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FIRST INTEGRALS FOR A CHARGED PARTICLE MOVING ON A PLANE UNDER THE ACTION OF A MAGNETIC FIELD ORTHOGONAL TO THIS PLANE

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Abstract. We characterize the polynomial (respectively analytical) first integrals of degree one in the variables u and v of the differential systems of the form $x' = u$, $y' = v$, $u' = B(x, y)v$ and $v' = -B(x, y)u$ where $B = B(x, y)$ is a polynomial (respectively analytic function) in the variables x and y. This differential system models a non–relativistic charge moving on a plane under the action of a magnetic field orthogonal to this plane.

1. Introduction and statement of the main results

One of the more classical problems in the qualitative theory of differential systems is to characterize the existence or not of first integrals.

We consider the differential equation of a non–relativistic charge moving on a plane under the action of a magnetic field orthogonal to this plane. The differential equations of this motion can be written as

$$\ddot{x} = B(x, y)\dot{y}, \quad \ddot{y} = -B(x, y)\dot{x},$$

where B is a non-zero function in the variables x and y. For more details see [2]. The dot denotes derivative with respect to the time t.

We consider the equivalent differential system of first order in $\mathbb{R}^4$

$$\dot{x} = u, \quad \dot{y} = v, \quad \dot{u} = B(x, y)v, \quad \dot{v} = -B(x, y)u. \tag{1}$$

Let $\dot{\mathbf{x}} = f(\mathbf{x})$ be a C^1 differential system in $\mathbb{R}^n$ and let $U \subset \mathbb{R}^n$ be an open set. We say that the non–constant function $H: \mathbb{R}^n \to \mathbb{R}$ is a *first integral* of $\dot{\mathbf{x}} = f(\mathbf{x})$ on U, if $H(x_1(t), \cdots, x_n(t)) =$ constant for all values of t for which the solution $(x_1(t), \cdots, x_n(t))$ of $\dot{\mathbf{x}} = f(\mathbf{x})$ is defined on U.

Let $H_k : U_k \to \mathbb{R}$ for $k = 1, \ldots, r$ be first integrals of $\dot{\mathbf{x}} = f(\mathbf{x})$. We say that they are *independent* on $U_1 \cap \cdots \cap U_r$ if their gradients are linearly independent over a full Lebesgue measure subset of $U_1 \cap \cdots \cap U_r$. We say that $\dot{\mathbf{x}} = f(\mathbf{x})$ is *completely integrable* if it has $n-1$ independent first integrals.

It is known that system (1) has the first integral

$$H_1 = u^2 + v^2.$$

A *polynomial first integral* is a first integral H which is a polynomial. An *analytic first integral* is a first integral H which is an analytic function.

We classify the integrable systems (1) for different classes of first integrals.

Theorem 1. *Let $b(z)$ be a polynomial in the variable z and let $T(x, y)$ be a polynomial in the variables x and y such that $\partial T/\partial x$ and $\partial T/\partial y$ are coprime.*

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(a) *System* (1) *with* $B(x,y) = b(T(x,y))$ *has a polynomial first integral independent with* H_1 *of degree one in the variables* u *and* v *if and only if*

$$T = a + bx + cy + d(x^2 + y^2), \tag{2}$$

with $a, b, c, d \in \mathbb{R}$. *Moreover a polynomial first integral independent with* H_1 *is*

$$H = H(x,y,u,v) = \int b(T)\,dT - cu + bv + 2d(xv - yu). \tag{3}$$

(b) *If* (a) *holds then system* (1) *is completely integrable with analytic first integrals.*

Theorem 2. *Let* $b(z)$ *be an analytic function in the variable* z *and let* $T(x,y)$ *be an analytic function in the variables* x *and* y *such that* $\partial T/\partial x$ *and* $\partial T/\partial y$ *are coprime.*

(a) *System* (1) *with* $B(x,y) = b(T(x,y))$ *has an analytic first integral independent with* H_1 *of degree one in the variables* u *and* v *if and only if* T *satisfies* (2). *Moreover* H *given in* (3) *is now an analytic first integral independent with* H_1.

