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PERIODIC ORBITS FOR PERTURBED NON-AUTONOMOUS DIFFERENTIAL EQUATIONS

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ABSTRACT. We consider non-autonomous differential equations, on the cylinder $(t,r)\in\mathcal{S}^1\times\mathbb{R}^d$, given by $dr/dt=f(t,r,\varepsilon)$ and having an open continuum of periodic solutions when $\varepsilon=0$. From the study of the variational equations of low order we obtain successive functions such that the simple zeroes of the first one that is not identically zero control the periodic orbits that persist for the unperturbed equation. We apply these results to several families of differential equations with $d=1,2,3$. They include some autonomous polynomial differential equations and some Abel type non-autonomous differential equations.

1. INTRODUCTION AND MAIN RESULTS

Consider the non-autonomous differential equation

$$\frac{dr}{dt}=f_0(t,r),\quad t\in[0,T],\, r\in\mathbb{R}^d, \tag{1}$$

where $d\in\mathbb{N}$ and f_0 is a real, smooth, T-periodic function in t. Assume that it has an open continuum of T-periodic solutions. That is, it has solutions $\varphi_0(t,\rho)$ such that $\varphi_0(T,\rho)=\varphi_0(0,\rho)=\rho$, for all ρ in a open non-empty neighborhood, $U\subset\mathbb{R}^d$. Consider perturbations of former differential equation, given by

$$\frac{dr}{dt}=f(t,r,\varepsilon)=f_0(t,r)+\sum_{i=1}^{m}f_i(t,r)\varepsilon^i+O(\varepsilon^{m+1}), \tag{2}$$

where $(t,x)\in[0,T]\times\mathbb{R}^d$, $m\in\mathbb{N}$, f_i, for each i, is also a smooth real T-periodic function in t, and $|\varepsilon|$ is small enough.

To determine which of the periodic solutions of equation (1) remain, for $\varepsilon\neq0$, as an isolated periodic orbit (limit cycle) of equation (2) and to fix bounds for this number of limit cycles is a problem of current interest, see for instance [1, 4, 8, 14, 21]. Several methods have been developed to approach these questions. For instance, Abelian integrals, Melnikov functions, averaging theory, Lyapunov constants or variational equations methods can be used, see [4, 5, 6, 13]. This paper is concerned with the former approach which in fact goes back to Poincaré. That is, by using variational equations, we obtain some functions $M_i(\rho), i\in\mathbb{N}$, such

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that their simple zeroes give rise to isolated T-periodic solutions of the perturbed system (2).

The expressions that we obtain of the functions $M_i(\rho)$ are based on the explicit knowledge of the flow of the unperturbed equation (1). Let $\varphi_\varepsilon(t,\rho)$ be the solution of equation (2) such that $\varphi_\varepsilon(0,\rho)=\rho\in\mathbb{R}^d$, and assume that it can be written as

$$\varphi_\varepsilon(t,\rho)=\varphi_0(t,\rho)+\sum_{i=1}^{k}u_i(t,\rho)\varepsilon^i+O(\varepsilon^{k+1}), \tag{3}$$

for some functions $u_i(t,\rho)$ such that $u_i(0,\rho)=0$. For $i>0$, when $M_1(\rho)\equiv\cdots M_{i-1}(\rho)\equiv 0$, we define $M_i(\rho)=u_i(T,\rho)$, and by way of notation $M_0(\rho)\equiv 0$. Note that $M_i:U\longrightarrow\mathbb{R}^d$.

As usual, given a smooth function $F:\mathbb{R}^d\to\mathbb{R}^d$, $D_\rho F$ denotes the Jacobian matrix of F and $D_{\rho\rho}F$ its Hessian matrix.

Our first result is the following:

Theorem 1. *Consider the differential equation* (2)*. Let $\varphi_\varepsilon(\theta,\rho)$ be the solution such that $\varphi_\varepsilon(0,\rho)=\rho$, written as* (3)*. Assume that $\varphi_0(T,\rho)=\rho$, for all $\rho\in U$. Hence,*

$$\begin{aligned}
M_1(\rho) &= \int_0^T (D_\rho\varphi_0(t,\rho))^{-1}f_1(t,\varphi_0(t,\rho))\,dt,\\
M_2(\rho) &= \int_0^T (D_\rho\varphi_0(t,\rho))^{-1}\Big(\frac{1}{2}u_1^{\intercal}(t,\rho)D_{\rho\rho}f_0(t,\varphi_0(t,\rho))u_1(t,\rho)\\
&\qquad + D_\rho f_1(t,\varphi_0(t,\rho))u_1(t,\rho)+f_2(t,\varphi_0(t,\rho))\Big)\,dt.
\end{aligned}$$

Furthermore, for each $i\in\{1,2\}$, if $M_{i-1}(\rho)\equiv 0$ then, for ε small enough, each simple zero of $M_i(\rho)$, $\rho^*\in U$, gives rise to an isolated T-periodic solution of system (2). This periodic solution tends, when ε goes to zero, to the solution of the unperturbed system passing through $\rho=\rho^*$.

In Remark 5 we also give the expression for $M_3(\rho)$. A similar result follows using this new function.

In the one dimensional case, $d=1$, and when $f_0(t,r)\equiv 0$, then $\varphi_0(t,\rho)=\rho$ and simpler expressions for M_i, $i=1,...,4$, are given in next proposition. In Remark 6 we present the corresponding expression for $M_5(\rho)$.

Proposition 2. *Consider equation (2) with $f_0(t,r)\equiv 0$ and $d=1$. Let f_i be, $i=1,\ldots,4$, $\mathcal{C}^3([0,T]\times U)$ functions, T-periodic in t, where U is some open subset* *of* $\mathbb{R}$. *Hence,*

$$
\begin{aligned}
M_1(\rho) &= \int_0^T f_1(t,\rho)\,dt,\\
M_2(\rho) &= \int_0^T \big(\frac{\partial f_1}{\partial \rho}(t,\rho)u_1(t,\rho) + f_2(t,\rho)\big)\,dt,\\
M_3(\rho) &= \int_0^T \big(\frac{1}{2}\frac{\partial^2 f_1}{\partial \rho^2}(t,\rho)u_1^2(t,\rho) + \frac{\partial f_1}{\partial \rho}(t,\rho)u_2(t,\rho) + \frac{\partial f_2}{\partial \rho}(t,\rho)u_1(t,\rho)\\
&\qquad + f_3(t,\rho)\big)\,dt,\\
M_4(\rho) &= \int_0^T \big(\frac{1}{6}\frac{\partial^3 f_1}{\partial \rho^3}(t,\rho)u_1^3(t,\rho) + \frac{\partial^2 f_1}{\partial \rho^2}(t,\rho)u_1(t,\rho)u_2(t,\rho) + \frac{\partial f_1}{\partial \rho}(t,\rho)u_3(t,\rho)\\
&\qquad + \frac{1}{2}\frac{\partial^2 f_2}{\partial \rho^2}(t,\rho)u_1^2(t,\rho) + \frac{\partial f_2}{\partial \rho}(t,\rho)u_2(t,\rho) + \frac{\partial f_3}{\partial \rho}(t,\rho)u_1(t,\rho) + f_4(t,\rho)\big)\,dt,
\end{aligned}
$$

Moreover, the same conclusions that in Theorem 1 hold.

