

# Simple examples of planar involutions with non-global Montgomery-Bochner linearizations

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## Abstract

We give two planar polynomial involutions, one preserving and the other one reversing orientation, for which the Montgomery-Bochner linearization is not a global diffeomorphism

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## 1 Introduction and results

A map  $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$  is said to be *periodic* if there exists  $p \in \mathbb{N}$  such that  $F^p = \text{Id}$ , where  $F^m = F \circ F^{m-1}$ . If  $p$  is the minimum positive integer satisfying  $F^p = \text{Id}$  then we will say that the map is  $p$ -periodic. As usual, 2-periodic maps are called *involutions*.

When  $n = 2$ , from the results of Kerékjártó (1920), it is known that if the map is  $p$ -periodic and continuous then has a fixed point and it is globally conjugated to a linear  $p$ -periodic map, see [3].

In general, if  $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$  is of class  $\mathcal{C}^k$ ,  $k \geq 1$ ,  $p$ -periodic and has a fixed point  $\mathbf{y}$ , then  $F$  is always locally conjugated, in a neighborhood of  $\mathbf{y}$ , to the linear map  $(DF)_{\mathbf{y}}$ . This result is known as Montgomery-Bochner Theorem ([4]). Moreover its proof is constructive and the conjugation is given by the map

$$\psi(\mathbf{x}) = \sum_{i=0}^{p-1} (DF)_{\mathbf{y}}^{-i} F^i(\mathbf{x}), \quad (1)$$

which satisfies  $\psi \circ F = (DF)_{\mathbf{y}} \circ \psi$  and is a diffeomorphism in a neighborhood of  $\mathbf{y}$  because  $(D\psi)_{\mathbf{y}} = p \text{Id}$ . We call the map  $\psi$  the *Montgomery-Bochner linearization*. Note that  $\psi$  has the same regularity that  $F$ .

In [2] it was proved that in several cases the Montgomery-Bochner linearization is in fact a global linearization. Moreover in [5] the authors give sufficient conditions to ensure that the Montgomery-Bochner linearization is a global diffeomorphism for some planar involutions. That paper also builds some  $\mathcal{C}^1$ -involutions for which the Montgomery-Bochner linearization fails to be a global diffeomorphism. The examples presented there rather than being explicit are either described in terms of geometrical properties or constructed gluing some suitable maps.

The goal of this note is to present two simple planar polynomial involutions, one preserving and the other one reversing orientation, for which the corresponding Montgomery-Bochner linearization  $\psi$  given in (1) is not a global diffeomorphism.

Notice that by Kerékjártó result, all planar involutions  $F$  can be written as

$$F = \phi \circ L \circ \phi^{-1}, \quad (2)$$

where  $L$  is a linear involution and  $\phi$  is a global homeomorphism.

The strategy for obtaining our polynomial examples will be to construct involutions of the form (2), with  $\phi$  being a *polynomial automorphism*. Recall that a *polynomial automorphism* is a polynomial map which has an inverse which is also polynomial. It is well-known that if  $\phi$  is a polynomial automorphism then  $\det((D\phi)_{\mathbf{x}})$  is a non-zero constant.

**Theorem 1.** *Let  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be given by*

$$F(x, y) = (x + 4xy + f(x, y), -y + 2(x^2 + y^2) - f(x, y)), \quad (3)$$

*where  $f(x, y) = 4(x + y)^2(y - x) - 4(x + y)^4$ . Then  $F$  is an involution that reverses orientation, has  $(0, 0)$  as a fixed point and its associated Montgomery-Bochner linearization  $\psi_1 = \text{Id} + (DF)_{(0,0)}^{-1} \circ F$  is not a global diffeomorphism.*

*Proof.* Consider the polynomial maps  $\phi, L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  defined by  $\phi(x, y) = (-y + x(1 - x), y + x^2)$  and  $L(x, y) = (x - 2y, -y)$ . The map  $\phi$  is a polynomial automorphism with inverse  $\phi^{-1}(x, y) = (x + y, y - (x + y)^2)$ . Then the map (3) is exactly  $F = \phi \circ L \circ \phi^{-1}$ . Since  $L$  is a linear involution which reverses orientation we get the same properties for  $F$ . The fact that  $\psi_1$  is not a diffeomorphism follows because  $\psi_1(0, 0) = \psi_1(0, 1) = (0, 0)$ .  $\square$

