PERIODIC POINT FREE CONTINUOUS SELF–MAPS ON GRAPHS AND SURFACES

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ABSTRACT. We prove the following three results. We denote by $\mathrm{Per}(f)$ the set of all periods of a self–map f.

Let $\mathbb{G}$ be a connected compact graph such that $\dim_{\mathbb{Q}} H_1(\mathbb{G}, \mathbb{Q}) = r$, and let $f : \mathbb{G} \to \mathbb{G}$ be a continuous map. If $\mathrm{Per}(f) = \emptyset$, then the eigenvalues of f_{*1} are 1 and 0, this last with multiplicity $r - 1$, where f_{*1} is the induced action of f on the first homological space.

Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and let $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map. The degree of f is d if $b = 0$. If $\mathrm{Per}(f) = \emptyset$, then the eigenvalues of f_{*1} are 1, d and 0, this last with multiplicity $2g - 2$ if $b = 0$; and 1 and 0, this last with multiplicity $2g + b - 2$ if $b > 0$.

Let $\mathbb{N}_{g,b}$ be a non–orientable connected compact surface of genus $g \geq 1$ with $b \geq 0$ boundary components, and let $f : \mathbb{N}_{g,b} \to \mathbb{N}_{g,b}$ be a continuous map. If $\mathrm{Per}(f) = \emptyset$, then the eigenvalues of f_{*1} are 1 and 0, this last with multiplicity $g + b - 2$.

The tools used for proving these results can be applied for studying the periodic point free continuous self–maps of many other compact absolute neighborhood retract spaces.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

A *discrete dynamical system* $(\mathbb{M}, f)$ is formed by a topological space $\mathbb{M}$ and a continuous map $f : \mathbb{M} \to \mathbb{M}$. A point x is called *fixed* if $f(x) = x$, and *periodic* of *period* k if $f^k(x) = x$ and $f^i(x) \neq x$ if $0 < i < k$. We denote the *set of periods* of all the periodic points of f by $\mathrm{Per}(f)$.

The set $\{x, f(x), f^2(x), \ldots, f^n(x), \ldots\}$ is called the *orbit* of the point $x \in \mathbb{M}$, where f^n means the composition of f with itself n times. To study *the dynamics of a map f* is to study all the different kind of orbits of f. If x is a periodic point of f of period k, then its orbit is $\{x, f(x), f^2(x), \ldots, f^{k-1}(x)\}$, and it is called a *periodic orbit*.

In general the periodic orbits play a main role in the dynamics of a discrete dynamical system, for studying them we can use topological information. Probably the best known results in this direction are the results contained in the seminal paper entitle *Period three implies chaos* for continuous self–maps on the interval, see [12].

All the non completely standard notions which will appear in the rest of the introduction will be defined in section 2.

In this note $\mathbb{M}$ will be either a connected compact graph, or a connected compact surface with or without boundary, orientable or not. Our aim is to study the continuous maps $f : \mathbb{M} \to \mathbb{M}$ having their $\mathrm{Per}(f) = \emptyset$, i.e. *periodic point free* continuous self–maps of $\mathbb{M}$.

Since the word "graph" has several different meanings in mathematics, we shall define here our graphs. A *graph* is a union of *vertices* (points) and *edges*, which are homeomorphic to the closed interval, and have mutually disjoint interiors. The endpoints of the edges are vertexes (not necessarily different) and the interiors of the edges are disjoint from the vertices.

An orientable connected compact surface without boundary of genus $g \geq 0$, $\mathbb{M}_g$, is homeomorphic to the sphere if $g = 0$, to the torus if $g = 1$, or to the connected sum of g copies of the torus if $g \geq 2$. An orientable connected compact surface with boundary of genus $g \geq 0$, $\mathbb{M}_{g,b}$, is homeomorphic to $\mathbb{M}_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closures. In what follows $\mathbb{M}_{g,0} = \mathbb{M}_g$.

