{[\gamma_c]} \omega_{ij}$ is a polynomial $P_{ij}(c)$.

Step 2. We are going to compute the upper bound for the degree of the polynomials $P_{ij}(c)$ by finding an upper bound for the positive integers $q_{s\mu}$.

The addition of (16) and $(l+1)$ times (17) yields

$$q_{s\mu}(k + r(l+1)) = \widetilde{N}_s + (l+1)\mu. \tag{18}$$

From (15) we then see that the right-hand side of (18) satisfies

$$\widetilde{N}_s + (l+1)\mu \leq -j + 2i + 2\mu + 1. \tag{19}$$

We can rewrite the right-hand side of (19) as $-2j + 2i + 2\mu + j + 1$. On the other hand we know that $\mu \leq j - i$ or in an equivalent form $-2j + 2i + 2\mu \leq 0$. We then obtain $-2j + 2i + 2\mu + j + 1 \leq j + 1$. Hence we have

$$\widetilde{N}_s + (l+1)\mu \leq j + 1. \tag{20}$$

From (18) and (20) it follows that

$$q_{s\mu}(k + r(l+1)) \leq j + 1,$$

whence we get

$$q_{s\mu} \leq \left[\frac{j+1}{k + r(l+1)}\right] = \left[\frac{j+1}{m+1}\right].$$

Since $j \leq n$, $P_{ij}(c)$ is a polynomial of degree at most $\hat{Z}(m,n) = \left[\frac{n+1}{m+1}\right]$. Therefore, $I(c) = \int_{[\gamma_c]} \omega = \sum_{j=1}^{n}\sum_{i=0}^{j-1} a_{ij}P_{ij}(c)$ is a polynomial of degree at most $\hat{Z}(m,n)$, which also is an upper bound for the number of isolated zeros of $I(c)$.

Step 3. Now, we will show that the upper bound $\hat{Z}(m,n)$ is optimal. As $l > 0$ and $k, r \in \mathbb{N} - \{0\}$, then any polynomial $H = x^k(x^ly - P(x))^r \in \mathcal{S}$ has degree

$k+r(l+1)\geq 3$. We consider the polynomial $H(x,y)=x(x^{m-1}y-1)\in\mathcal{S}$ of degree $m+1\geq 3$ (thus $l=m-1>0$). The generic fiber $\mathcal{L}_c$ of H is parameterized by

$$\begin{array}{rcl}\varphi_c:\mathbb{C}^* & \to & \mathcal{L}_c\\ z & \mapsto & \left(z,\dfrac{c+z}{z^m}\right).\end{array} \tag{21}$$

For each positive integer n we define the polynomial 1-form

$$\Omega_n^m := \left(y^n+\hat{Z}(m,n)\left(x^{m\hat{Z}(m,n)-1}y^{\hat{Z}(m,n)}-x^{m-2}y\right)\right)dx.$$

Clearly Ω_n^m is of degree n. To study the complete Abelian integral $I(c)$ defined by H and Ω_n^m we will consider two possibilities $n=1$ and $n\geq 2$.

i) If $n=1$, then $\Omega_1^m=ydx$. Hence, by using the parametrization (21), we get

$$\int_{[\gamma_c]}\Omega_1^m=\int_{[\gamma_c]}ydx=\int_\alpha\frac{c+z}{z^m}dz=\begin{cases}2\pi\sqrt{-1} & \text{if } m=2,\\ 0 & \text{if } m>2.\end{cases} \tag{22}$$

ii) If $n\geq 2$, then by using the parametrization (21), we obtain

$$\int_{[\gamma_c]}y^ndx=\int_\alpha\frac{(c+z)^n}{z^{mn}}dz=\sum_{\mu=0}^{n}\binom{n}{\mu}\left(\int_\alpha\frac{dz}{z^{(m-1)n+\mu}}\right)c^\mu=0 \tag{23}$$

because $(m-1)n+\mu\geq n+\mu\geq n\geq 2$, and

$$\int_{[\gamma_c]}x^{m\hat{Z}(m,n)-1}y^{\hat{Z}(m,n)}dx=\sum_{\mu=0}^{\hat{Z}(m,n)}\binom{\hat{Z}(m,n)}{\mu}\left(\int_\alpha z^{\hat{Z}(m,n)-\mu-1}dz\right)c^\mu. \tag{24}$$

The integral $\int_\alpha z^{\hat{Z}(m,n)-\mu-1}dz$ in (24) is different from zero if and only if $\mu=\hat{Z}(m,n)$. Hence

$$\int_{[\gamma_c]}x^{m\hat{Z}(m,n)-1}y^{\hat{Z}(m,n)}dx=\left(\int_\alpha z^{-1}dz\right)c^{Z(m,n)}=\left(2\pi\sqrt{-1}\right)c^{\hat{Z}(m,n)}. \tag{25}$$

Moreover we have

$$\int_{[\gamma_c]}x^{m-2}ydx=\int_\alpha\frac{c+z}{z^2}dz=2\pi\sqrt{-1}. \tag{26}$$

Therefore from (23), (25) and (26) we obtain

$$\int_{[\gamma_c]}\Omega_n^m=\left(2\pi\sqrt{-1}\right)\hat{Z}(m,n)\left(c^{\hat{Z}(m,n)}-1\right). \tag{27}$$

Hence from (22) and (27) $\int_{[\gamma_c]}\Omega_n^m$ is a polynomial of degree $\hat{Z}(m,n)$. In addition, the zeros of $\int_{[\gamma_c]}\Omega_n^m$ are all different. □

Remark 8. We have proved that any complete Abelian integral $I(c)$ defined by $H\in\mathcal{S}$ and a polynomial 1-form ω is a polynomial on $\mathbb{C}-\Sigma_H=\mathbb{C}-\{0\}$ and, of course, $I(c)$ extends to the whole $\mathbb{C}$. We know that $c=0$, the unique singular value of $H=(x^2+y^2)/2$ or equivalently of $\widetilde{H}=xy\in\mathcal{S}$, is always a zero of $I(c)$. Hence a natural question is when the unique singular value $c=0$ of $H=x^k(x^ly-P(x))^r\in\mathcal{S}$ is a zero of $I(c)$? There are two answers.

1. If $l=0$, then $c=0$ is always a zero of $I(c)$. This assertion follows from step 1 in the case $l=0$ of the previous proof.
2. If $l>0$, then $c=0$ may not be a zero of $I(c)$. See for instance (25) and (27).

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