The expressions presented in the above results are equivalent and quite similar to the ones obtained using averaging theory and given for instance in [4, 17, 18, 19]. We remark that there are some differences in the rational numbers appearing in the expressions of $M_i(\rho), i \geq 3$, because in these works the solution (3) is written as

$$\varphi_\varepsilon(t,\rho) = \varphi_0(t,\rho) + \sum_{i=1}^{k} \frac{1}{i!}u_i(t,\rho)\varepsilon^i + O(\varepsilon^{k+1}).$$

As we have seen, in Theorem 1 and Proposition 2 a key hypothesis is that all the zeroes of the corresponding $M_i(\rho)$ are simple. We also present a general result that allows to check this hypothesis when the system $M_i(\rho) = 0$ is equivalent to a polynomial system in $\mathbb{R}^d$. For simplicity we state it for $d = 2$, although it can be easily extended to higher dimensions. As usual given two polynomials $P(x)$ and $Q(x)$, $\operatorname{Res}(P(x), Q(x), x)$ denotes the resultant of them with respect to x, see [10]. Recall that the resultant vanishes if and only if both polynomials have a common zero (real or complex).

Proposition 3. *Consider a planar polynomial system*

$$P(x,y) = 0, \quad Q(x,y) = 0,$$

and compute

$$
\begin{aligned}
J(x,y) &= \frac{\partial P(x,y)}{\partial x}\frac{\partial Q(x,y)}{\partial y} - \frac{\partial P(x,y)}{\partial y}\frac{\partial Q(x,y)}{\partial x},\\
R^x(y) &= \operatorname{Res}(P(x,y), J(x,y), x), \quad S^x(y) = \operatorname{Res}(Q(x,y), J(x,y), x),\\
R^y(x) &= \operatorname{Res}(P(x,y), J(x,y), y), \quad S^y(x) = \operatorname{Res}(Q(x,y), J(x,y), y),\\
T_1 &= \operatorname{Res}(R^x(y), S^x(y), y) \in \mathbb{C}, \quad T_2 = \operatorname{Res}(R^y(x), S(x), x) \in \mathbb{C}.
\end{aligned}
$$

If $T_1 \neq 0$ or $T_2 \neq 0$ then all the solutions (real or complex) of the system are simple.

To illustrate the above result we present a simple example. Consider the system,

$$x^2 + y^2 - 1 = 0, \quad ax + by + c = 0.$$

Then $T_1 = -16b^2(b^2 + a^2)(a^2 - c^2 + b^2)$ and $T_2 = -16a^2(b^2 + a^2)(a^2 - c^2 + b^2)$. Hence when $(b^2 + a^2)(a^2 + b^2 - c^2) \neq 0$ all its solutions are simple.

Remark 4. *Sometimes can be useful to decompose the polynomials R^x, R^y, S^x and S^y in factors. Using these decompositions it can be proved that, in a certain region, all the solutions of the system are simple.*

As far as we know, most papers face the above question solving first the system of equations and then proving that the solutions are simple, computing the Jacobian on them. Our method is simple and algebraic and works independently of how complicated is the system, assuming of course that it is given by polynomial functions. We will apply this approach in Section 4.2.

Section 2 is devoted to the proof of Theorem 1 and Propositions 2 and 3.

In Section 3 we apply our results to find limit cycles for two families of polynomial autonomous systems. As a first example we consider the planar system

$$\begin{cases} \dot{x} = -y + x^2/2 + \varepsilon P(x,y), \\ \dot{y} = \quad x + xy/2 + \varepsilon Q(x,y), \end{cases}$$

with P and Q quadratic polynomials and prove that we can choose P and Q such that for ε small enough it has at least two limit cycles. This result is already known, see [15], but our proof is different.

As a second family we consider the 3-dimensional polynomial vector field of degree n,

$$\begin{cases} \dot{x} &= -y + \varepsilon a(x^2), \\ \dot{y} &= x + \varepsilon b(z), \\ \dot{z} &= \varepsilon x c(z) + \varepsilon^2 d(z), \end{cases} \tag{4}$$

where a, b, c and d are real polynomials with respective degrees $[n/2]$, n, $n-1$ and n, and ε is a small parameter. We will show that we can choose a, b, c and d such that system (4) has $([n/2]-1)(2n-1)$ limit cycles bifurcating from the continuum of periodic orbits existing for $\varepsilon = 0$, see Proposition 8. Some related results are given in [9, 17].

The above system gives a simple 3-dimensional example for which we can apply Theorem 1 and for which all the computations can be done easily, without the need of using algebraic manipulators. We want to remark that it is not difficult to construct 3-dimensional polynomial vector fields of degree n having more limit cycles. For instance, by considering the two dimensional example of degree n, say $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$, given in [7] which has $O(n^2 \log n)$ limit cycles, we can construct the uncoupled 3-dimensional system

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \quad \dot{z} = R(z),$$

with R a polynomial of degree n having n different real roots, $z = z_i$, $i = 1, \ldots, n$. It has $O(n^3 \log n)$ limit cycles, all of them on planes of the form $\{z = z_i\}$ with $R(z_i) = 0$, $i = 1, \ldots, n$. In fact there are examples of polynomial systems in $\mathbb{R}^3$ having infinitely many limit cycles, see [3, 11].

In Section 4 we present a couple of examples dealing with generalized Abel equations. First we study the number of limit cycles arising from the equation

$$\frac{dr}{dt} = \varepsilon \sum_{j=0}^{N} f_{1,j}(t) r^j + \varepsilon^2 \sum_{j=0}^{N} f_{2,j}(t) r^j + \varepsilon^3 \sum_{j=0}^{N} f_{3,j}(t) r^j,$$

where $f_{i,j}$ is a smooth real T-periodic function in t, and $N \geq 3$ is an arbitrary positive integer number. Note that if $N \in \{0, 1\}$, then this equation is linear and it is well known that linear equations have either a continuum of periodic solutions or at most one limit cycle. In case $N = 2$ it is a Riccati equation and it has at most two limit cycles, see for instance [16, 20]. When $N = 3$, equation (16) is called Abel equation. There is no upper bound for the number of limit cycles of Abel equations and for equations that are polynomial in r of degree N, with $N \geq 3$, see [12, 16, 22].

In Proposition 10 we show that for $N \geq 3$, $M_i(\rho)$, $i = 1, 2, 3$ are polynomials in ρ of degree at most N, $2N - 2$ or $3N - 3$, respectively, and that these degrees are attained. Moreover, when $N \leq 6$, it is possible to find examples having all the roots real and simple. This result also helps to understand why for polynomial equations in r of degree $N \geq 3$ the number of limit cycles is unbounded. Each higher order of perturbation in ε gives rise to more limit cycles.