A non–orientable connected compact surface without boundary of genus $g \geq 1$, $\mathbb{N}_g$, is homeomorphic to the real projective plane if $g = 1$, or to the connected sum of g copies of the real projective plane if $g > 1$. A non–orientable connected compact surface with boundary of genus $g \geq 1$, $\mathbb{N}_{g,b}$, is homeomorphic to $\mathbb{N}_g$ minus a finite number $b > 0$ of open discs having pairwise disjoint closures. In what follows $\mathbb{N}_{g,0} = \mathbb{N}_g$.

We denote by $H_k(\mathbb{M}, \mathbb{Q})$ the homological spaces with coefficients in $\mathbb{Q}$. Of course, $k = 0, 1$ if $\mathbb{M}$ is a graph, or $k = 0, 1, 2$ if $\mathbb{M}$ is a surface. Each of these spaces is a finite dimensional linear space over $\mathbb{Q}$. Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ it induces linear maps $f_{*k} : H_k(\mathbb{M}, \mathbb{Q}) \to H_k(\mathbb{M}, \mathbb{Q})$ on the homological spaces of $\mathbb{M}$. Every linear map f_{*k} is given by an $n_k \times n_k$ matrix with integer entries, where n_k is the dimension of $H_k(\mathbb{M}, \mathbb{Q})$.

Our main result is the following one.

Theorem 1. *The following three statements hold.*

(a) *Let $\mathbb{G}$ be a connected compact graph such that $\dim_{\mathbb{Q}} H_1(\mathbb{G}, \mathbb{Q}) = r$, and let $f : \mathbb{G} \to \mathbb{G}$ be a continuous map. If $Per(f) = \emptyset$, then the eigenvalues of f_{*1} are 1 and 0, this last with multiplicity $r - 1$.*

(b) *Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map. The degree of f is d if $b = 0$. If $Per(f) = \emptyset$, then the eigenvalues of f_{*1} are 1, d and 0, this last with multiplicity $2g - 2$ if $b = 0$; and 1 and 0, this last with multiplicity $2g + b - 2$ if $b > 0$.*

(c) *Let $\mathbb{N}_{g,b}$ be a non–orientable connected compact surface of genus $g \geq 1$ with $b \geq 0$ boundary components, and let $f : \mathbb{N}_{g,b} \to \mathbb{N}_{g,b}$ be a continuous map. If $Per(f) = \emptyset$, then the eigenvalues of f_{*1} are 1 and 0, this last with multiplicity $g + b - 2$.*

Theorem 1 is proved in section 2. The way that we prove this theorem can be used for studying the periodic point free continuous self–maps of many other compact polyhedra, or absolute neighborhood retract spaces which support the existence of Lefschetz numbers (see for more details [5]).

Note that Theorem 1 provide necessary conditions in order that a continuous self–map of $\mathbb{M}$ be periodic point free, but such conditions are not sufficient as it is well–known, because for instance we can construct easily continuous self–map of the circle $\mathbb{S}^1$ and of the 2–dimensional torus $\mathbb{T}^2$ having fixed or periodic points and satisfying the necessary conditions of Theorem 1 in order to have $Per(f) = \emptyset$.

From Theorem 1 we obtain the following well–known corollary.

Corollary 2. *The following two statements hold.*

(a) *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a continuous map of degree d. If $Per(f) = \emptyset$, then $d = 1$.*

(b) *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a continuous map of degree d. If $Per(f) = \emptyset$, then the eigenvalues of f_{*1} are 1 and d.*

For different proofs of the ones that we will provide here of statement (a) of Corollary 2 see for instance [2], and of statement (b) see [1].

There are several papers studying different classes of periodic point free self–maps on the annulus see [8, 10], or on the 2–dimensional torus see [4, 9, 11], or in some connected compact manifolds see [7], ... But in general to characterize the periodic point free maps on a connected compact polyhedron $\mathbb{M}$ is a very hard problem. Here our results are easy to obtain because we only provide necessary conditions for having periodic point free continuous self–maps, which in general are not sufficient.

