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ON THE PERIODIC SOLUTIONS OF THE STATIC, SPHERICALLY SYMMETRIC EINSTEIN-YANG-MILLS EQUATIONS

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Abstract. We prove that the static, spherically symmetric Einstein-Yang-Mills equations do not have periodic solutions.

1. Introduction

The static, spherically symmetric Einstein-Yang-Mills equations with a cosmological constant $a \in \mathbb{R}$ are

$$
\begin{aligned}
\dot{r} &= rN,\\
\dot{W} &= rU,\\
\dot{N} &= (k-N)N - 2U^2,\\
\dot{k} &= s(1-2ar^2) + 2U^2 - k^2,\\
\dot{U} &= sWT + (N-k)U,\\
\dot{T} &= 2UW - NT,
\end{aligned} \tag{1}
$$

where $(r, W, N, k, U, T) \in \mathbb{R}^6$, $s \in \{-1, 1\}$ refers to regions where t is a time-like respectively space-like, and the dot denotes a derivative with respect to t. See for instanced [2] and the references quoted therein for additional details on these equations.

Let $f = 2kN - N^2 - 2U^2 - s(1 - T^2 - ar^2)$. Then it holds that

$$\frac{df(t)}{dt} = -2N(t)f(t).$$

Hence $f = 0$ is an invariant hypersurface under the flow of system (1), i.e. if a solution of system (1) has a point in $f = 0$, then the whole solution is contained in $f = 0$.

We observe that system (1) correspond to the original symmetric reduced Einstein-Yang-Mills equations only if it is restricted to the hypersurfaces $f = 0$ and $rT - W^2 = -1$. It is easy to verify that $rT - W^2$ is a first integral of system (1). Moreover the physicists are mainly interested in the solutions of the differential system (1) with $r > 0$, see the middle of the page 573 of [2]. *We shall prove that system* (1) *has no periodic solutions when* $r > 0$.

Due to its the physical origin we must study the orbits of system (1) on the hypersurface $f = 0$. Defining the variables $x_1 = r$, $x_2 = W$, $x_3 = N$, $x_4 = k$, $x_5 = U$, $x_6 = T$, we obtain that system (1)

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on $f = 0$ is equivalent to the homogeneous polynomial differential system

$$\begin{aligned}\dot{x}_1 &= x_1 x_3,\\ \dot{x}_2 &= x_1 x_5,\\ \dot{x}_3 &= (x_4 - x_3)x_3 - 2x_5^2,\\ \dot{x}_4 &= -(x_4 - x_3)^2 + s(-a x_1^2 + x_6^2),\\ \dot{x}_5 &= s x_2 x_6 + (x_3 - x_4)x_5,\\ \dot{x}_6 &= 2x_2 x_5 - x_3 x_6,\end{aligned} \tag{2}$$

of degree 2 in $\mathbb{R}^6$.

There are several papers studying the dynamics of the static, spherically symmetric EYM system, see for instance [1, 2, 3, 4, 5, 6, 7, 8]. In the paper [6] the authors prove that there are no periodic orbits for system (2) in some invariant set of codimension one. Here in this work we prove the following result.

Theorem 1. *If the differential system* (2) *has a periodic solution then the following statements hold.*

- (a) *This solution must be contained in* $x_1 = 0$ *and* $x_2 = c \neq 0$.
- (b) *The parameter* $s = 1$.
- (c) *The first integral* $H = 2x_3x_4 - x_3^2 + x_6^2 - 2x_5^2$ *of system* (2) *restricted to* $x_1 = 0$, $x_2 = c$ *and* $s = 1$ *is positive on the periodic orbit taking the value* h.
- (d) *Due to the symmetries of the problem, it must be a periodic solution* $(x_1(t) = 0, x_2(t) = c, x_3(t), x_4(t), x_5(t), x_6(t))$ *satisfying* $c > 0$, $x_3(t) < 0$, $x_4(t) - x_3(t) < 0$, $x_5(t)x_6(t) < 0$, $x_4(t) = (h - x_3^2(t) + 2x_5^2(t) - x_6^2(t))/4$ *and being* $(x_3(t), x_5(t), x_6(t))$ *a periodic solution of*

