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THE GENERALIZED VAN DEL WAALS HAMILTONIAN: PERIODIC ORBITS AND $\mathcal{C}^1$ NON–INTEGRABILITY

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ABSTRACT. The aim of this paper is to study the periodic orbits of the generalized van der Waals Hamiltonian system. The tool for studying such periodic orbits is the averaging theory. Moreover, for this Hamiltonian system we provide information on its $\mathcal{C}^1$ non–integrability, i.e., on the existence of a second first integral of class $\mathcal{C}^1$.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

We study the generalized van der Waals problem given by the Hamiltonian

$$\mathcal{H} = \frac{1}{2}(P_1^2 + P_2^2 + P_3^2) - \frac{1}{\sqrt{Q_1^2 + Q_2^2 + Q_3^2}} + (Q_1^2 + Q_2^2 + \beta^2 Q_3^2), \tag{1}$$

depending on the parameter $\beta \in \mathbb{R}$. This Hamiltonian is a generalization of the Hamiltonian which studies the classical dynamics of a hydrogen atom in the presence of uniform magnetic and quadrupolar electric field. Doing some restrictions the motion of the system is described by a Hamiltonian system with two degrees of freedom. For more details see [6, 7, 8, 9] and the references quoted therein.

Particular cases connected with problems of physical interest are $\beta = 0$ the Zeeman effect, and $\beta = \sqrt{2}$ which corresponds to the Van der Waals effect, see Elipe et al. [4] and references therein. For the values $\beta^2 = 1/4, 1, 4$ the Hamiltonian system is integrable, see Farrelly et al. [5] and Ferrer et al. [3, 12].

Introducing the canonical change of coordinates given by the cylindrical coordinates $Q_1 = R\cos\theta$, $Q_2 = R\sin\theta$, $Q_3 = Z$ the Hamiltonian (1) becomes

$$\mathcal{H} = \frac{1}{2}(P_R^2 + \frac{P_\theta^2}{R^2} + P_Z^2) - \frac{1}{\sqrt{R^2 + Z^2}} + (R^2 + \beta^2 Z^2). \tag{2}$$

Since the momentum P_θ is a first integral of the Hamiltonian system associated to the Hamiltonian (2), this Hamiltonian system can be reduced to a system with two degrees of freedom. The dynamics of the so called *polar problem* (see Elipe [4]) is considered when $P_\theta = 0$. In this case the Hamiltonian (2)

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reduces to

$$\mathcal{H} = \frac{1}{2}(P_R^2 + P_Z^2) - \frac{1}{\sqrt{R^2+Z^2}} + (R^2 + \beta^2 Z^2), \tag{3}$$

Our main objective will be to prove analytically the existence of periodic solutions of the Hamiltonian system associated to the Hamiltonian (3), and as a corollary to provide information about the $\mathcal{C}^1$–non integrability of such Hamiltonian system.

In this work we shall use as a main tool the averaging method of first order to find analytically periodic orbits of the Hamiltonian system associated to the Hamiltonian (2) with $P_\theta = 0$. See the appendix for more details on the averaging theory, see also some recent applications of this method to other Hamiltonian systems like the ones studied in [10, 11]. One of the main difficulties in practice for applying the averaging method is to express the differential system into the normal form for applying the averaging theory, see the appendix. The use of adequate variables in each situation can allow the application of the averaging theory for finding periodic orbits.

For the Hamiltonian system associated to the Hamiltonian (2) with $P_\theta = 0$ we have the following results.

Theorem 1. *For every $h < 0$ the Hamiltonian system associated to the generalized van der Waals Hamiltonian $\mathcal{H}$ with $P_\theta = 0$ given by* (3) *has a periodic solution in the energy level $\mathcal{H} = h + \sqrt{-2/h}$ if $\beta \notin \{\pm 2, \pm 1/2\}$. Moreover, this periodic solution is linear stable if $\beta \in (-\infty, -2) \cup (-1/2, 1/2) \cup (2, \infty)$, and unstable if $\beta \in (-2, -1/2) \cup (2, 1/2)$.*

Using the periodic orbits found in Theorem 1 we study the $\mathcal{C}^1$ non–integrability in the Liouville–Arnold sense of the polar generalized van der Waals Hamiltonians.