The proof of [16] showing that there are Abel equations with an arbitrarily high number of limit cycles is based on studying the non-autonomous differential equation of the form

$$\dot{r} = a(t) r^2 + \varepsilon f(t) r^3,$$

for suitable trigonometric polynomials a and f. Later on, this result has been extended to differential equations of the form

$$\dot{r} = a(t) r^n + \varepsilon f(t) r^m,$$

for most natural numbers n and m, see [12]. Both works use the first order variational equation and compute $M_1(\rho)$. In our second example of Section 4 we will extend this approach to study systems of coupled generalized Abel equations of the form,

$$\begin{cases} \dot{r} = a(t) r^2 + \varepsilon f(t) r^n s^m, \\ \dot{s} = b(t) s^2 + \varepsilon g(t) r^p s^q, \end{cases}$$

for $n + m \geq 3$, $p + q \geq 3$ and a, b, f and g trigonometric polynomials.

2. Proof of the main results

We introduce some notations. Given $x \in \mathbb{R}^d$, a $d \times d$ matrix $A = (a_{i,j})$ and a $d \times d \times d$ matrix $B = (b_{i,j,k})$ then

$$x^\intercal A x = \sum_i \sum_j a_{i,j} x_i x_j, \quad x^\intercal (Bx)\, x = \sum_i \sum_j \sum_k b_{i,j,k} x_i x_j x_k,$$

where recall that $x^\intercal$ stands for the transposed vector of x.

Our proof of Theorem 1 also allows to get the expression of $M_3(\rho)$ which is given in next remark. Using the same approach it is not difficult of obtain expressions for $M_i(\rho)$, $i \geq 4$ that we omit for the sake of simplicity.

Remark 5. *Assuming the hypotheses of Theorem 1 and the notations of this section, when additionally* $f_0 \in \mathcal{C}^3([0,T]\times U)$,

$$\begin{aligned} M_3(\rho) \;\; = \;\; & \int_0^T (D_\rho\varphi_0(t,\rho))^{-1}\Big(u_1^\intercal(t,\rho)D_{\rho\rho}f_0(t,\varphi_0(t,\rho))u_2(t,\rho) \\ & +\frac{1}{6}u_1^\intercal(t,\rho)\left(D_{\rho\rho\rho}f_0(t,\varphi_0(t,\rho))u_1(t,\rho)\right)u_1(t,\rho) \\ & +D_\rho f_1(t,\varphi_0(t,\rho))u_2(t,\rho)+\frac{1}{2}u_1^\intercal(t,\rho)D_{\rho\rho}f_1(t,\varphi_0(t,\rho))u_1(t,\rho) \\ & +D_\rho f_2(t,\varphi_0(t,\rho))u_1(t,\rho)+f_3(t,\varphi_0(t,\rho))\Big)dt. \end{aligned}$$

Proof of Theorem 1 and Remark 5. By imposing that $\varphi_\varepsilon(t,\rho)$, given in (3), is a solution of equation (2) we get the identity

$$\begin{aligned} \frac{\partial\varphi_0(t,\rho)}{\partial t} + \sum_{i=1}^{3}\frac{\partial u_i(t,\rho)}{\partial t}\varepsilon^i + O(\varepsilon^4) = {} & f_0(t,\varphi_0(t,\rho)) + D_\rho f_0(t,\varphi_0(t,\rho))h(t,\rho,\varepsilon) \\ & + \frac{1}{2}h^\intercal(t,\rho,\varepsilon)D_{\rho\rho}f_0(t,\varphi_0(t,\rho))h(t,\rho,\varepsilon) \\ & + \frac{1}{6}h^\intercal(t,\rho,\varepsilon)\left(D_{\rho\rho\rho}f_0(t,\varphi_0(t,\rho))h(t,\rho,\varepsilon)\right)h(t,\rho,\varepsilon) + O(\varepsilon^4) \\ & + \Bigg(f_1(t,\varphi_0(t,\rho)) + D_\rho f_1(t,\varphi_0(t,\rho))h(t,\rho,\varepsilon) \\ & + \frac{1}{2}h^\intercal(t,\rho,\varepsilon)D_{\rho\rho}f_1(t,\varphi_0(t,\rho))h(t,\rho,\varepsilon) + O(\varepsilon^3)\Bigg)\varepsilon \\ & + \Big(f_2(t,\varphi_0(t,\rho)) + D_\rho f_2(t,\varphi_0(t,\rho))h(t,\rho,\varepsilon) + O(\varepsilon^2)\Big)\varepsilon^2 \\ & + \Big(f_3(t,\varphi_0(t,\rho)) + O(\varepsilon)\Big)\varepsilon^3, \end{aligned} \tag{5}$$

where $h(t,\rho,\varepsilon) = \varepsilon u_1(t,\rho) + \varepsilon^2 u_2(t,\rho) + O(\varepsilon^3)$, and if $f_m = (f_{m,k})_{k=1,\ldots,d}$, $m = 0,\ldots,3$, then $D_\rho f_m$ and $D_{\rho\rho} f_m$ stand for the Jacobian and the Hessian matrix, respectively, of f_m

$$D_\rho f_m = \left(\frac{\partial f_{m,k}}{\partial\rho_i}\right)_{k=1,\ldots,d}, \quad D_{\rho\rho} f_m = \left(\left(\frac{\partial^2 f_{m,k}}{\partial\rho_i\partial\rho_j}\right)\right)_{k=1,\ldots,d}$$

and $D_{\rho\rho\rho} f_0$ denotes the third order derivative matrix of f_0 written as

$$D_{\rho\rho\rho} f_0 = \left(\left(\frac{\partial^3 f_{0,k}}{\partial\rho_i\partial\rho_j\partial\rho_l}\right)\right)_{k=1,\ldots,d}.$$

By collecting terms in ε, ε^2 and ε^3 into expression (5), we obtain the following three linear non-homogeneous differential equations for the unknown functions

$u_i(t\rho)$, $i=1,2,3$,

$$
\begin{aligned}
\frac{\partial u_1}{\partial t}(t,\rho) &= D_\rho f_0(t,\varphi_0(t,\rho))u_1(t,\rho)+f_1(t,\varphi_0(t,\rho)), \qquad (6)\\
\frac{\partial u_2}{\partial t}(t,\rho) &= D_\rho f_0(t,\varphi_0(t,\rho))u_2(t,\rho)+\frac{1}{2}u_1^{\intercal}(t,\rho)D_{\rho\rho}f_0(t,\varphi_0(t,\rho))u_1(t,\rho)\\
&\quad +D_\rho f_1(t,\varphi_0(t,\rho))u_1(t,\rho)+f_2(t,\varphi_0(t,\rho)), \qquad (7)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial u_3}{\partial t}(t,\rho) &= D_\rho f_0(t,\varphi_0(t,\rho))u_3(t,\rho)+u_1^{\intercal}(t,\rho)D_{\rho\rho}f_0(t,\varphi_0(t,\rho))u_2(t,\rho)\\
&\quad +\frac{1}{6}u_1^{\intercal}(t,\rho)\left(D_{\rho\rho\rho}f_0(t,\varphi_0(t,\rho))u_1(t,\rho)\right)u_1(t,\rho)\\
&\quad +D_\rho f_1(t,\varphi_0(t,\rho))u_2(t,\rho)+\frac{1}{2}u_1^{\intercal}(t,\rho)D_{\rho\rho}f_1(t,\varphi_0(t,\rho))u_1(t,\rho)\\
&\quad +D_\rho f_2(t,\varphi_0(t,\rho))u_1(t,\rho)+f_3(t,\varphi_0(t,\rho)). \qquad (8)
\end{aligned}
$$