(b) *If* (a) *holds then system* (1) *is completely integrable with analytic first integrals.*

The proofs of Theorems 1 and 2 are given in section 3.

In [2] the authors obtained the first integral $H = xv - yu + \frac{1}{2}\int B(x^2+y^2)\,d(x^2+y^2)$ when $B(x,y) = B(x^2+y^2)$, and the first integral $H = u - \int B(y)\,dy$ when $B(x,y) = B(y)$. They are both independent with H_1. We note that these are two particular cases of Theorems 1 and 2 which corresponds to take $a = b = c = 0$ and $d = 1$, i.e. $T = x^2 + y^2$ in the first case; and $a = b = d = 0$ and $c = 1$, i.e. $T = y$ in the second case. Providing in both cases the first integrals of [2].

2. Auxiliary result

The objective of this section is to prove the next auxiliary result.

Theorem 3. *System* (1) *is completely integrable if and only if there exists an analytic first integral* H_2 *independent with* $H_1 = u^2 + v^2$.

In the proof we need the following known result stated in the next theorem.

Assume that we have a C^1 differential system $\dot{\mathbf{x}} = f(\mathbf{x})$ in $\mathbb{R}^n$, i.e. $\mathbf{x} \in \mathbb{R}^n$ and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function. Let ϕ_t be its flow; i.e. $\phi_t(\mathbf{x})$ is the solution of $\dot{\mathbf{x}} = f(\mathbf{x})$ which pass through the point $\mathbf{x} \in \mathbb{R}^n$ at time zero after the time t, assuming that the solution $\phi_t(\mathbf{x})$ exists for such a time t. The ordinary differential system $\dot{\mathbf{x}} = f(\mathbf{x})$ on $\mathbb{R}^n$ is called *analytic* if the function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is analytic.

A non-negative C^1 function $M\colon \mathbb{R}^n \to \mathbb{R}$ and non–identically zero on any open subset of $\mathbb{R}^n$, is called a *Jacobi multiplier* of the differential system $\dot{\mathbf{x}} = f(\mathbf{x})$, if for any open set $D \subset \mathbb{R}^n$, and any $t \in \mathbb{R}$ for which $\phi_t(D)$ is defined, we have

$$\int_D M(\mathbf{x})\,d\mathbf{x} = \int_{\phi_t(D)} M(\mathbf{x})\,d\mathbf{x}.$$

Under good conditions Jacobi multipliers can be used for constructing an additional first integral, as it is stated in the following result (for a proof see for example Theorem 2.7 in [3]).

Theorem 4 (Jacobi)**.** *Assume that the analytic differential system* $\dot{\mathbf{x}} = f(\mathbf{x})$ *on* $\mathbb{R}^n$ *admits an analytic Jacobi multiplier* M *and* $n-2$ *analytic first integrals. Then system* $\dot{\mathbf{x}} = f(\mathbf{x})$ *admits an additional analytic first integral independent of the previous ones.*

We recall that the *divergence* of a differential system $\dot{\mathbf{x}} = f(\mathbf{x})$ of $\mathbb{R}^n$ with $f = (f_1, \ldots, f_n)$ is

$$\frac{\partial f_1}{\partial x_1} + \cdots + \frac{\partial f_n}{\partial x_n}.$$

Proof of Theorem 3. We remark that the divergence of system (1) is zero. Then, by the Liouville formula (see for instance [1]), the flow ϕ_t of system (1) preserves the volume, i.e. the volume of the subset D of $\mathbb{R}^n$ and the volume of $\phi_t(D)$ are equal, assuming that $\phi_t(D)$ is well-defined. So, from the definition of Jacobi multiplier it follows that the constant function $M = 1$ is a Jacobi multiplier.