Theorem 2. *For the generalized van der Waals Hamiltonian $\mathcal{H}$ with $P_\theta = 0$ given by* (3) *and $\beta \notin \{\pm 2, \pm 1/2\}$ its associated Hamiltonian system cannot have a $\mathcal{C}^1$ second first integral G such that the gradients of $\mathcal{H}$ and G are linearly independent at each point of the periodic orbits found in Theorem 1.*

Many times the study of the periodic orbits of a Hamiltonian system is made numerically. In general to prove analytically the existence of periodic solutions of a Hamiltonian system is a very difficult task, many times impossible to do. Here, with the averaging theory we reduce this difficult problem for the Hamiltonian system associated to the Hamiltonian (3) to find the zeros of a non–linear system of two equations and two unknowns. We must mention that the averaging theory for finding periodic solutions in general does not provide all the periodic solutions of the system. For more information about the averaging theory see the Appendix and the references quoted there.

The way that in this article we study the periodic orbits of a Hamiltonian system, is very general and can be applied to arbitrary Hamiltonian systems. Theorem 5 of the Appendix due to Poincaré gives the information about the existence of a $\mathcal{C}^1$ second first integral obtained once we know some periodic orbits of a Hamiltonian system. This tool works for an arbitrary Hamiltonian system.

We remark that there are very good theories for studying the existence of a second meromorphic first integral in a Hamiltonian system like, Ziglin's theory [17] and the Morales–Ramis's theory [13], but as far as we know the unique result about the existence of a second $\mathcal{C}^1$ first integral is the one due to Poincaré used in this paper.

The rest of the paper is dedicated to prove the previous two theorems.

2. Proof of the results

Proof of Theorem 1. The Hamiltonian (3) can be written as

$$\mathcal{H} = \frac{1}{2}(P_1^2 + P_2^2) - \frac{1}{\sqrt{Q_1^2 + Q_2^2}} + (Q_1^2 + \beta^2 Q_2^2). \tag{4}$$

To avoid the difficulties due to the collision (i.e. $Q_1 = Q_2 = 0$) we do the Levi–Civita regularization, doing the change of variables in the positions given by

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$

then the induced change in the conjugate momenta is

$$\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} = \frac{2}{q_1^2 + q_2^2} \begin{pmatrix} q_1 & -q_2 \\ q_2 & q_1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.$$

To complete the regularization it is necessary to rescale the time t taking as the new time τ through $d\tau = 4dt/(q_1^2 + q_2^2)$. We apply these changes of variables to the energy level of the Hamiltonian $\mathcal{H} = h$ with $h < 0$, and we introduce the new Hamiltonian

$$\mathcal{H}^* = \frac{1}{4}(q_1^2 + q_2^2)(\mathcal{H} - h), \tag{5}$$

i.e.

$$\mathcal{H}^* = \frac{1}{2}\left(p_1^2 + p_2^2\right) - \frac{h}{2}\frac{q_1^2 + q_2^2}{2} + \frac{1}{4}\left(q_1^2 + q_2^2\right)\left((q_1^2 - q_2^2)^2 + 4\beta^2 q_1^2 q_2^2\right).$$

We note that we choose $h < 0$ because the Kepler problem given by the Hamiltonian

$$\mathcal{H} = \frac{1}{2}(P_1^2 + P_2^2) - \frac{1}{\sqrt{Q_1^2 + Q_2^2}},$$

has its periodic orbits in the negative energy levels, and for small values of Q_1 and Q_2 the Hamiltonian (3) is close to the Kepler one.