Recall that the solution of the Cauchy problem for the linear differential equation

$$\frac{dx}{dt}=a(t)x+b(t), \quad x(0)=0,$$

is given by $x(t)=A(t)\int_0^t A^{-1}(s)b(s)ds$, where $A(t)$ is a fundamental matrix of the homogeneous equation. It is well known that $\frac{\partial\varphi_0}{\partial\rho}(t,\rho)$ is a fundamental matrix of the homogeneous linear equation

$$\frac{\partial u_i}{\partial t}(t,\rho) = D_\rho f_0(t,\varphi_0(t,\rho))u_i(t,\rho).$$

for $i=1,2,3$. This holds because $\frac{\partial\varphi_0}{\partial t}(t,\rho)=f_0(t,\varphi_0(t,\rho))$, and then

$$\frac{\partial}{\partial t}\left(\frac{\partial\varphi_0}{\partial\rho}(t,\rho)\right)=D_\rho f_0(t,\varphi_0(t,\rho))\frac{\partial\varphi_0}{\partial\rho}(t,\rho).$$

Hence, by imposing the periodicity condition $\varphi_0(T,\rho)=\rho$, we obtain the solutions of equations (6)-(8) at $t=T$, in the form

$$u_i(T,\rho)=\int_0^T (D_\rho\varphi_0(s,\rho))^{-1}b_i(s)ds,$$

where $b_i(s)$ is the non-homogeneous part of equations (6)-(8), respectively, as we wanted to prove.

To end the proof, let us show that each simple zero of $M_1(\rho)$ gives rise, for ε small enough, to an isolated T-periodic solution of system (2). From (3), $\varphi_\varepsilon(T,\rho)=\rho+\varepsilon M_1(\rho)+O(\varepsilon^2)$. Then,

$$\Pi(\rho,\varepsilon):=\frac{\varphi_\varepsilon(T,\rho)-\rho}{\varepsilon}=M_1(\rho)+O(\varepsilon).$$

Since $\varphi_0(0,\rho)=\varphi_0(T,\rho)=\rho$, we have that

$$\Pi(\rho,0)=0 \quad \text{and} \quad \left.\frac{\partial\Pi(\rho,\varepsilon)}{\partial\rho}\right|_{\varepsilon=0}=M_1'(\rho).$$

Then, from the implicit function Theorem the result follows.

In general, if we assume $M_i(\rho) \equiv 0$, for $i = 1, ..., k-1$, then we can apply the same argument to

$$\frac{\varphi_\varepsilon(T,\rho)-\rho}{\varepsilon^k} = M_k(\rho) + O(\varepsilon).$$

$\square$

The proof of Proposition 2 follows using again the above arguments. In next result we also give the expression of M_5.

Remark 6. *Under the hypotheses and notations of Proposition 2, when additionally* $f_4 \in \mathcal{C}^4([0,T]\times U)$,

$$\begin{aligned} M_5(\rho) = \int_0^T &\Big(\frac{1}{24}\frac{\partial^4 f_1}{\partial\rho^4}(t,\rho)u_1^3(t,\rho) + \frac{1}{2}\frac{\partial^3 f_1}{\partial\rho^3}(t,\rho)u_1^2(t,\rho)u_2(t,\rho) \\ &+ \frac{\partial^2 f_1}{\partial\rho^2}(t,\rho)\big(\frac{1}{2}u_2^2(t,\rho) + u_1(t,\rho)u_3(t,\rho)\big) + \frac{\partial f_1}{\partial\rho}(t,\rho)u_4(t,\rho) \\ &+ \frac{1}{6}\frac{\partial^3 f_2}{\partial\rho^3}(t,\rho)u_1^3(t,\rho) + \frac{\partial^2 f_2}{\partial\rho^2}(t,\rho)u_1(t,\rho)u_2(t,\rho) + \frac{\partial f_2}{\partial\rho}(t,\rho)u_3(t,\rho) \\ &+ \frac{1}{2}\frac{\partial^2 f_3}{\partial\rho^2}(t,\rho)u_1^2(t,\rho) + \frac{\partial f_3}{\partial\rho}(t,\rho)u_2(t,\rho) + \frac{\partial f_4}{\partial\rho}(t,\rho)u_1(t,\rho) + f_5(t,\rho)\Big)\,dt. \end{aligned}$$

To end this section we will prove Proposition 3.

Proof of Proposition 3. Let $(x_0, y_0) \in \mathbb{C}^2$ be a multiple solution of the system. Recall that at any multiple solution the Jacobian function $J(x,y)$ vanishes. Therefore $P(x_0,y_0) = Q(x_0,y_0) = J(x_0,y_0) = 0$ we have that $R^x(y_0) = S^x(y_0) = R^y(x_0) = S^y(x_0) = 0$. Hence $T_1 = T_2 = 0$. Therefore the result follows. $\square$

3. Applications to autonomous systems

In this section we apply Theorem 1 to two families of autonomous polynomial differential equations. The first one is a planar system and the second one is a three dimensional polynomial system.

3.1. **A family of planar vector fields.** Consider the family of differential equations

$$\begin{cases} \dot{x} = -y + xH_{n-1}(x,y) + \varepsilon P(x,y), \\ \dot{y} = x + yH_{n-1}(x,y) + \varepsilon Q(x,y), \end{cases} \tag{9}$$

where H_{n-1} is a homogeneous polynomial of degree $n-1$ and P and Q are polynomials of degree n. When $\varepsilon = 0$ and $\int_0^{2\pi} H_{n-1}(\cos s, \sin s)\,ds = 0$, the above family is formed by isochronous centers. Passing to (r,θ)-polar coordinates, it writes equivalently as

$$\begin{aligned} \frac{dr}{d\theta} &= \frac{H_{n-1}(\cos\theta,\sin\theta)r^n + \varepsilon R(r\cos\theta, r\sin\theta)}{1+\varepsilon S(r\cos\theta, r\sin\theta)} \\ &= H_{n-1}(\cos\theta,\sin\theta)r^n + \varepsilon T(r\cos\theta, r\sin\theta) + O(\varepsilon^2), \end{aligned} \tag{10}$$

where

$$R(r\cos\theta, r\sin\theta) = \cos\theta P(r\cos\theta, r\sin\theta) + \sin\theta Q(r\cos\theta, r\sin\theta),$$
$$S(r\cos\theta, r\sin\theta) = \left(\cos\theta Q(r\cos\theta, r\sin\theta) - \sin\theta P(r\cos\theta, r\sin\theta)\right)/r,$$
$$T(r\cos\theta, r\sin\theta) = R(r\cos\theta, r\sin\theta) - S(r\cos\theta, r\sin\theta) H_{n-1}(\cos\theta, \sin\theta) r^n.$$

For $\varepsilon = 0$ it is is easy to obtain its solution as

$$\varphi_0(\theta,\rho) = \frac{\rho}{\sqrt[n-1]{1-(n-1)\rho^{n-1}\int_0^\theta H_{n-1}(\cos s, \sin s)\, ds}}. \tag{11}$$

This expression proves the isochronism of the unperturbed system and allows us to get the function $M_1(\rho)$ associated to (10). To illustrate its applicability we present a simple quadratic example, with $n = 2$ and $H_1(x, y) = x/2$, having two limit cycles, already studied in [15] using Abelian integrals.

