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POLYNOMIAL AND RATIONAL INTEGRABILITY OF POLYNOMIAL HAMILTONIAN SYSTEMS

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ABSTRACT. We study the relationship between the existence of Darboux polynomials and additional polynomial or rational first integrals for polynomial Hamiltonian systems satisfying certain symmetries.

1. Introduction and statement of the main results

One of the most natural class of canonical Hamiltonian systems which appear in Mechanics is given by Hamiltonians expressed as the a sum of the kinetic and potential energy. Maciejewski, Nakagawa and Przybylska in [6], Maciejewski and Przybylska in [5], and and later on Garcia, Grau and Llibre in [2] studied the integrability of canonical Hamiltonian systems in $\mathbb{C}^{2m}$ where the Hamiltonian is given by a polynomial expression of the form

$$H(q,p) = \frac{1}{2}\sum_{i=1}^{m} \mu_i p_i^2 + V(q), \tag{1}$$

where $q, p \in \mathbb{C}^{2m}$, the potential energy $V(q)$ is a polynomial, and $\mu_i \in \mathbb{C}$ for $i = 1, \ldots, m$.

In this paper, first we extend results on Hamiltonians of the form (1) as obtained in the aforementioned papers to time reversible Hamiltonian systems in $\mathbb{C}^{2m}$ with an arbitrary polynomial Hamiltonian $H(q,p)$. For such systems, under convenient assumptions, we deduce the existence of a second polynomial first integral independent of the Hamiltonian.

Second, we consider invariant polynomial Hamiltonian systems in $\mathbb{C}^{2m}$ under an involution acting on $(q, p, H(q,p))$. In this case, provided some additional assumptions are satisfied, we obtain a second polynomial or rational first integral independent of the Hamiltonian.

A canonical Hamiltonian system defined on $\mathbb{C}^{2m}$ with m degrees of freedom and Hamiltonian $H(q,p)$ is given by

$$\frac{dq_i}{dt} = \frac{\partial H(q,p)}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H(q,p)}{\partial q_i}, \quad \text{for } i = 1, \ldots, m, \tag{2}$$

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where $q = (q_1, \ldots, q_m) \in \mathbb{C}^m$ and $p = (p_1, \ldots, p_m) \in \mathbb{C}^m$ are the generalized coordinates and momenta, respectively.

We denote by X_H the associated Hamiltonian vector field in $\mathbb{C}^{2m}$ to the Hamiltonian system (2), i.e.,

$$X_H = \sum_{i=1}^{m} \frac{\partial H(q,p)}{\partial p_i} \frac{\partial}{\partial q_i} - \sum_{i=1}^{m} \frac{\partial H(q,p)}{\partial q_i} \frac{\partial}{\partial p_i}. \tag{3}$$

We define the involution $\tau : \mathbb{C}^{2m} \times \mathbb{R} \to \mathbb{C}^{2m} \times \mathbb{R}$ as $\tau(q, p, t) = (q, -p, -t)$, i.e. τ is a diffeomorphism such that τ^2 is the identity. The vector field $\mathcal{X}_H$ on $\mathbb{C}^{2m}$ is said to be *τ–reversible* if $\tau_*(\mathcal{X}_H) = -\mathcal{X}_{\mathcal{H}}$, where τ_* is the push–forward associated to the diffeomorphism τ. We recall that the push–forward is defined as $\tau_*(\mathcal{X}) = (D\tau\, \mathcal{X}) \circ \tau^{-1}$. In our case $\tau^{-1} = \tau$ because τ is an involution.

Let U be an open subset of $\mathbb{C}^{2m}$, such that its closure is $\mathbb{C}^{2m}$. Then, a function $I : U \to \mathbb{C}^{2m}$ constant on the orbits of the Hamiltonian vector field $\mathcal{X}_H$ contained in U is called a *first integral* of $\mathcal{X}_H$, i.e. $\mathcal{X}_H I \equiv 0$ on U. Of course, H is a first integral of the vector field $\mathcal{X}_H$.

A first integral $I(q, p)$ of the Hamiltonian vector field $\mathcal{X}_H$ is called an *additional first integral* when H and I are functionally independent, i.e., when the gradient vectors of $H(q, p)$ and $I(q, p)$ are linearly independent in $\mathbb{C}^{2m}$ except perhaps **on** a zero Lebesgue measure set.

A non–constant polynomial $F \in \mathbb{C}[q, p]$ is a *Darboux polynomial* of the polynomial Hamiltonian vector field $\mathcal{X}_H$ if there exists a polynomial $K \in \mathbb{C}[q, p]$, called the *cofactor* of F such that $\mathcal{X}_H F = KF$. We say that F is a *proper* Darboux polynomial if its cofactor is not zero, i.e. if F is not a polynomial first integral of $\mathcal{X}_H$.