If we do the canonical change of variables $(q_1, q_2, p_1, p_2) \to (x, y, X, Y)$ given by

$$q_1 = 2c^{1/4}x, \qquad q_2 = 2c^{1/4}y, \qquad p_1 = 2c^{3/4}X, \qquad p_2 = 2c^{3/4}Y,$$

with $c = -h/2 > 0$ we obtain the regularized Hamiltonian

$$\mathcal{H}_1^* = \frac{1}{2}(X^2+Y^2+x^2+y^2) + 4\left(x^2+y^2\right)\left((x^2-y^2)^2+4\beta^2x^2y^2\right). \tag{6}$$

Doing the rescaling $(x, y, X, Y) \to (\mu x, \mu y, \mu X, \mu Y)$ and the denoting by $\varepsilon = \mu^4$ the Hamiltonian (6) becomes

$$\mathcal{H}_2^* = \frac{1}{2}(X^2+Y^2+x^2+y^2) + \varepsilon 4\left(x^2+y^2\right)\left((x^2-y^2)^2+4\beta^2x^2y^2\right). \tag{7}$$

Finally doing the non–canonical change of variables

$$x = R_1\cos\theta_1,\ X = R_1\sin\theta_1,\ y = R_2\cos(\theta_1+\theta_2),\ Y = R_2\sin(\theta_1+\theta_2),$$

the Hamiltonian (7) becomes the first integral

$$\begin{aligned} \mathcal{H}_3^* = {} & \frac{1}{2}(R_1^2+R_2^2) + \varepsilon\big(R_1^2\cos^2\theta_1 + R_2^2\cos^2(\theta_1+\theta_2)\big)\Big(R_1^4\cos^4\theta_1 \\ & +2R_1^2R_2^2(2\beta^2-1)\cos^2(\theta_1+\theta_2)\cos^2\theta_1 + R_2^4\cos^4(\theta_1+\theta_2)\Big). \end{aligned} \tag{8}$$

of the following equations of motion

$$\begin{aligned}
\dot R_1 = {} & -\varepsilon 8R_1\Big(3R_1^4\cos^4\theta_1 + 2R_1^2R_2^2\left(4\beta^2-1\right)\cos^2\left(\theta_1+\theta_2\right)\cos^2\theta_1 \\
& +R_2^4(4\beta^2-1)\cos^4\left(\theta_1+\theta_2\right)\Big)\sin\theta_1\cos\theta_1, \\
\dot\theta_1 = {} & -1-\varepsilon 8\Big(3R_1^4\cos^4\theta_1 + 2R_1^2R_2^2(4\beta^2-1)\cos^2(\theta_1+\theta_2)\cos^2\theta_1 \\
& +R_2^4\left(4\beta^2-1\right)\cos^4\left(\theta_1+\theta_2\right)\Big)\cos^2\theta_1, \\
\dot R_2 = {} & -\varepsilon 8R_2\Big(R_1^4\left(4\beta^2-1\right)\cos^4\theta_1 + 2R_1^2R_2^2\left(4\beta^2-1\right)\cos^2\left(\theta_1+\theta_2\right)\cos^2\theta_1 \\
& +3R_2^4\cos^4\left(\theta_1+\theta_2\right)\Big)\sin\left(\theta_1+\theta_2\right)\cos(\theta_1+\theta_2), \\
\dot\theta_2 = {} & \varepsilon 8\Big(3R_1^4\cos^6\theta_1 - R_1^2\left(R_1^2-2R_2^2\right)\left(4\beta^2-1\right)\cos^2\left(\theta_1+\theta_2\right)\cos^4\theta_1 \\
& +R_2^2\left(R_2^2-2R_1^2\right)\left(4\beta^2-1\right)\cos^4\left(\theta_1+\theta_2\right)\cos^2\theta_1 - 3R_2^4\cos^6\left(\theta_1+\theta_2\right)\Big).
\end{aligned}$$