$$\begin{aligned}\dot{x}_3 &= \frac{1}{2}(h - x_3^2 - 2x_5^2 - x_6^2),\\ \dot{x}_5 &= \frac{1}{2x_3}(-h x_5 + 2c x_3 x_6 + x_3^2 x_5 - 2x_5^3 + x_5 x_6^2),\\ \dot{x}_6 &= 2c x_5 - x_3 x_6.\end{aligned} \tag{3}$$

Theorem 1 is proved in section 2.

Since $x_1 = r$, a direct consequence of Theorem 1 is the following result.

Corollary 2. *The static, spherically symmetric Einstein-Yang-Mills equations* (1) *has no periodic solutions in the region* $r > 0$.

It is an open problem to know if the differential system (2) has periodic solutions. Note that due to statement (d) of Theorem 1 the study of the existence of periodic solutions for system (2) has been reduced to study the existence of periodic solutions for system (3) with $c > 0$, in the region $x_3 < 0$ and $x_5 x_6 < 0$.

2. Proof of Theorem 1

We shall prove some auxiliary results.

Lemma 3. *If* Γ *is a periodic orbit of system* (2) *then* Γ *does not intersect the hyperplane* $\{x \in \mathbb{R}^6 : x_3 = 0\}$.

Proof. Let $\Gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ be a periodic solution of system (2). Assume that there exists $t = t_1$ such that $x_3(t_1) = 0$. We claim that there are only two possibilities: either (i) $\dot{x}_3(t_1) < 0$ or (ii) $\dot{x}_3(t_1) = 0$, $\ddot{x}_3(t_1) = 0$ and $\dddot{x}_3(t_1) < 0$. Now we shall prove the claim.

By the third equation of (2), we have that $\dot{x}_3(t_1) = -2(x_5(t_1))^2 \leq 0$. Consider the case $x_5(t_1) = 0$. Computing the second derivative of x_3 with respect to t we get

$$\ddot{x}_3 = (\dot{x}_4 - \dot{x}_3)x_3 + (x_4 - x_3)\dot{x}_3 - 4x_5\dot{x}_5.$$

Evaluating in $t = t_1$, and using that $x_3(t_1) = x_5(t_1) = \dot{x}_3(t_1) = 0$ we get $\ddot{x}_3(t_1) = 0$. Now, computing the third derivative of x_3 with respect to t we get

$$\dddot{x}_3 = (\ddot{x}_4 - \ddot{x}_3)x_3 + (\dot{x}_4 - \dot{x}_3)\dot{x}_3 + (\dot{x}_4 - \dot{x}_3)\dot{x}_3 + (x_4 - x_3)\ddot{x}_3 - 4\dot{x}_5\dot{x}_5 - 4x_5\ddot{x}_5.$$

Evaluating in $t = t_1$, and using that $x_3(t_1) = x_5(t_1) = \dot{x}_3(t_1) = \ddot{x}_3(t_1) = 0$ we get $\dddot{x}_3(t_1) = -4s^2(x_2(t_1))^2(x_6(t_1))^2$. Now we shall prove that $x_2(t_1) \neq 0$ and $x_6(t_1) \neq 0$.