Proposition 7. *For ε small enough, there are systems of the form*

$$\begin{cases} \dot{x} = -y + x^2/2 + \varepsilon P(x, y), \\ \dot{y} = x + xy/2 + \varepsilon Q(x, y), \end{cases}$$

with P and Q quadratic polynomials having at least two limit cycles.

Proof. Take $P(x, y)$ and $Q(x, y)$ arbitrary quadratic polynomials. From (11) and Theorem 1, we get that

$$M_1(\rho) = \frac{1}{4}\int_0^{2\pi} (2 - \rho\sin s)^2 f_1(s, \varphi_0(s,\rho))\, ds,$$

where

$$f_1(s, r) = (\cos s\, P + \sin s\, Q) - (\cos s\, Q - \sin s\, P)\frac{r\cos s}{2},$$

being $P = P(r\cos s, r\sin s)$, $Q = Q(r\cos s, r\sin s)$ and $\varphi_0(s,\rho) = 2\rho/(2-\rho\sin s)$ and $0 < \rho < 2$. Hence, after some computations,

$$M_1(\rho) = \frac{p_1(\rho^2) + q_1(\rho^2)\sqrt{4-\rho^2}}{\rho},$$

where p_1 and q_1 are linear polynomials satisfying $p_1(0)+2q_1(0) = 0$. By introducing the new variable $\tau^2 = 4 - \rho^2$, for $0 < \tau < 2$, we get that $\rho M_1(\rho)$ writes as

$$N(\tau) = (\tau - 2)p_2(\tau),$$

being p_2 an arbitrary polynomial of degree 2. Hence, taking P and Q such that p_2 has two simple zeroes in the interval $(0, 2)$, we get that the corresponding quadratic system has two limit cycles, for ε small enough, as we wanted to prove. □

3.2. **A three dimensional polynomial example.**

Proposition 8. *There exist polynomials a, b, c and d such that, for ε small enough, the system* (4) *has* $([n/2] - 1)(2n - 1)$ *limit cycles.*

Proof. Using polar coordinates, (r,θ), for the variables (x,y), system (4) writes as

$$(\frac{dr}{d\theta},\frac{dz}{d\theta}) = \varepsilon f_1(\theta,r,z) + \varepsilon^2 f_2(\theta,r,z) + O(\varepsilon^3)$$

where $f_i = (f_{i,1}, f_{i,2})$, $i = 1,2$ and

$$\begin{aligned}
f_{1,1}(\theta,r,z) &= \cos\theta\, a(r^2\cos^2\theta) + \sin\theta\, b(z),\\
f_{1,2}(\theta,r,z) &= r\cos\theta\, c(z),\\
f_{2,1}(\theta,r,z) &= \frac{1}{r}\left(\cos\theta\, a(r^2\cos^2\theta) + \sin\theta\, b(z)\right)\left(\sin\theta\, a(r^2\cos^2\theta) - \cos\theta\, b(z)\right),\\
f_{2,2}(\theta,r,z) &= d(z) - c(z)\left(\cos^2\theta\, b(z) - \sin\theta\cos\theta\, a(r^2\cos^2\theta)\right).
\end{aligned}$$

From Theorem 1, if we write $\rho = (\rho_1,\rho_2)$ and $u_i = (u_{i,1}, u_{i,2})$, $i = 1,2$, then

$$u_{1,1}(\theta,\rho) = \int_0^\theta f_{1,1}(\psi,\rho)\, d\psi = \int_0^\theta \cos\psi\, a(\rho_1^2\cos^2\psi)\, d\psi + (1-\cos\theta)\, b(\rho_2).$$

Since

$$\int_0^\theta \cos^{2k+1}\psi\, d\psi = \sin\theta\, P(\cos^2\theta), \tag{12}$$

for some polynomial function P, by taking $M_i = (M_{i,1}, M_{i,2})$, $i = 1,2$, we easily obtain that $M_{1,1}(\rho) \equiv 0$. Similarly

$$u_{1,2}(\theta,\rho) = \int_0^\theta f_{1,2}(\psi,\rho)\, d\psi = \rho_1\, c(\rho_2)\, \sin\theta,$$

and so $M_{1,2}(\rho) \equiv 0$. Now we proceed by computing $M_2(\rho)$. Again by Theorem 1, we get

$$M_2(\rho) = \int_0^{2\pi} \left(D_\rho f_1(\theta,\rho)\, u_1(\theta,\rho) + f_2(\theta,\rho)\right)\, d\theta.$$

Note that

$$D_\rho f_1(\theta,\rho)\, u_1(\theta,\rho) = \begin{pmatrix} a'(\rho_1^2\cos^2\theta)\, 2\rho_1\, \cos^3\theta & \sin\theta\, b'(\rho_2) \\ \cos\theta\, c(\rho_2) & \rho_1\, \cos\theta\, c'(\rho_2) \end{pmatrix} \begin{pmatrix} u_{1,1}(\theta,\rho) \\ \rho_1\, c(\rho_2)\, \sin\theta \end{pmatrix}.$$

Hence

$$
\begin{aligned}
M_{2,1}(\rho) = & \int_0^{2\pi} \Big(a'(\rho_1^2\cos^2\theta) 2\rho_1 \cos^3\theta \, u_{1,1}(\theta,\rho) + \rho_1\, c(\rho_2)\, b'(\rho_2)\, \sin^2\theta \\
& + \frac{1}{\rho_1} \big(\cos\theta\, a(\rho_1^2\cos^2\theta) + \sin\theta\, b(\rho_2)\big) \big(\sin\theta\, a(\rho_1^2\cos^2\theta) - \cos\theta\, b(\rho_2)\big) \Big)\, d\theta \\
= & \int_0^{2\pi} \Big(a'(\rho_1^2\cos^2\theta) 2\rho_1 \cos^3\theta \int_0^\theta \cos\psi\, a(\rho_1^2\cos^2\psi)\, d\psi \\
& + a'(\rho_1^2\cos^2\theta) 2\rho_1\, \cos^3\theta\, (1-\cos\theta)\, b(\rho_2) \Big)\, d\theta + \rho_1\, c(\rho_2)\, b'(\rho_2)\pi \\
& + b(\rho_2) \int_0^{2\pi} \frac{1}{\rho_1} (\sin^2\theta - \cos^2\theta)\, a(\rho_1^2\cos^2\theta)\, d\theta \\
= & \rho_1\, c(\rho_2)\, b'(\rho_2)\pi \\
& + b(\rho_2) \int_0^{2\pi} \Big(\frac{(1-2\cos^2\theta)\, a(\rho_1^2\cos^2\theta)}{\rho_1} - 2\rho_1 a'(\rho_1^2\cos^2\theta)\cos^4\theta \Big)\, d\theta,
\end{aligned}
$$