Taking the variable θ_1 as the new time these four differential equations reduce to the three equations

$$\begin{aligned}
R_1' = {} & \varepsilon 8R_1\Big(3R_1^4\cos^4\theta_1 + 2R_1^2R_2^2\left(4\beta^2-1\right)\cos^2\left(\theta_1+\theta_2\right)\cos^2\theta_1 \\
& +R_2^4(4\beta^2-1)\cos^4\left(\theta_1+\theta_2\right)\Big)\sin\theta_1\cos\theta_1 + O(\varepsilon^2), \\
R_2' = {} & \varepsilon 8R_2\Big(R_1^4\left(4\beta^2-1\right)\cos^4\theta_1 + 2R_1^2R_2^2\left(4\beta^2-1\right)\cos^2\left(\theta_1+\theta_2\right)\cos^2\theta_1 \\
& +3R_2^4\cos^4\left(\theta_1+\theta_2\right)\Big)\sin\left(\theta_1+\theta_2\right)\cos(\theta_1+\theta_2) + O(\varepsilon^2), \\
\theta_2' = {} & -\varepsilon 8\Big(3R_1^4\cos^6\theta_1 - R_1^2\left(R_1^2-2R_2^2\right)\left(4\beta^2-1\right)\cos^2\left(\theta_1+\theta_2\right)\cos^4\theta_1 \\
& +R_2^2\left(R_2^2-2R_1^2\right)\left(4\beta^2-1\right)\cos^4\left(\theta_1+\theta_2\right)\cos^2\theta_1 - 3R_2^4\cos^6\left(\theta_1+\theta_2\right)\Big) \\
& +O(\varepsilon^2),
\end{aligned}$$

where the prime denotes derivative with respect to θ_1.

Substituting the variable $R_2 = \sqrt{2h^* - R_1^2} + O(\varepsilon)$ isolated from the first integral level $\mathcal{H}_3^* = h^* > 0$ into the previous three differential equations we obtain a new reduction to the following two differential equations:

(9)

$$\begin{aligned} R_1' = \;& \varepsilon 8\sqrt{2h^* - R_1^2}\cos(\theta_1+\theta_2)\Big(R_1^4\left(4\beta^2-1\right)\cos^4\theta_1 - \\ & 2R_1^2\left(R_1^2-2h^*\right)\left(4\beta^2-1\right)\cos^2(\theta_1+\theta_2)\cos^2\theta_1 + \\ & 3\left(R_1^2-2h^*\right)^2\cos^4(\theta_1+\theta_2)\Big)\sin(\theta_1+\theta_2) + O(\varepsilon^2) \\ = \;& \varepsilon F_{11}(R_1,\theta_2,\theta_1) + O(\varepsilon^2), \\ \theta_2' = \;& -\varepsilon 8\Big(3R_1^4\cos^6\theta_1 + R_1^2\left(4h^*-3R_1^2\right)\left(4\beta^2-1\right)\cos^2(\theta_1+\theta_2)\cos^4\theta_1 + \\ & \left(3R_1^4 - 8h^*R_1^2 + 4h^{*2}\right)\left(4\beta^2-1\right)\cos^4(\theta_1+\theta_2)\cos^2\theta_1 - \\ & 3\left(R_1^2-2h^*\right)^2\cos^6(\theta_1+\theta_2)\Big) + O(\varepsilon^2) \\ = \;& \varepsilon F_{12}(R_1,\theta_2,\theta_1) + O(\varepsilon^2). \end{aligned}$$

For $\varepsilon \neq 0$ sufficiently small this differential system is in the normal form (16) for applying to it the averaging theory described in Theorem 3. So using the notation of Theorem 3 we have

$$f_{11} = \frac{1}{2\pi}\int_0^{2\pi} F_{11}(R_1,\theta_2,\theta_1)d\theta_1 = 2h^*R_1^2\sqrt{2h^*-R_1^2}(4\beta^2-1)\sin 2\theta_2,$$

$$f_{12} = \frac{1}{2\pi}\int_0^{2\pi} F_{12}(R_1,\theta_2,\theta_1)d\theta_1 = -4h^*(h^*-R_1^2)(6\beta^2-9+(4\beta^2-1)\cos 2\theta_2).$$