2. PROOF OF THEOREM 1

One of the main contributions of Lefschetz was to link the homology class of a given map with an earlier work on the indices of Brouwer on

the continuous self–maps on compact simplicial complexes. These two notions provide equivalent definitions for the Lefschetz numbers, and from their comparison, can be obtained information about the existence of fixed points.

Let n be the topological dimension of a compact polyhedron $\mathbb{M}$. Given a continuous map $f : \mathbb{M} \to \mathbb{M}$ its *Lefschetz number* $L(f)$ is defined as

$$(1) \qquad L(f) = \sum_{k=0}^{n} (-1)^k \operatorname{trace}(f_{*k}).$$

One of the main results connecting the algebraic topology with the fixed point theory is the *Lefschetz Fixed Point Theorem* which establishes the existence of a fixed point if $L(f) \neq 0$, see for instance [5]. If we consider the Lefschetz number of f^m, in general, it is not true that $L(f^m) \neq 0$ implies that f has a periodic point of period m; it only implies the existence of a periodic point of period a divisor of m.

From the Lefschetz Fixed Point Theorem it follows immediately the next result.

Proposition 3. *Let $\mathbb{M}$ be a polyhedron. A necessary condition in order that a map $f : \mathbb{M} \to \mathbb{M}$ be periodic point free is that all Lefschetz numbers $L(f^m)$ be zero for $m = 1, 2, 3, \ldots$.*

We define that a continuous self–map f of $\mathbb{M}$ is *Lefschetz periodic point free* if $L(f^m) = 0$ for $m = 1, 2, 3, \ldots$ More information on the sequence $L(f^m)$ can be found in [3].

The Lefschetz zeta function $\mathcal{Z}_f(t)$ simplifies the study of the periodic points of f, and also facilitates the determination of the maps f which are Lefschetz periodic point free. The function $\mathcal{Z}_f(t)$ is a generating function for all the Lefschetz numbers of all iterates of f. More precisely the *Lefschetz zeta function* of f is defined as

$$(2) \qquad \mathcal{Z}_f(t) = \exp\left(\sum_{m \geq 1} \frac{L(f^m)}{m} t^m \right).$$

This function keeps the information of the Lefschetz number for all the iterates of f, so this function gives information about the set of periods of f. From its definition the Lefschetz zeta function is only a formal power series.

When $\mathbb{M}$ is a polyhedron there is the following alternative way to compute it

$$(3) \qquad \mathcal{Z}_f(t) = \prod_{k=0}^{n} \det(Id_k - t f_{*k})^{(-1)^{k+1}},$$

where $n = \dim \mathbb{M}$ and Id_k is the identity map of $H_k(\mathbb{M}, \mathbb{Q})$, and by convention $\det(Id_k - tf_{*k}) = 1$ if $n_k = 0$, for more details see [6]. Note that from (3) the Lefschetz zeta function is a rational function with integers coefficients, so the power series defining it converges. Moreover, with a finite number of integers (the coefficients of the rational function) we keep the information of the infinite sequence $\{L(f^m)\}_{m \in \mathbb{N}}$ for $m = 1, 2, \ldots$.

From the definition of Lefschetz zeta function and Proposition 3 it follows immediately the next result.

Proposition 4. *A necessary condition in order that a map $f : \mathbb{M} \to \mathbb{M}$ be periodic point free is that the Lefschetz zeta function $\mathcal{Z}_f(t) = 1$.*

We shall use Proposition 4 as a key point for proving our Theorem 1.