Now we assume that system (1) has a second independent analytic first integral with respect to the analytic first integral H_1. Then Theorem 4 implies that there exists also a third independent analytic first integral. □

3. Proof of Theorem 1

To prove Theorem 1 we introduce the change of variables

$$X = x + iy, \quad Y = x - iy, \quad U = u + iv, \quad V = u - iv. \tag{4}$$

With this change of variables system (1) becomes

$$\begin{aligned} X' &= U, \\ Y' &= V, \\ U' &= -i\, B\left(\frac{X+Y}{2}, \frac{X-Y}{2i}\right)U, \\ V' &= i\, B\left(\frac{X+Y}{2}, \frac{X-Y}{2i}\right)V. \end{aligned} \tag{5}$$

In these variables we have that H_1 is

$$H_1 = UV.$$

We write

$$\tilde{B}(X,Y) = B\left(\frac{X+Y}{2}, \frac{X-Y}{2i}\right),$$

and system (5) can be written as

$$\begin{aligned} X' &= U, \\ Y' &= V, \\ U' &= -i\, \tilde{B}(X,Y)U, \\ V' &= i\, \tilde{B}(X,Y)V. \end{aligned} \tag{6}$$

The vector field associated to system (6) is

$$\mathcal{X} = U\frac{\partial}{\partial X} + V\frac{\partial}{\partial Y} - i\, \tilde{B}(X,Y)U\frac{\partial}{\partial U} + i\, \tilde{B}(X,Y)V\frac{\partial}{\partial V}.$$

We first prove the following result.

Theorem 5. *Let $b(z)$ be a polynomial in the variable z and let $T(X,Y)$ be a polynomial in the variables X and Y such that $\partial T/\partial x$ and $\partial T/\partial y$ are coprime. System (6) with $\tilde{B}(X,Y) = b(T(X,Y)) = b(T)$ has a polynomial first integral independent with H_1 and of degree one in the variables U and V if and only if T satisfies*

$$\frac{\partial^2 T}{\partial X^2} = \frac{\partial^2 T}{\partial Y^2} = 0. \tag{7}$$

Moreover a polynomial first integral independent with H_1 is

$$H = i\int b(T)\, dT + U\frac{\partial T}{\partial X} - V\frac{\partial T}{\partial Y}. \tag{8}$$

Proof. It is easy to check that if $\tilde{B}(X,Y) = b\big(T(X,Y)\big) = b(T)$, where T is a polynomial in the variables X and Y satisfying (7) then

$$H = i\int b(T)\,dT + U\frac{\partial T}{\partial X} - V\frac{\partial T}{\partial Y}.$$

is a polynomial first integral of system (6) independent with H_1. Indeed, it is clear that it is independent of H_1. Moreover,

$$\begin{aligned}\mathcal{X}H &= U\frac{\partial H}{\partial X} + V\frac{\partial H}{\partial Y} - i\,b(T)U\frac{\partial H}{\partial U} + i\,b(T)V\frac{\partial H}{\partial V}\\ &= U i\,b(T)\frac{\partial T}{\partial X} - UV\frac{\partial^2 T}{\partial Y\partial X} + V i\,b(T)\frac{\partial T}{\partial Y} + VU\frac{\partial^2 T}{\partial X\partial Y} - i\,b(T)U\frac{\partial T}{\partial X} - i\,b(T)V\frac{\partial T}{\partial Y}\\ &= 0.\end{aligned}$$

Now we prove the converse. Assume that system (6) has a polynomial first integral H independent of H_1. Then H satisfies

$$\mathcal{X}H = U\frac{\partial H}{\partial X} + V\frac{\partial H}{\partial Y} - i\,b(T)U\frac{\partial H}{\partial U} + i\,b(T)V\frac{\partial H}{\partial V} = 0. \tag{9}$$

Then, if $H = H(U,V)$ we get, after simplifying by $ib(T)$, that

$$-U\frac{\partial H}{\partial U} + V\frac{\partial H}{\partial V} = 0,$$

that is H is any polynomial function in the variable $UV = H_1$, and hence it is dependent with H_1, a contradiction. Therefore $H = H(X,Y,U,V)$ and we write it as a polynomial in the variables U and V as follows

$$H = \sum_{j,k=0}^{n} H_{jk}(X,Y)U^jV^k, \tag{10}$$

where each $H_{jk}(X,Y)$ is a polynomial in the variables X and Y.