where in the last equality we have used again property (12). It is well known that

$$
\int_0^{2\pi} \cos^{2m}\theta\, d\theta = 2\pi \frac{(2m-1)!!}{(2m)!!},
$$

where $0!! = 1!! = 1$ and $2!! = 2$. Hence, by taking $a(x) = \sum_{i=0}^{[n/2]} a_i x^i$, some simple computations give that

$$
M_{2,1}(\rho) = \pi\rho_1 \Big(c(\rho_2)\, b'(\rho_2) - 4b(\rho_2) \sum_{j=1}^{[n/2]} j a_j\, \frac{(2j-1)!!}{(2j)!!} \rho_1^{2j-2} \Big). \tag{13}
$$

The computation of $M_{2,2}(\rho)$ is similar. We have

$$
\begin{aligned}
M_{2,2}(\rho) = \int_0^{2\pi} \Big(& \cos\theta\, u_{1,1}(\theta,\rho) c(\rho_2) + \rho_1^2 \sin\theta\cos\theta\, c(\rho_2) c'(\rho_2) + d(\rho_2) \\
& - c(\rho_2)(\cos^2\theta\, b(\rho_2) - \sin\theta\cos\theta\, a(\rho_1^2\cos^2\theta)) \Big)\, d\theta.
\end{aligned}
$$

Hence,

$$
M_{2,2}(\rho) = \pi(2d(\rho_2) - b(\rho_2)\, c(\rho_2)). \tag{14}
$$

Thus, from (13) and (14), we get the explicit expression of $M_2(\rho)$.

By taking $b(z) = z^n$ we obtain that $M_{2,2}(\rho)$ can be any polynomial in ρ_2 of degree $2n-1$. Fix c and d such that $M_{2,2}(\rho)$ has $2n-1$ non-zeros real roots. Then for each one of these roots $\rho_2 = \rho_{2,i}$, $i = 1, \ldots, 2n-1$, consider the values $k_i = (c(\rho_{2,i})\, b'(\rho_{2,i}))/(4b(\rho_{2,i})) = nc(\rho_{2,i})/(4\rho_{2,i})$. It is possible to choose the numbers a_j such that for each $i = 1, \ldots, 2n-1$ the equation

$$
\sum_{j=1}^{[n/2]} j a_j \frac{(2j-1)!!}{(2j)!!} \rho_1^{2j-2} = k_i, \tag{15}
$$

has exactly $[n/2]-1$ simple positive solutions, say $\rho_{1,i_1},\ldots,\rho_{1,i_{[n/2]-1}}$. This can be seen by noting that, taking suitable a_j, the left hand side of equation (15) can be taken as an arbitrary polynomial in the variable ρ_1^2.

Thus, system (4) has $(2n-1)([n/2]-1)$ limit cycles that tend, when ε goes to zero, to the periodic orbits $r=\rho_{1,i_j}$, $z=\rho_{2,i}$, $i=1,\ldots,2n-1$, $j=1,\ldots,[n/2]-1$. □

4. Applications to non-autonomous systems

In this section we apply our results to two families of non-autonomous differential equations of Abel type.

4.1. **A generalized Abel equation.** In this section we study the number of limit cycles arising from polynomial differential equations of the form,

$$\frac{dr}{dt}=\varepsilon\sum_{j=0}^{N}f_{1,j}(t)r^j+\varepsilon^2\sum_{j=0}^{N}f_{2,j}(t)r^j+\varepsilon^3\sum_{j=0}^{N}f_{3,j}(t)r^j, \tag{16}$$

where $f_{i,j}$ is a real and smooth T-periodic function in t, and $N\geq 3$.

We introduce some notation. Given two integrable function f and g, define

$$\widetilde{f}(t):=\int_0^t f(s)\,ds, \qquad \widetilde{g\widetilde{f}}\,(t):=\int_0^t f(s)\int_0^s g(w)\,dw\,ds.$$

Then the relations given in the following lemma hold.

Lemma 9. *Let f and g smooth T-periodic functions in t, $t\in[0,T]$, such that $\widetilde{f}(T)=\widetilde{g}(T)=0$. If $I=\int_0^T g(t)(\widetilde{f}(t))^2\,dt$, then:*

(i) $\int_0^T f(t)\widetilde{f(t)}\widetilde{g}(t)\,dt=-I/2$, *(ii)* $\int_0^T f(t)\widetilde{f\widetilde{g}}(t)\,dt=I/2$,
(iii) $\int_0^T g(t)\widetilde{f\widetilde{f}}(t)\,dt=I/2$, *(iv)* $\int_0^T f(t)\widetilde{g\widetilde{f}}(t)\,dt=-I$,
(v) $\int_0^T f(t)\widetilde{f\widetilde{f}}(t)\,dt=0$.

We prove:

Proposition 10. *Consider equation (16) with $N\geq 3$. Then, the function $M_i(\rho)$ is a polynomial in ρ of degree at most N, $2N-2$ or $3N-3$ when $i=1,2$ or 3, respectively, and these upper bounds are sharp. Moreover, for $N\leq 6$, there are suitable choices of the functions $f_{i,j}(t)$ for which all roots of $M_i(\rho)$ are real and simple.*

Proof. From Proposition 2, we have

$$M_1(\rho)=u_1(T,\rho)=\sum_{j=0}^{N}\int_0^T f_{1,j}(t)\,dt\,\rho^j.$$

To get $M_2(\rho)$ we impose that $M_1(\rho) \equiv 0$, i.e. that $\int_0^T f_{1,j}(t)\,dt = 0$, for all $j = 1,\dots,N$. From Proposition 2, we have that

$$M_2(\rho) = \left(N\int_0^T f_{1,N}(t)\widetilde{f_{1,N}}(t)\,dt\right)\rho^{2N-1} + \left(N\int_0^T f_{1,N}(t)\widetilde{f_{1,N-1}}(t)\,dt \right.$$
$$\left. + (N-1)\int_0^T f_{1,N-1}(t)\widetilde{f_{1,N}}(t)\,dt\right)\rho^{2N-2} + Q_{2N-3}(\rho),$$

where the coefficient of ρ^{2N-1} is zero, because $\int_0^T f_{1,N}(t)\,dt = 0$, and $Q_{2N-3}(\rho)$ is a polynomial of degree $2N-3$ in the variable ρ. To obtain $M_3(\rho)$ we impose that $M_2(\rho) \equiv 0$. In particular, from the coefficient of ρ^{2N-2}, we need to have