**Remark 2.** *The polynomial automorphism  $\phi$  used in the proof of Theorem 1 is constructed in such a way that  $\phi^{-1}$  sends the parabola  $\{(x(1 - x), x^2) : x \in \mathbb{R}\}$  into the straight line  $\{y = 0\}$ , which is the line of symmetry of the family of linear involutions  $L_k(x, y) = (x + ky, -y)$ . Then  $k$  is selected imposing that  $(D(\phi \circ L_k \circ \phi^{-1}))_{(0,0)}(x, y) = (x, -y)$ .*

**Theorem 3.** *Consider  $G = F \circ (-\text{Id}) \circ F$ , where  $F$  is the polynomial involution given in (3). Then  $G$  is a polynomial involution that preserves orientation, has  $(0, 0)$  as a fixed*

point and its associated Montgomery-Bochner linearization  $\psi_2 = \text{Id} - G$  is not a global diffeomorphism.

*Proof.* Recall that  $F$  is a polynomial involution and so  $F^{-1} = F$ . Then  $G$  is also a polynomial involution that preserves orientation and has  $(0, 0)$  as a fixed point because it is globally conjugated to  $-\text{Id}$ . Moreover  $(DG)_{(0,0)} = -\text{Id}$  and so its associated Montgomery-Bochner linearization is  $\psi_2 = \text{Id} - G$ . Note that

$$\psi_2 = F \circ F^{-1} - F \circ (-\text{Id}) \circ F^{-1} = (F - F \circ (-\text{Id})) \circ F^{-1}.$$

Therefore to prove that  $\psi_2$  is not a global diffeomorphism is equivalent to prove that the polynomial automorphism  $F$  is such that its "odd part",  $\tilde{F} := F - F \circ (-\text{Id})$ , is not an automorphism. Then

$$\tilde{F}(x, y) = (2x + 8(x + y)^2(y - x), -2y - 8(x + y)^2(y - x)),$$

and it is not an automorphism because, for instance,  $\tilde{F}(3/8, 1/8) = \tilde{F}(1/8, -1/8) = (1/4, 1/4)$ .  $\square$

**Remark 4.** The explicit expression of the map  $G = (G_1, G_2)$  constructed in Theorem 3 is

$$G_1 = -x - 8xy + 16x(x^2 - y^2) + 8(3x^4 + 20x^3y - 6x^2y^2 + 4xy^3 - 5y^4) - g(x, y)$$

$$G_2 = -y + 4(x^2 + y^2) + 16y(x^2 - y^2) - 8(x^4 + 12x^3y - 18x^2y^2 - 4xy^3 - 7y^4) + g(x, y),$$

where  $g(x, y) = 32(3x^3 - 15x^2y + 9xy^2 - 5y^3)(x + y)^2 + 384(x - y)^2(x + y)^4 + 512(x - y)(x + y)^6 + 256(x + y)^8$ .

Using next result it is also easy to give a different proof that the maps  $\psi_1$  and  $\tilde{F}$ , appearing in the proofs of Theorems 1 and 3, respectively, are not injective.

**Theorem 5** ([1]). Assume that  $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is a  $\mathcal{C}^1$ -map such that at some point  $\mathbf{y}$ ,  $\det(DF)_{\mathbf{y}} = 0$  and at any neighborhood of  $\mathbf{y}$ , the function  $\mathbf{x} \rightarrow \det(DF)_{\mathbf{x}}$  takes both positive and negative values. Then, in any small enough neighborhood of  $\mathbf{y}$ , the map  $F$  is not one to one.

This alternative proof holds because the functions  $\det(D\psi_1)_{(x,y)} = 4 - 16(x^2 - y^2) - 32(x + y)^3$  and  $\det(D\tilde{F})_{(x,y)} = -4 + 64(x^2 - y^2)$  change sign.

**Remark 6.** (i) The tools introduced in this note can also be used to search  $p$ -periodic polynomial maps with  $p > 2$  for which the Montgomery-Bochner linearization (1) is not a global diffeomorphism.

(ii) From a  $p$ -periodic example  $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ ,  $F = (F_1, F_2)$ , for which the Montgomery-Bochner linearization (1) is not a global diffeomorphism it is very easy to construct an example in  $\mathbb{R}^n$ ,  $n > 2$ , satisfying the same property. It suffices to consider the extended map

$$\hat{F}(x_1, x_2, \dots, x_n) = (F_1(x_1, x_2), F_2(x_1, x_2), x_3, \dots, x_n).$$

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