It is easy to check directly from the definition of a Darboux polynomial F that the hypersurface $F(q, p) = 0$ defined by a Darboux polynomial is invariant by the flow of $\mathcal{X}_H$, i.e., if an orbit of the vector field $\mathcal{X}_H$ has a point on that hypersurface, then the whole orbit is contained in it.

The Darboux polynomials where introduced by Darboux [1] in 1878 for studying the existence of first integrals in the polynomial differential systems in $\mathbb{C}^m$. His original ideas have been developed by many authors, see the survey [3] and the paper [4] with the references therein for the more recent result on the Darboux theory of integrability.

We say that a function $G(q, p)$ is *even* with respect to the variable q if $G(q, p) = G(-q, p)$, and we say that it is *odd* with respect to the variable q if $G(q, p) = -G(-q, p)$.

In this paper as in the articles [5, 6, 2] we extend some of the ideas of the Darboux theory of integrability for polynomial differential systems to polynomial Hamiltonian systems. Our main result is the following:

Theorem 1. *Consider a polynomial Hamiltonian $H(q, p)$ such that its corresponding Hamiltonian vector field* (3) *is τ–reversible. Let $F(q, p)$ be a proper Darboux polynomial of the Hamiltonian vector field $\mathcal{X}_H$ with a cofactor $K(q, p)$ which is an even* *function with respect to the variable p. Then $F(q,p)F(q,-p)$ is a polynomial first integral of $\mathcal{X}_H$.*

Theorem 1 is proved in section 2.

Corollary 2. *Consider a polynomial Hamiltonian $H(q,p)$ given by* (1)*. Let $F(q,p)$ be a proper Darboux polynomial of the Hamiltonian vector field $\mathcal{X}_H$. Then $F(q,p)F(q,-p)$ is a polynomial first integral of $\mathcal{X}_H$.*

We do not prove Corollary 2 since in fact this result is not new. If the polynomial $V(q)$ of the Hamiltonian (1) has even degree, then Corollary 2 was proved in [5], if it has odd degree than it was proved in [2].

Theorem 1 may be extended to any σ–involution defined by a diffeomorphism $\sigma : \mathbb{C}^{2m} \times \mathbb{R} \to \mathbb{C}^{2m} \times \mathbb{R}$ which can be written as $\sigma(p,q,t) = (\bar{\sigma}(q,p), -t)$ where $\bar{\sigma} : \mathbb{C}^{2m} \to \mathbb{C}^{2m}$ is such that $\bar{\sigma}^2$ is the identity. Let $F : \mathbb{C}^{2m} \to \mathbb{C}^{2m}$ be a function **and** define $F^{\bar{\sigma}} = \bar{\sigma}_* F = F \circ \bar{\sigma}$. We have:

Theorem 3. *Consider a polynomial Hamiltonian $H(q,p)$ such that its corresponding Hamiltonian vector field* (3) *is σ–reversible. Let $F(q,p)$ be a proper Darboux polynomial of the Hamiltonian vector field $\mathcal{X}_H$ with a cofactor $K(q,p)$ such that $K^{\bar{\sigma}} = K$. Then $FF^{\bar{\sigma}}$ is a polynomial first integral of $\mathcal{X}_H$.*

The proof of Theorem 3 is similar to the proof of Theorem 1.

We define the involution $\tau^*\colon \mathbb{C}^{2m} \to \mathbb{C}^{2m}$ as $\tau^*(q,p) = (-q,p)$, i.e., τ^* is a diffeomorphism such that $(\tau^*)^2$ is the identity.

We also define $\sigma^*\colon \mathbb{C}^{2m} \times \mathbb{C} \to \mathbb{C}^{2m} \times \mathbb{C}$ as

$$\sigma^*(q,p,H(q,p)) = \big(\tau^*(q,p), -H(\tau^*(q,p))\big). \tag{4}$$

We note that $(\sigma^*)^2$ is also the identity. The vector field X_H or the Hamiltonian system (2) on $\mathbb{C}^{2m}$ is σ^*-*equivariant* if the Hamiltonian system (2) is invariant under σ^*, that is

$$\sigma^*_*(X_H) = X_H.$$

For Hamitonian vector fields as given by (3) this is equivalent to

$$\frac{\partial H(-q,p)}{\partial p_i} = -\frac{\partial H(q,p)}{\partial p_i} \quad \text{and} \quad \frac{\partial H(-q,p)}{\partial q_i} = -\frac{\partial H(q,p)}{\partial q_i}, \tag{5}$$

for $i = 1, \ldots, n$; i.e., the functions $\partial H(q,p)/\partial p_i$ and $\partial H(q,p)/\partial q_i$ must be odd with respect to the variable q.