Using (10) and computing the coefficients of degree one in (9) of the variables U and V we get

$$\frac{\partial H_{00}}{\partial X} - i\,b(T)H_{10} = 0 \quad\text{and}\quad \frac{\partial H_{00}}{\partial Y} + i\,b(T)H_{01} = 0, \tag{11}$$

respectively. Therefore $i\,b(T)$ is an integrating factor of the system

$$\dot{X} = H_{01}(X,Y), \quad \dot{Y} = H_{10}(X,Y), \tag{12}$$

and consequently

$$\frac{\partial(b(T)H_{01})}{\partial X} = -\frac{\partial(b(T)H_{10})}{\partial Y}. \tag{13}$$

Then it is easy to check using (11) and (13) that

$$H_{00} = i\int b(T)H_{10}\,dX = -i\int b(T)H_{01}\,dY, \tag{14}$$

is first integral of (12) and

$$\frac{\partial T}{\partial X}H_{01} = -\frac{\partial T}{\partial Y}H_{10}.$$

Taking into account that $\partial T/\partial x$ and $\partial T/\partial y$ are coprime we have that

$$H_{01} = -\frac{\partial T}{\partial Y}G \quad\text{and}\quad H_{10} = \frac{\partial T}{\partial X}G, \tag{15}$$

for some polynomial $G = G(X,Y)$.

Now using (10) and computing the coefficients of degree two in (9) of the variables U and V we obtain

$$\frac{\partial H_{10}}{\partial X} - 2i\, b(T)H_{20} = 0, \quad \frac{\partial H_{01}}{\partial X} + \frac{\partial H_{10}}{\partial Y} = 0 \quad \text{and} \quad \frac{\partial H_{01}}{\partial Y} + 2i\, b(T)H_{02} = 0. \tag{16}$$

Since the degree of H is one in the variables U and V we have that $H_{20} = H_{02} = 0$. Then

$$\frac{\partial H_{10}}{\partial X} = 0 \quad \text{that is} \quad H_{10} = H_{10}(Y),$$

and

$$\frac{\partial H_{01}}{\partial Y} = 0 \quad \text{that is} \quad H_{01} = H_{01}(X).$$

Then it follows from (15) that G must be constant. Therefore, again from (15) it follows that

$$\frac{\partial^2 T}{\partial Y^2} = \frac{\partial^2 T}{\partial X^2} = 0.$$

This concludes the proof of the theorem. □

Proof of Theorem 1. It follows from Theorem 5 that system (6) with $\tilde{B}(X,Y) = b(T)$ and $T(X,Y)$ being polynomials, has a polynomial first integral of degree one in U and V, if and only if T satisfies (7). Since T is a polynomial we can write it as

$$T = a_0 + a_1 X + a_2 Y + a_3 XY, \quad a_0, a_1, a_2, a_3 \in \mathbb{C}.$$

Taking

$$a_1 = \frac{b - ci}{2}, \quad a_2 = \frac{b + ci}{2} \quad \text{and} \quad a_3 = d, \quad b, c, d \in \mathbb{R},$$

and using the change of variables (4) we get

$$T = a_0 + (a_1 + a_2)x + i(a_1 - a_2)y + a_3(x^2 + y^2) = a + bx + cy + d(x^2 + y^2).$$

Furthermore, again using (4) the first integral (8) becomes

$$\begin{aligned} &i \int b(T)\, dT + (a_1 - a_2)u + i(a_1 + a_2)v + 2ia_3(xv - yu) \\ &= i\Big(\int b(T)\, dT - cu + bv + 2d(xv - yu)\Big). \end{aligned}$$

This completes the proof of statement (a) of Theorem 1

The proof of statement (b) follows immediately from Theorem 3 and statement (a). □

Proof of Theorem 2. This proof is analogous to the proof of Theorem 1 using a similar result of Theorem 5 for analytic functions. □

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