Solving the system $f_{11}(R_1,\theta_2) = f_{12}(R_1,\theta_2) = 0$ we get the solutions (R_1^*,θ_2^*) given by

$$\left(0, \pm\frac{1}{2}\arccos\frac{3(2\beta^2-3)}{1-4\beta^2}\right) \quad \text{if and only if} \quad \beta \in [-2,-1]\cup[1,2], \tag{10}$$

$$(\sqrt{h^*},0), \quad (\sqrt{h^*},\pi), \tag{11}$$

$$\left(\sqrt{h^*},\frac{\pi}{2}\right), \quad \left(\sqrt{h^*},\frac{3\pi}{2}\right), \tag{12}$$

$$\left(\sqrt{2h^*}, \pm\frac{1}{2}\arccos\frac{3(2\beta^2-3)}{1-4\beta^2}\right) \quad \text{if and only if} \quad \beta \in [-2,-1]\cup[1,2]. \tag{13}$$

If we compute the determinant

$$\det\left(\left.\frac{\partial(f_1^1,f_1^2)}{\partial(R_1,\theta_2)}\right|_{(R_1,\theta_2)=(R_1^*,\theta_2^*)}\right) \neq 0, \tag{14}$$

on the solutions (10), (11), (12) and (13) we obtain respectively

$$\begin{gathered} 0, \\ -320h^{*4}(\beta^2-1)(4\beta^2-1), \\ 64h^{*4}(\beta^2-4)(4\beta^2-1), \\ \infty \quad \text{if} \quad (\beta^2-1)(\beta^2-4)\neq 0. \end{gathered}$$

By Theorem 3 only the solutions (11), (12) and (13) provide periodic solutions of the differential system (9) when the corresponding determinant is non–zero. We note that the solutions in (11) (respectively (12)) define a unique periodic orbit, both orbits are different but their projection into the plane (x, X) is a circle of radius $\sqrt{h^*}$ and into the plane (y, Y) also is a circle of radius $\sqrt{h^*}$. The solutions in (13) define the same periodic orbit which projected into the plane (x, X) is a circle of radius $\sqrt{2h^*}$ and into the plane (y, Y) is projected into the origin.

Now we shall study what of these three periodic solutions of the differential system (9) provide periodic solutions for the van der Waals Hamiltonian system with Hamiltonian (3).

The equality (5) writes as

$$1 = \sqrt{-\frac{h}{2}}\cos^2\theta_1(\mathcal{H} - h),$$

$$1 = \sqrt{-\frac{h}{2}}(\mathcal{H} - h),$$

$$1 = \sqrt{-2h}\cos^2\theta_1(\mathcal{H} - h),$$

evaluated on the periodic solutions of the differential system (9) provided by (11), (12) and (13) respectively. Therefore, only the periodic solution of the differential system (9) given by (12) becomes a solution for the van der Waals Hamiltonian system associated to the Hamiltonian $\mathcal{H}$ given by (3), because it is the unique such that the Hamiltonian $\mathcal{H}$ is constant on it taken the value $h + \sqrt{-2/h}$, recall that $h < 0$.

Since the eigenvalues of the Jacobian matrix which appears in (14) are

$$\pm 8(h^*)^2\sqrt{(\beta^2 - 4)(1 - 4\beta^2)}, \tag{15}$$

by Theorem 3(c) it follows that the periodic solution given by (12) is linear stable if $\beta \in (-\infty, -2) \cup (-1/2, 1/2) \cup (2, \infty)$, and unstable if $\beta \in (-2, -1/2) \cup (2, 1/2)$. This completes the proof of the theorem. □

Proof of Theorem 2. By Theorem 1 we know that the generalized van der Waals Hamiltonian system (3) at the energy levels $h + \sqrt{-2/h}$ for all $h < 0$ has a periodic solution if $\beta \notin \{\pm 2, \pm 1/2\}$ whose eigenvalues (15) or multipliers are different from 1 almost for all h, see for more details the part of the appendix between the two theorems that it contains. Hence, by Theorem 5 the proof of the theorem follows. □

3. Appendix

Now we shall present the basic results from averaging theory that we need for proving the results of this paper.