$$\int_0^T \widetilde{f}_{1,N}(t)f_{1,N-1}(t)\,dt = 0. \tag{17}$$

From Proposition 2 the coefficients of the highest order terms in ρ in $M_3(\rho)$ are

$$M_3(\rho) = \left(\tfrac{N(N-1)}{2}\int_0^T f_{1,N}(t)(\widetilde{f}_{1,N}(t))^2\,dt + N^2\int_0^T f_{1,N}(t)\widetilde{f_{1,N}\widetilde{f}_{1,N}}(t)\,dt\right)\rho^{3N-2}$$
$$+ \left(N(N-1)\int_0^T f_{1,N}(t)\widetilde{f}_{1,N}(t)\widetilde{f}_{1,N-1}(t)\,dt\right.$$
$$+ \tfrac{(N-1)(N-2)}{2}\int_0^T (\widetilde{f}_{1,N}(t))^2 f_{1,N-1}(t)\,dt + N^2\int_0^T f_{1,N}(t)\widetilde{f_{1,N}\widetilde{f}_{1,N-1}}(t)\,dt$$
$$+N(N-1)\int_0^T f_{1,N-1}(t)\widetilde{f_{1,N}\widetilde{f}_{1,N}}(t)\,dt$$
$$\left. + N(N-1)\int_0^T f_{1,N}(t)\widetilde{f_{1,N-1}\widetilde{f}_{1,N}}(t)\,dt\right)\rho^{3N-3} + Q_{3N-4}(\rho),$$

where $Q_{3N-4}(\rho)$ is a polynomial of degree $3N-4$ in the variable ρ. From Lemma 9 and since $\int_0^T f_{1,N}(t)\,dt = 0$, $M_3(\rho)$ writes as

$$M_3(\rho) = \frac{2-N}{2}\int_0^T f_{1,N-1}(t)(\widetilde{f}_{1,N}(t))^2\,dt\,\rho^{3N-3} + Q_{3N-4}(\rho).$$

We note that condition $\int_0^T f_{1,N-1}(t)(\widetilde{f}_{1,N}(t))^2\,dt \neq 0$ is compatible with condition (17), i.e. it is possible to get functions $f_{1,N-1}$ and $f_{1,N}$ such that both conditions are satisfied. This fact proves the first part of the proof.

To prove that the functions $M_i(\rho)$ can be chosen arbitrarily, let us take $f_{1,j}(t) = \alpha_j + \beta_j\cos t + \gamma_j(\sin^2 t - 1/2)$, $f_{2,j}(t) = \delta_j + \eta_j\cos t$ and $f_{3,j} = \lambda_j$, where α_j, β_j, γ_j, δ_j, η_j and λ_j are arbitrary real numbers, $j = 0,\dots,N-1$, $f_{1,N} = \alpha_N + \sin^4 t + \phi\sin t - 3/8$, $f_{2,N} = \delta_N + \sin t$ and $f_{3,N} = \lambda_N$ is an arbitrary real constant. We note that these functions are 2π-periodic.

For simplicity, we only present the details for the case $N = 3$. The other cases can be studied similarly. To simplify calculations, we take: $\beta_0 = \beta_1 = \gamma_0 = 0$.

From Proposition 2 we have that,

$$\begin{aligned}
M_1(\rho) &= 2\pi \sum_{i=0}^{3} \alpha_i \rho^i,\\
M_2(\rho) &= \frac{\beta_2\phi}{2}\rho^4 + \delta_3\rho^3 + \delta_2\rho^2 + \delta_1\rho + \delta_0,\\
M_3(\rho) &= \rho^6 + (128\beta_2^2 - 2)\rho^5 + (5\gamma_1 + 512\beta_2)\rho^4 + (-4\gamma_1^2 + 1024\lambda_3)\rho^3\\
&\quad + 1024\lambda_2\rho^2 + 1024\lambda_1\rho + 1024\lambda_0.
\end{aligned}$$

Recall that to obtain expression $M_3(\rho)$, it is necessary to have $M_2(\rho) \equiv 0$. To achieve this condition, $\beta_2\phi = 0$, and we have fixed $\phi = 0$. In order to finish the proof, we observe that all the coefficients of the first two polynomials can be arbitrarily chosen. Additionally, taking β_2 such that $128\beta_2^2 - 2 = 0$, $M_3(\rho)$ is an arbitrary monic polynomial of degree six without the term ρ^5. By means of a suitable translation, it is easy to see that any polynomial of degree six can be written in this way. Hence the result follows. □

4.2. **A planar non-autonomous system.** Consider the system of coupled generalized Abel equations:

$$\begin{cases} \dot r = a(t)\, r^2 + \varepsilon f(t)\, r^n s^m,\\ \dot s = b(t)\, s^2 + \varepsilon g(t\,) r^p s^q, \end{cases} \tag{18}$$

for $n+m \geq 3$, $p+q \geq 3$ and a, b, f and g, T-periodic trigonometric polynomials.

For any $\rho = (\rho_1, \rho_2)$, let us denote by $\varphi_0(t,\rho)$ the solution of equation (18), for $\varepsilon = 0$, such that $\varphi_0(0,\rho) = (\rho_1, \rho_2)$. In this case,

$$\varphi_0(t,\rho) = \left(\frac{\rho_1}{1 - A(t)\rho_1}, \frac{\rho_2}{1 - B(t)\rho_2} \right),$$

where $A(t) = \int_0^t a(s)\, ds$ and $B(t) = \int_0^t b(s)\, ds$. We assume that $A(T) = B(T) = 0$ to ensure that the unperturbed system has a continuum of periodic orbits. Applying Theorem 1 we need to compute

$$(D_{(\rho)}\varphi_0(t,\rho))^{-1} = \begin{pmatrix} (1 - A(t)\rho_1)^2 & 0\\ 0 & (1 - B(t)\rho_2)^2 \end{pmatrix}.$$

Hence

$$M_1(\rho) = \begin{pmatrix} \rho_1^n \rho_2^m \int_0^T \dfrac{f(t)}{(1 - A(t)\rho_1)^{n-2}(1 - B(t)\rho_2)^m}\, dt\\ \rho_1^p \rho_2^q \int_0^T \dfrac{g(t)}{(1 - A(t)\rho_1)^{p}(1 - B(t)\rho_2)^{q-2}}\, dt \end{pmatrix}. \tag{19}$$

Proposition 11. *There are systems of the form*

$$\begin{cases} \dot r = (\cos t)\, r^2 + \varepsilon(F_1 + F_2 \sin t + \sin^2 t)\, r\, s^2,\\ \dot s = (\cos t)\, s^2 + \varepsilon(G_1 + G_2 \sin t + \sin^2 t)\, r^2 s, \end{cases} \tag{20}$$

where F_i and G_i, $i = 1, 2$, are real constants and ε is small enough, having four isolated limit cycles.