Observe that the set $\{x \in \mathbb{R}^6 : x_2 = x_3 = x_5 = 0\}$ is an invariant manifold to system (2), i.e. if a solution of (2) has a point in $\{x \in \mathbb{R}^6 : x_2 = x_3 = x_5 = 0\}$ then the whole solution is contained in $\{x \in \mathbb{R}^6 : x_2 = x_3 = x_5 = 0\}$. So, if $x_2(t_1) = 0$ then $x_2(t) = x_3(t) = x_5(t) = 0$ for all $t \in \mathbb{R}$. From the first and sixth equation of (2), and using that $x_3(t) = x_5(t) = 0$, we get that there exist constants $b, c \in \mathbb{R}$ such that $x_1(t) = b$ and $x_6(t) = c$ for all $t \in \mathbb{R}$. The real function $x_4(t)$ is a periodic function that is solution of the equation $\dot{x}_4 = -x_4^2 + s(-ab^2 + c^2)$. It is known that any periodic solution of a differential equation in dimension one must be constant. So, there exists $d \in \mathbb{R}$ such that $x_4(t) = d$ for all $t \in \mathbb{R}$. In this case Γ is constant and not a periodic solution. So we have proved that $x_2(t_1) \neq 0$.

Consider the case $x_6(t_1) = 0$. By using the fact that the set $\{x \in \mathbb{R}^6 : x_3 = x_5 = x_6 = 0\}$ is an invariant manifold to system (2) we get that $x_3(t) = x_5(t) = x_6(t) = 0$ for all $t \in \mathbb{R}$. From the first and second equation of (2) we get that $x_1(t)$ and $x_2(t)$ are constant. So, $x_4(t)$ also is constant and Γ is constant. Hence we have proved that $x_6(t_1) \neq 0$.

In short, the claim that either (i) $\dot{x}_3(t_1) < 0$ or (ii) $\dot{x}_3(t_1) = 0$, $\ddot{x}_3(t_1) = 0$ and $\dddot{x}_3(t_1) < 0$ is proved. This implies that in all zeroes of $x_3(t)$, this function is decreasing. But this is a contradiction because $x_3(t)$ is a real periodic function. □

Lemma 4. *If there exists Γ a periodic orbit for system (2) then there exists $c \in \mathbb{R} \setminus \{0\}$, such that the periodic orbit is contained in the set $\{x \in \mathbb{R}^6 : x_1 = 0$ and $x_2 = c\}$.*

Proof. Since the hyperplane $\{x \in \mathbb{R}^6 : x_1 = 0\}$ is invariant for the system (2), if $\Gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ is a periodic solution of system (2) then $x_1(t)$ does not change sign. From Lemma 3 we have that $x_3(t)$ also does not change sign. By the first equation of (2), using that $x_1(t)$ is a real periodic function and $x_1(t)x_3(t)$ does not change sign we get that $x_1(t) = 0$ for all $t \in \mathbb{R}$. Substituting $x_1(t) = 0$ in the second equation of (2) we get that there exists $c \in \mathbb{R}$ such that $x_2(t) = c$ for all $t \in \mathbb{R}$.

It remains to show that $c \neq 0$. Suppose that $c = 0$. From the sixth equation of (2) we get $\dot{x}_6(t) = -x_3(t)x_6(t)$. From Lemma 3 we have either $x_3(t) > 0$ for all t, or $x_3(t) < 0$ for all t. In the first case we have that the real function $x_6(t)$ is an increasing function in the set $\{t \in \mathbb{R} : x_6(t) < 0\}$, and a decreasing function in the set $\{t \in \mathbb{R} : x_6(t) > 0\}$. Therefore, this is impossible that $c = 0$, because $x_6(t)$ is a periodic function except if $x_6(t) \equiv 0$. In the second case $x_6(t)$ is an increasing function in the set $\{t \in \mathbb{R} : x_6(t) > 0\}$ and a decreasing function in the set $\{t \in \mathbb{R} : x_6(t) < 0\}$. Again it is impossible that $c = 0$ except if $x_6(t) \equiv 0$. Consequently, if $c = 0$ then $x_6(t) = 0$ for all $t \in \mathbb{R}$. Substituting $x_6(t) = 0$ in the fourth equation of (2), and using that $x_4(t)$ is periodic, we have that there exists $d \in \mathbb{R}$ such that $x_3(t) = x_4(t) = d$ for all t. Now from the third equation of (2) we get that $x_5(t) = 0$ for all t. This is a contradiction because $\Gamma(t)$ is a constant solution instead of a periodic solution. □