Theorem 4. *Consider a polynomial Hamiltonian $H(q,p)$ such that its corresponding Hamiltonian vector field* (3) *is σ^*-equivariant. Let $F(q,p)$ be a proper Darboux polynomial of the Hamiltonian vector field X_H with a cofactor $K(q,p)$. Then the following statements hold.*

(a) *If $K(q,p)$ is an even function with respect to the variable q, then $F(-q,p)F(q,p)$ is a polynomial first integral of X_H.*
(b) *If $K(q,p)$ is an odd function with respect to the variable q, then $F(-q,p)/F(q,p)$ is a rational first integral of X_H.*

Theorem 4 is proved in section 3.

Theorem 4 can be extended to any $\tilde{\sigma}^*\colon \mathbb{C}^{2m}\times\mathbb{C}\to\mathbb{C}^{2m}\times\mathbb{C}$ defined as

$$\tilde{\sigma}^*(q,p,H(q,p)) = (\tilde{\tau}^*(q,p), -H(\tilde{\tau}^*(q,p)))$$

where $\tilde{\tau}^*\colon \mathbb{C}^{2m}\to\mathbb{C}^{2m}$ is such that $(\tilde{\tau}^*)^2$ is the identity. Let $F\colon \mathbb{C}^{2m}\to\mathbb{C}^{2m}$ be a function. We define $F^{\tilde{\tau}^*} = F\circ\tilde{\tau}^*$.

Theorem 5. *Consider a polynomial Hamiltonian $H(q,p)$ such that its corresponding Hamiltonian vector field* (3) *is $\tilde{\sigma}^*$-equivariant. Let $F(q,p)$ be a proper Darboux polynomial of the Hamiltonian vector field X_H with a cofactor K. Then the following statements hold.*

(a) *If K is such that $K^{\tilde{\tau}^*} = K$, then $F^{\tilde{\tau}^*}F$ is a polynomial first integral of X_H.*
(b) *If K is such that $K^{\tilde{\tau}^*} = -K$, then $F^{\tilde{\tau}^*}/F$ is a rational first integral of X_H.*

The proof of Theorem 5 is the same as the proof of Theorem 4 and hence it is omitted.

Corollary 6. *Consider a polynomial Hamiltonian $H(q,p)$ given by* (1)*, and assume that the potential energy $V(q)$ is even with respect to the variable q. Let $F(q,p)$ be a proper Darboux polynomial of the Hamiltonian vector field X_H with cofactor K. Then the following statements hold.*

(a) *If K is an even function in the variable q, then $F(-q,p)F(q,p)$ is a polynomial first integral of X_H.*
(b) *If K is an odd function in the variable q, then $F(-q,p)/F(q,p)$ is a rational first integral of X_H.*

Corollary 6 is also proved in section 3.

2. Proof of Theorem 1

We shall need the following auxiliary result.

Lemma 7. *Under the assumptions of Theorem 1, we have that $F(q,-p)$ is another proper Darboux polynomial of $\mathcal{X}_H$ with cofactor $-K(q,-p)$.*

Proof. In what follows we use the push–forward notation. Since

$$\mathcal{X}_H F(q,p) = K(q,p)F(q,p),$$

we have

$$\tau_*(\mathcal{X}_H F(q,p)) = \tau_*(K(q,p)F(q,p)),$$

or equivalently

$$\tau_*(\mathcal{X}_H)\tau_*(F(q,p)) = \tau_*(K(q,p))\tau_*(F(q,p)),$$

that is

$$-\mathcal{X}_H F(q,-p) = K(q,-p)F(q,-p).$$

Finally we have that

$$\mathcal{X}_H F(q,-p) = -K(q,-p)F(q,-p).$$

So $F(q,-p)$ is a proper Darboux polynomial of $\mathcal{X}_H$ with cofactor $-K(q,-p)\neq 0$, because $K(q,p)\neq 0$ due to the fact that $F(q,p)$ is a proper Darboux polynomial. □

Proof of Theorem 1. Under the assumptions of Theorem 1 we have that $\mathcal{X}_H F(q,p) = K(q,p)F(q,p)$ with $K(q,p) \neq 0$.