The next theorem provides a first order approximation for the periodic solutions of a periodic differential system, for the proof see Theorems 11.5 and 11.6 of Verhulst [16].

Consider the differential equation

$$\dot{x} = \varepsilon F_1(t,x) + \varepsilon^2 R(t,x,\varepsilon),\ x(0) = x_0, \tag{16}$$

with $x \in D$ where D is an open subset of $\mathbb{R}^n$, and $t \geq 0$. Moreover we assume that $F_1(t,\mathrm{x})$ is T periodic in t. Separately we consider in D the averaged differential equation

$$\dot{y} = \varepsilon f_1(y),\ y(0) = x_0,, \tag{17}$$

where

$$f_1(y) = \frac{1}{T}\int_0^T F_1(t,y)dt.$$

Under certain conditions, see the next theorem, equilibrium solutions of the averaged equation turn out to correspond with T–periodic solutions of equation (17).

Theorem 3. *Consider the two initial value problems* (16) *and* (17). *Suppose:*

(i) *F_1, its Jacobian $\partial F_1/\partial x$, its Hessian $\partial^2 F_1/\partial x^2$ are defined, continuous and bounded by an independent constant ε in* $[0,\ \infty)\times D$ *and* $\varepsilon \in (0,\ \varepsilon_0]$.
(ii) *F_1 is T–periodic in t (T independent of ε).*
(iii) *$y(t)$ belongs to D on the interval of time* $[0, 1/\varepsilon]$.

Then the following statements hold.

(a) *For $t \in [0, 1/\varepsilon]$ we have that $x(t) - y(t) = O(\varepsilon)$, as $\varepsilon \to 0$.*
(b) *If p is a singular point of the averaged equation* (17) *and*

$$\det\left(\frac{\partial f_1}{\partial y}\right)\bigg|_{\mathrm{y}=p} \neq 0,$$

then there exists a T–periodic solution $\varphi(t,\varepsilon)$ of equation (16) *such that $\varphi(0,\varepsilon) \to p$ as $\varepsilon \to 0$.*
(c) *The stability or instability of the periodic solution $\varphi(t,\varepsilon)$ is given by the stability or instability of the singular point p of the averaged system* (17). *In fact, the singular point p has the stability behavior of the Poincaré map associated to the periodic solution $\varphi(t,\varepsilon)$.*

We point out the main facts in order to prove Theorem 3(c), for more details see section 6.3 and 11.8 in [16]. Suppose that $\varphi(t,\varepsilon)$ is a periodic solution of (16) corresponding to $y = p$ an equilibrium point of the averaged system (17). Linearizing equation (16) in a neighborhood of the periodic solution $\varphi(t,\varepsilon)$ we obtain a linear equation with T-periodic coefficients

$$\dot{x} = \varepsilon A(t,\varepsilon)x, \quad A(t,\varepsilon) = \frac{\partial}{\partial x}[F(_1(t,x) - F_2(t,x,\varepsilon)]|_{x=\varphi(t,\varepsilon)}. \tag{18}$$

We introduce the T-periodic matrices

$$B(t) = \frac{\partial F_1}{\partial x}(t,p), \quad B_1 = \frac{1}{T}\int_0^T B(t)dt, \quad C(t) = \int_0^t (B(s) - B_1)ds.$$

From Theorem 3 we have

$$\lim_{\varepsilon \to 0} A(t, \varepsilon) = B(t),$$

and it is clear that B_1 is the matrix of the linearized averaged equation. The matrix C has average zero. The near identity transformation

$$x \longmapsto y = (I - \varepsilon C(t))x, \tag{19}$$

permits to write (18) as

$$\dot{y} = \varepsilon B_1 y + \varepsilon[A(t, \varepsilon) - B(t)]y + O(\varepsilon^2). \tag{20}$$