Lemma 5. *For $s = -1$ system* (2) *has no periodic orbits.*

Proof. In [6] the authors prove that for $s = -1$ system (2) restricted to the hyperplane $\{x \in \mathbb{R}^6 : x_1 = 0\}$ has no periodic orbits. The proof that for $s = -1$ system (2) has no periodic orbits follows from this fact and from Lemma 4. □

Lemma 6. *If there exists a periodic orbit for system* (2)*, with $s = 1$, restricted to the hyperplane $\{x \in \mathbb{R}^6 : x_1 = 0\}$, then it is contained in the set $\{x \in \mathbb{R}^6 : \; x_3(x_4 - x_3) > 0\}$.*

Proof. Assume that $\Gamma(t) = (0, x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ is a periodic solution of (2), with $s = 1$, restricted to the hyperplane $\{x \in \mathbb{R}^6 : x_1 = 0\}$. From Lemma 3 we know that $x_3(t)$ does not change sign. So either $x_3(t) > 0$ for all $t \in \mathbb{R}$, or $x_3(t) < 0$ for all $t \in \mathbb{R}$. Now we will prove that either $x_3(t) - x_4(t) > 0$ for all $t \in \mathbb{R}$, or $x_3(t) - x_4(t) < 0$ for all $t \in \mathbb{R}$.

Note that if $x_3(t_0) - x_4(t_0) = 0$, then from the third and fourth equation of (2) we get $\dot{x}_3(t_0) - \dot{x}_4(t_0) = -2(x_5(t_0))^2 - (x_6(t_0))^2 \leq 0$. Using that $\dot{x}_3 - \dot{x}_4$ is periodic we get that there exists at least $t = t_1$ such that $x_5(t_1) = x_6(t_1) = 0$. By using the fact that $\{x \in \mathbb{R}^6 : x_5 = x_6 = 0\}$ is an invariant manifold for system (2), we get that $x_5(t) = x_6(t) = 0$ for all $t \in \mathbb{R}$. Substituting $x_1(t) = x_6(t) = 0$ in the fourth equation of (2) and using the fact that $x_4(t)$ is periodic we get that there exists $b \in \mathbb{R}$ such that $x_3(t) = x_4(t) = b$ for all $t \in \mathbb{R}$. Substituting $x_5(t) = 0$ in the second equation of (2) we have that $x_2(t)$ is constant. So, Γ is constant and this is a contradiction with the fact that Γ is a periodic solution. Hence, it is proved that either $x_3(t) - x_4(t) > 0$ for all $t \in \mathbb{R}$, or $x_3(t) - x_4(t) < 0$ for all $t \in \mathbb{R}$.

Now we prove that $\Gamma(t)$ cannot be in $\{x \in \mathbb{R}^6 : \; x_3(x_4 - x_3) < 0\}$. If the orbit associated to $\Gamma(t)$ is contained in $\{x \in \mathbb{R}^6 : \; x_3(x_4 - x_3) \leq 0\}$, then from the third equation of system (2) we have that $\dot{x}_3(t) \leq 0$ for all t. It is impossible because $x_3(t)$ is a real periodic function. □

Lemma 7. *Let $\Gamma(t)$ be a periodic solution of system* (2)*. The function $H = 2x_3x_4 - x_3^2 + x_6^2 - 2x_5^2$ is a first integral of system* (2) *restricted to $x_1 = 0$, $x_2 = c$ and $s = 1$, and there exists $h \in \mathbb{R}$, $h > 0$, such that $H(\Gamma(t)) = h$ for all t.*

Proof. System (2) restricted to $x_1 = 0$, $x_2 = c$ and $s = 1$ is given by

$$
\begin{aligned}
\dot{x}_3 &= (x_4 - x_3)x_3 - 2x_5^2, \\
\dot{x}_4 &= -(x_4 - x_3)^2 + x_6^2, \\
\dot{x}_5 &= cx_6 + (x_3 - x_4)x_5, \\
\dot{x}_6 &= 2cx_5 - x_3x_6.
\end{aligned} \tag{4}
$$