Proof of statement (a) of Theorem 1. Let $\mathbb{G}$ be a connected compact graph such that $\dim_{\mathbb{Q}} H_1(\mathbb{G}, \mathbb{Q}) = r$, and let $f : \mathbb{G} \to \mathbb{G}$ be a continuous map. Then, f_{*1} is an $r \times r$ matrix, and f_{*0} is the 1×1 matrix (1) because $\mathbb{G}$ is connected (for more details see [13, 14]). Therefore, if $p(\lambda)$ is the characteristic polynomial of the matrix f_{*1}, from (2) we have

$$\mathcal{Z}_f(t) = \frac{\det(Id - tf_{*1})}{1 - t} = \frac{(-1)^r t^r p\left(\frac{1}{t}\right)}{1 - t}.$$

If $\mathrm{Per}(f) = \emptyset$, by Proposition 4 we must have $\mathcal{Z}_f(t) = 1$. Therefore, from the previous equalities the characteristic polynomial must be $p(\lambda) = (-1)^r \lambda^{r-1}(\lambda - 1)$. Clearly the zeros of this characteristic polynomial are 1 and 0, this last with multiplicity $r-1$. Hence the statement is proved. $\square$

Proof of statement (b) of Theorem 1. Let $\mathbb{M}_{g,b}$ be an orientable connected compact surface of genus $g \geq 0$ with $b \geq 0$ boundary components, and $f : \mathbb{M}_{g,b} \to \mathbb{M}_{g,b}$ be a continuous map. The degree of f is d if $b = 0$. We recall the homological spaces of $\mathbb{M}_{g,b}$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{M}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\ldots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = 2g$ if $b = 0$, $n_1 = 2g + b - 1$ if $b > 0$, $n_2 = 1$ if $b = 0$, and $n_2 = 0$ if $b > 0$; and the induced linear maps $f_{*0} = (1)$, $f_{*2} = (d)$ if $b = 0$, and $f_{*2} = 0$ if $b > 0$ (see for additional details [13, 14]).

Assume $b = 0$. If $p(\lambda)$ is the characteristic polynomial of the matrix f_{*1}, from (2) we have

$$\mathcal{Z}_f(t) = \frac{\det(Id - tf_{*1})}{(1 - t)(1 - dt)} = \frac{t^{2g} p\left(\frac{1}{t}\right)}{(1 - t)(1 - dt)}.$$

Since $\mathrm{Per}(f) = \emptyset$, by Proposition 4 we must have $\mathcal{Z}_f(t) = 1$. Therefore, from the previous equalities the characteristic polynomial must be $p(\lambda) =$

$\lambda^{2g-2}(\lambda-1)(\lambda-d)$. Then, the zeros of this characteristic polynomial are 1, d and 0, this last with multiplicity $2g - 2$. Hence the statement is proved if $b = 0$.

Assume $b > 0$. Then

$$\mathcal{Z}_f(t) = \frac{\det(Id - tf_{*1})}{1 - t} = \frac{(-1)^{2g+b-1}t^{2g+b-1}p\left(\frac{1}{t}\right)}{1 - t}.$$

Clearly, the rest of the proof follows as in the proof of statement (a), obtaining that the roots of the characteristic polynomial are 1 and 0, this last with multiplicity $2g + b - 2$. This completes the proof of statement (b). $\qquad\square$

Proof of statement (c) *of Theorem 1.* Let $\mathbb{N}_{g,b}$ be a non–orientable connected compact surface of genus $g \geq 1$ with $b \geq 0$ boundary components, and let $f : \mathbb{N}_{g,b} \to \mathbb{N}_{g,b}$ be a continuous map. We recall the homological spaces of $\mathbb{N}_g$ with coefficients in $\mathbb{Q}$, i.e.

$$H_k(\mathbb{N}_{g,b}, \mathbb{Q}) = \mathbb{Q} \oplus \overset{n_k}{\cdots} \oplus \mathbb{Q},$$

where $n_0 = 1$, $n_1 = g + b - 1$ and $n_2 = 0$; and the induced linear map $f_{*0} = (1)$ (see again for additional details [13, 14]). So, the action on homology for the surface $\mathbb{N}_{g,b}$ is the same than the action on the homology of a graph $\mathbb{G}$ with $\dim_{\mathbb{Q}} H_1(\mathbb{G}, \mathbb{Q}) = g + b - 1$. Therefore, the rest of the proof follows as in the proof of statement (a). $\qquad\square$

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