By Lemma 7 we have that $\mathcal{X}_H F(q,-p) = -K(q,-p)F(q,-p)$. Therefore

$$\begin{aligned}\mathcal{X}_H(F(q,p)F(q,-p)) = &\ \mathcal{X}_H(F(q,p))F(q,-p) + F(q,p)\mathcal{X}_H(F(q,-p)) \\ = &\ K(q,p)F(q,p)F(q,-p) + F(q,p)(-K(q,-p)F(q,-p)) \\ = &\ (K(q,p) - K(q,-p))F(q,p)F(q,-p).\end{aligned}$$

This last expression is zero due to the fact that the cofactor $K(q,p)$ is an even function in the variable p. So $F(q,p)F(q,-p)$ is a polynomial first integral of Hamiltonian vector field $\mathcal{X}_H$. □

3. Proofs of the remaining results

We first prove Theorem 4. We use the following auxiliary result.

Lemma 8. *Under the assumptions of Theorem 4 we have that $F(-q,p)$ is another proper Darboux polynomial of X_H with cofactor $-K(-q,p)$.*

Proof. From the definition of τ_*^* it follows that $\tau_*^*(X_H) = -X_H$. This implies that

$$\tau_*^*(X_H F) = -X_H \tau^*(F) = -X_H F(-q,p). \tag{6}$$

Moreover, we have that $X_H F = KF$ and thus

$$\tau_*^*(X_H F) = \tau_*^*(KF) = \tau_*^*(K)\tau_*^*(F) = K(-q,p)F(-q,p). \tag{7}$$

Combining equations (6) and (7) we get

$$X_H F(-q,p) = -K(-q,p)F(-q,p).$$

Therefore $F(-q,p)$ is a proper Darboux polynomial of X_H with cofactor $-K(-q,p)$. We note that $K(-q,p) \neq 0$ due to the fact that $F(-q,p)$ is a proper Darboux polynomial and consequently $K(q,p) \neq 0$. □

Proof of Theorem 4. Under the assumptions of Theorem 4 we have that $X_H F(q,p) = K(q,p)F(q,p)$ with $K(q,p) \neq 0$.

By Lemma 8 we have that $X_H F(-q,p) = -K(-q,p)F(-q,p)$. Therefore

$$\begin{aligned}&X_H(F(-q,p)F(q,p)) = X_H(F(-q,p))F(q,p) + F(-q,p)X_H(F(q,p)) \\ &= -K(-q,p)F(-q,p)F(q,p) + F(-q,p)K(q,p)F(q,p) \\ &= (-K(-q,p) + K(q,p))F(q,-p)F(q,p).\end{aligned}$$

If K is an even function in the variable q, the last expression is zero. So, in this case, $F(-q,p)F(q,p)$ is a polynomial first integral of the Hamiltonian vector field X_H. This completes the proof of statement (a).

On the other hand,

$$X_H(F(-q,p)/F(q,p)) = \frac{X_H(F(-q,p))F(q,p) - F(-q,p)X_H(F(q,p))}{F(q,p)^2}$$
$$= \frac{-K(-q,p)F(-q,p)F(q,p) - F(-q,p)K(q,p)F(q,p)}{F(q,p)^2}$$
$$= -(K(-q,p) + K(q,p))\frac{F(-q,p)}{F(q,p)}.$$

If K is an odd function in the variable q, the last expression is zero. So, in this case, $F(-q,p)/F(q,p)$ is a rational first integral of the Hamiltonian vector field X_H. This completes the proof of the theorem. □

To prove Corollary 6 we recall the following result whose proof can be found in [2].

Lemma 9. *Let $F(q,p)$ be a proper Darboux polynomial of the Hamiltonian vector field X_H associated to the Hamiltonian H given by* (1). *Then its cofactor is a polynomial of the form $K(q)$.*

Proof of Corollary 6. It is immediate to check that the Hamiltonian vector field associated to the Hamiltonian (1)

$$X_H = \sum_{i=1}^{m} \mu_i p_i \frac{\partial}{\partial q_i} - \sum_{i=1}^{m} \frac{\partial V}{\partial q_i}\frac{\partial}{\partial p_i},$$

with V an even function in the variable q, is σ^*-equivariant, i.e. (5) is satisfied.

If $F(q,p)$ is a proper Darboux polynomial of the Hamiltonian vector field X_H, by Lemma 9 we have that its cofactor is of the form $K(q)$. Then, if K is an even function in the variable q then the Hamiltonian vector field X_H satisfies all the assumptions of Theorem 4(a), and consequently $F(-q,p)F(q,p)$ is a polynomial first integral of X_H. On the other hand, if K is an odd function in the variable q then the Hamiltonian vector field X_H satisfies all the assumptions of Theorem 4(b), and consequently $F(-q,p)/F(q,p)$ is a rational first integral of X_H. This completes the proof. □

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