Clearly H is a first integral of (4), because it satisfies

$$\dot{H} = \sum_{i=3}^{6} \frac{\partial H}{\partial x_i} \dot{x}_i = 0.$$

This means that H is constant along the solutions of (4). So, there exists $h \in \mathbb{R}$ such that $H(\Gamma(t)) = h$ for all t. It remains to show that $h > 0$. From $2x_3x_4 - x_3^2 + x_6^2 - 2x_5^2 = h$ we get

$$x_4 = \frac{1}{2x_3}(h - x_3^2 + 2x_5^2 - x_6^2). \tag{5}$$

Substituting this expression in the first equation of (4) we obtain $\dot{x}_3 = (h - x_3^2 - 2x_5^2 - x_6^2)/2$. The fact that function $x_3(t)$ is periodic implies that $\dot{x}_3$ must be zero at some point. So $h > 0$ because $x_3(t) \neq 0$ for all t. □

Lemma 8. *Let $\Gamma(t) = (0, c, x_3(t), x_4(t), x_5(t), x_6(t))$ be a periodic solution of system (2), and $h = H(\Gamma(t))$, where H is given in Lemma 7. The cooordinates of $\Gamma(t)$ satisfy $c > 0$, $x_3(t) < 0$, $x_4(t) - x_3(t) < 0$, $x_5(t)x_6(t) < 0$, $x_4(t)$ is given by (5), and $(x_3(t), x_5(t), x_6(t))$ is a periodic solution of*

$$\begin{aligned} \dot{x}_3 &= \frac{1}{2}(h - x_3^2 - 2x_5^2 - x_6^2), \\ \dot{x}_5 &= \frac{1}{2x_3}(-hx_5 + 2cx_3x_6 + x_3^2x_5 - 2x_5^3 + x_5x_6^2), \\ \dot{x}_6 &= 2cx_5 - x_3x_6. \end{aligned} \tag{6}$$

Proof. Since $x_2 = c$, due to the fact that the symmetry

$$(x_1, x_2, x_3, x_4, x_5, x_6, t) \mapsto (-x_1, -x_2, -x_3, -x_4, -x_5, -x_6, -t)$$

leaves the differential system (2) invariant, we can assume that $c > 0$.

From the proof of Lemma 7 it is clear that $x_4(t)$ is given by (5). Substituting (5) in system (4) and eliminating the second equation we get system (6). So, it is clear that $(x_3(t), x_5(t), x_6(t))$ is a periodic solution of system (6).

We observe that system (6) is symmetric with respect to $(x_3, x_5, x_6, t) \mapsto (-x_3, x_5, -x_6, -t)$, and from Lemma 3 we have that $x_3(t)$ does not change sign. So, we can assume that the periodic orbit lives in $x_3 < 0$. By Lemma 6 we get $x_4(t) - x_3(t) < 0$ for all t. So, $x_4(t) < 0$ for all t.

From system (2) we get

$$\frac{d}{dt}(x_5x_6) = c(x_6^2 + 2x_5^2) - x_4x_5x_6. \tag{7}$$

It means that in all points $t = t_0$ where $x_5(t_0)x_6(t_0) = 0$ we have that $\frac{d}{dt}(x_5x_6)|_{t=t_0}$ has the same sign of c, i.e., positive sign. But it is impossible because $x_5(t)x_6(t)$ is a periodic real function. This implies that $x_5(t)$ and $x_6(t)$ never change sign. From (7), and since the function $x_5(t)x_6(t)$ is periodic and $x_4(t) < 0$ for all t, we get $x_5(t)x_6(t) < 0$ for all t. □

Proof of Theorem 1. Statements (a), (b), (c) and (d) follow from lemmas 4, 5, 7 and 8 respectively. □

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