Proof. By way of notation, let us write $F = (F_1, F_2)$ and $G = (G_1, G_2)$. For system (20), if $M_1(\rho) = (M_{1,1}(\rho), M_{1,2}(\rho))$ then, by (19),

$$M_{1,1}(\rho) = \rho_1\rho_2^2 \int_0^{2\pi} \frac{(F_1 + F_2 \sin t + \sin^2 t)(1 - \rho_1 \sin t)}{(1 - \rho_2 \sin t)^2}\, dt.$$

Hence, by assuming that $|\rho_2| < 1$,

$$M_{1,1}(\rho) = \frac{2\rho_1\pi}{\rho_2(1-\rho_2^2)^{3/2}} \left(P_F(\rho_1,\rho_2) + Q_F(\rho_1,\rho_2)\sqrt{1-\rho_2^2} \right),$$

where

$$P_F(\rho_1,\rho_2) = 2\rho_1 - \rho_2 - 3\rho_1\rho_2^2 + 2\rho_2^3 + \left(\rho_2^3 - \rho_1\rho_2^4\right) F_1 + \left(\rho_1\rho_2 - 2\rho_1\rho_2^3 + \rho_2^4\right) F_2,$$
$$Q_F(\rho_1,\rho_2) = -2\rho_1 + \rho_2 + 2\rho_1\rho_2^2 - \rho_2^3 + \left(-\rho_1\rho_2F_2 + \rho_1\rho_2^3\right) F_2.$$

We note that $M_{1,2}(\rho_2,\rho_1) = M_{1,1}(\rho_1,\rho_2)$, by replacing F_i by G_i, $i = 1, 2$, in $M_{1,1}$ and assuming that $|\rho_1| < 1$.

Also we observe that, if $\rho_1 = 0$ (resp.: $\rho_2 = 0$), then $M_1(\rho_1,\rho_2) = 0$, for all ρ_2 (resp.: ρ_1). Hence, simple solutions of the system

$$\frac{(1-\rho_2^2)^{3/2}}{2\pi\rho_1} M_{1,1}(\rho) = 0, \qquad \frac{(1-\rho_1^2)^{3/2}}{2\pi\rho_2} M_{1,2}(\rho) = 0, \tag{21}$$

in $\mathcal{D} := \{(\rho_1,\rho_2) \in \mathbb{R}^2 : |\rho_1| < 1, |\rho_2| < 1, \rho_1\rho_2 \neq 0\}$, give rise to limit cycles of system (20), for $|\varepsilon|$ small enough.

The simple zeroes of system (21) in $\mathcal{D}$ are zeroes of the polynomial system

$$\begin{cases} P_F^2(\rho_1,\rho_2) - Q_F^2(\rho_1,\rho_2)(1-\rho_2^2) = 0, \\ P_G^2(\rho_2,\rho_1) - Q_G^2(\rho_2,\rho_1)(1-\rho_1^2) = 0, \end{cases} \tag{22}$$

in the same domain.

For many values of F and G, system (22) only has one solution in $\mathcal{D}$. We have found several examples with four solutions. For instance, we fix

$$F = \overline{F} = (-1/2, -1/10) \quad \text{and} \quad G = \overline{G} = (-3/5, 3/10).$$

For these values of F and G, system (22) writes as

$$\begin{cases} \Phi_1(\rho_1,\rho_2) := P_{\overline{F}}^2(\rho_1,\rho_2) - Q_{\overline{F}}^2(\rho_1,\rho_2)(1-\rho_2^2) = 0, \\ \Phi_2(\rho_1,\rho_2) := P_{\overline{G}}^2(\rho_2,\rho_1) - Q_{\overline{G}}^2(\rho_2,\rho_1)(1-\rho_1^2) = 0. \end{cases}$$

We will use the Gröbner basis approach to solve it. Using the lexicographic order, we get that the ideal generated by Φ_1 and Φ_2 is the same that the ideal generated by four functions, say $B_i(\rho_1,\rho_2), i = 1, \ldots, 4$. In fact,

$$B_1(\rho_1,\rho_2) = B_1(\rho_2) = k\,\mathrm{Res}\,(\Phi_1(\rho_1,\rho_2), \Phi_2(\rho_1,\rho_2), \rho_1) = \rho_2^{23}(\rho_2^2-1)^6 q_{23}(\rho_2),$$
$$B_2(\rho_1,\rho_2) = p_{10}(\rho_2)\rho_1 + p_{39}(\rho_2),$$
$$B_3(\rho_1,\rho_2) = p_3(\rho_2)\rho_1^2 + p_9(\rho_2)\rho_1 + p_{39}(\rho_2),$$
$$B_4(\rho_1,\rho_2) = p_0(\rho_2)\rho_1^8 + \sum_{i=3}^{7} p_{2,i}(\rho_2)\rho_1^i + p_9(\rho_2)\rho_1 + p_{39}(\rho_2),$$

where $k \in \mathbb{Q}$, p_j or $p_{j,m}$ denote polynomials in $\mathbb{Q}[\rho_2]$, of degree j, not necessarily equals.

Since solving $\Phi_1(\rho_1,\rho_2) = \Phi_2(\rho_1,\rho_2) = 0$ is equivalent to solve $B_i(\rho_1,\rho_2) = 0, i = 1,\ldots,4$, we will deal with this second system, which notice that has a triangular structure. The first equation gives us the ρ_2-coordinates of the possible solutions. Using the other equations we obtain the first coordinate of the solutions. It turns out that the system has exactly 10 solutions in $\mathcal{D}$.

Next we show that all these solutions are simple. To prove this we will apply Proposition 3. Write $J(\rho_1,\rho_2) = \det\left(D\,\Phi(\rho_1,\rho_2)\right)$. Then

$$\begin{aligned}\operatorname{Res}(\Phi_1(\rho_1,\rho_2), J(\rho_1,\rho_2),\rho_1) &= \rho_2^{42}(\rho_2^2-1)^7 p_{34}(\rho_2),\\ \operatorname{Res}(\Phi_2(\rho_1,\rho_2), J(\rho_1,\rho_2),\rho_1) &= \rho_2^{19}(\rho_2^2-1)^7 p_{32}(\rho_2),\end{aligned}$$

and $\widetilde{T}_1 := \operatorname{Res}(p_{34}(\rho_2), p_{32}(\rho_2), \rho_2) \neq 0$, where we denote by p_k a suitable polynomial of degree k with integer coefficients. Notice that instead of computing T_1 we have only used some factors of the former functions for computing $\widetilde{T}_1$, see Remark 4. Since the lines $\rho_2 = 0$ and $\rho_2^2 = 1$ are not in $\mathcal{D}$ we have already proved that the 10 solutions are simple. Finally, we have to check which of them are also zeros of $M_1(\rho)$, or in other words we have to discard the ones introduced by the squaring process. It is not difficult to see that only 4 of them remain. Their approximate values are:

$$\begin{aligned}&(0.5687347144, 0.4908073644), \quad (0.4399109306, 0.1414187169),\\ &(-0.6057078106, 0.2243188311), \quad (-0.7297882979, -0.4973407037).\end{aligned}$$

The fact that they are also simple zeroes for $M_1(\rho)$ is because the transformations made to convert the equations into polynomial ones are local diffeomorphisms in the corresponding domains. □

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