Notice that $A(t, \varepsilon) - B(t) \to 0$ as $\varepsilon \to 0$, and also the characteristic exponents of equation (20) depend continuously on the small parameter ε. It follows that, for ε sufficiently small, if the determinant of B_1 is not zero, then 0 is not an eigenvalue of the matrix B_1 and then it is not a characteristic exponent of (20). By the near-identity transformation we obtain that system (18) has not multipliers equal to 1.

We shall summarize some facts on the Liouville–Arnold integrability theory for Hamiltonian systems and on the theory of periodic orbits of differential equations, for more details see [1] and subsection 7.1.2 of [2], respectively. Here we only present these results for Hamiltonian systems of two degrees of freedom.

A Hamiltonian system with Hamiltonian $\mathcal{H}$ of two degrees of freedom is called *integrable in the sense of Liouville–Arnold* if it has a first integral $\mathcal{G}$ independent of $\mathcal{H}$ (i.e. the gradient vectors of $\mathcal{H}$ and $\mathcal{G}$ are independent in all the points of the phase space except perhaps in a set of zero Lebesgue measure), and in *involution* with $\mathcal{H}$ (i.e. the parenthesis of Poisson of $\mathcal{H}$ and $\mathcal{G}$ is zero).

A flow defined on a subspace of the phase space is *complete* if its solutions are defined for all time.

Now are ready for stating the Liouville–Arnold's Theorem restricted to Hamiltonian systems of two degrees of freedom.

Theorem 4 (Lioville–Arnold)**.** *Suppose that a Hamiltonian system with two degrees of freedom defined on the phase space M has its Hamiltonian $\mathcal{H}$ and the function $\mathcal{G}$ as two independent first integrals in involution. If $I_{hc} = \{\ p \in M : H(p) = h$ and $C(p) = c\} \neq \emptyset$ and $(h,\ c)$ is a regular value of the map $(\mathcal{H},\ \mathcal{G})$, then the following statements hold.*

(a) *I_{hc} is a two dimensional submanifold of M invariant under the flow of the Hamiltonian system.*
(b) *If the flow on a connected component I^*_{hc} of I_{hc} is complete, then I^*_{hc} is diffeomorphic either to the torus $\mathbb{S}^1 \times \mathbb{S}^1$, or to the cylinder $\mathbb{S}^1 \times \mathbb{R}$, or to the plane $\mathbb{R}^2$. If I^*_{hc} is compact, then the flow on it is always complete and $I^*_{hc} \approx \mathbb{S}^1 \times \mathbb{S}^1$.*
(c) *Under the hypothesis (b) the flow on I^*_{hc} is conjugated to a linear flow either on $\mathbb{S}^1 \times \mathbb{S}^1$, on $\mathbb{S}^1 \times \mathbb{R}$, or on $\mathbb{R}^2$.*

For an autonomous differential system, one of the multipliers is always 1, and its corresponding eigenvector is tangent to the periodic orbit.

A periodic orbit of an autonomous Hamiltonian system always has two multipliers equal to one. One multiplier is 1 because the Hamiltonian system is autonomous, and the other has again value 1 due to the existence of the first integral given by the Hamiltonian.

Theorem 5 (Poincaré). *If a Hamiltonian system with two degrees of freedom and Hamiltonian H is Liouville–Arnold integrable, and G is a second first integral such that the gradients of H and G are linearly independent at each point of a periodic orbit of the system, then all the multipliers of this periodic orbit are equal to 1.*

Theorem 5 is due to Poincaré [15], see also [14]. It gives us a tool to study the non Liouville–Arnold integrability, independently of the class of differentiability of the second first integral. The main problem for applying this result in a negative way is to find periodic orbits having multipliers different from 1.

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