ON THE PERIODIC SOLUTIONS
OF THE STATIC, SPHERICALLY SYMMETRIC
EINSTEIN-YANG-MILLS EQUATIONS

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Abstract. We prove that the static, spherically symmetric Einstein-Yang-Mills equations do not have periodic solutions.

1. Introduction

The static, spherically symmetric Einstein-Yang-Mills equations with a cosmological constant \( a \in \mathbb{R} \) are

\[
\begin{align*}
\dot{r} &= rN, \\
\dot{W} &= rU, \\
\dot{N} &= (k - N)N - 2U^2, \\
\dot{k} &= s(1 - 2ar^2) + 2U^2 - k^2, \\
\dot{U} &= sWT + (N - k)U, \\
\dot{T} &= 2UW - NT,
\end{align*}
\]

where \((r, W, N, k, U, T) \in \mathbb{R}^6, s \in \{-1, 1\}\) refers to regions where \( t \) is a time-like respectively space-like, and the dot denotes a derivative with respect to \( t \). See for instance [2] and the references quoted therein for additional details on these equations.

Let \( f = 2kN - N^2 - 2U^2 - s(1 - T^2 - ar^2) \). Then it holds that

\[
\frac{df(t)}{dt} = -2N(t)f(t).
\]

Hence \( f = 0 \) is an invariant hypersurface under the flow of system (1), i.e. if a solution of system (1) has a point in \( f = 0 \), then the whole solution is contained in \( f = 0 \).

We observe that system (1) correspond to the original symmetric reduced Einstein-Yang-Mills equations only if it is restricted to the hypersurfaces \( f = 0 \) and \( rT - W^2 = -1 \). It is easy to verify that \( rT - W^2 \) is a first integral of system (1). Moreover the physicists are mainly interested in the solutions of the differential system (1) with \( r > 0 \), see the middle of the page 573 of [2]. We shall prove that system (1) has no periodic solutions when \( r > 0 \).

Due to its the physical origin we must study the orbits of system (1) on the hypersurface \( f = 0 \). Defining the variables \( x_1 = r, x_2 = W, x_3 = N, x_4 = k, x_5 = U, x_6 = T \), we obtain that system (1)

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on \( f = 0 \) is equivalent to the homogeneous polynomial differential system
\[
\begin{align*}
\dot{x}_1 &= x_1x_3, \\
\dot{x}_2 &= x_1x_5, \\
\dot{x}_3 &= (x_4 - x_3)x_3 - 2x_5^2, \\
\dot{x}_4 &= -(x_4 - x_3)^2 + s(-ax_1^2 + x_6^2), \\
\dot{x}_5 &= sx_2x_6 + (x_3 - x_4)x_5, \\
\dot{x}_6 &= 2x_2x_5 - x_3x_6,
\end{align*}
\]
(2)
of degree 2 in \( \mathbb{R}^6 \).

There are several papers studying the dynamics of the static, spherically symmetric EYM system, see for instance [1, 2, 3, 4, 5, 6, 7, 8]. In the paper [6] the authors prove that there are no periodic orbits for system (2) in some invariant set of codimension one. Here in this work we prove the following result.

**Theorem 1.** If the differential system (2) has a periodic solution then the following statements hold.

(a) This solution must be contained in \( x_1 = 0 \) and \( x_2 = c \neq 0 \).

(b) The parameter \( s = 1 \).

(c) The first integral \( H = 2x_3x_4 - x_3^2 + x_6^2 - 2x_5^2 \) of system (2) restricted to \( x_1 = 0 \), \( x_2 = c \) and \( s = 1 \) is positive on the periodic orbit taking the value \( h \).

(d) Due to the symmetries of the problem, it must be a periodic solution \( (x_1(t) = 0, x_2(t) = c, x_3(t), x_4(t), x_5(t), x_6(t)) \) satisfying \( c > 0 \), \( x_3(t) < 0 \), \( x_4(t) - x_3(t) < 0 \), \( x_5(t)x_6(t) < 0 \), \( x_4(t) = (h - x_3^2(t) + 2x_5^2(t) - x_6^2(t))/4 \) and being \( (x_3(t), x_5(t), x_6(t)) \) a periodic solution of
\[
\begin{align*}
\dot{x}_3 &= \frac{1}{2}(h - x_3^2 - 2x_5^2 - x_6^2), \\
\dot{x}_5 &= \frac{1}{2x_3}(-hx_5 + 2cx_3x_6 + x_3^2x_5 - 2x_3^2 + x_5x_6^2), \\
\dot{x}_6 &= 2cx_5 - x_3x_6.
\end{align*}
\]

Theorem 1 is proved in section 2.

Since \( x_1 = r \), a direct consequence of Theorem 1 is the following result.

**Corollary 2.** The static, spherically symmetric Einstein-Yang-Mills equations (1) has no periodic solutions in the region \( r > 0 \).

It is an open problem to know if the differential system (2) has periodic solutions. Note that due to statement (d) of Theorem 1 the study of the existence of periodic solutions for system (2) has been reduced to study the existence of periodic solutions for system (3) with \( c > 0 \), in the region \( x_3 < 0 \) and \( x_5x_6 < 0 \).

## 2. Proof of Theorem 1

We shall prove some auxiliary results.

**Lemma 3.** If \( \Gamma \) is a periodic orbit of system (2) then \( \Gamma \) does not intersect the hyperplane \( \{x \in \mathbb{R}^6 : x_3 = 0\} \).
Proof. Let $\Gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ be a periodic solution of system (2). Assume that there exists $t = t_1$ such that $x_3(t_1) = 0$. We claim that there are only two possibilities: either (i) $\dot{x}_3(t_1) < 0$ or (ii) $\dot{x}_3(t_1) = 0$, $\ddot{x}_3(t_1) < 0$. Now we shall prove the claim.

By the third equation of (2), we have that $\dot{x}_3(t_1) = \frac{d}{dt}x_3(t_1) = -2(x_5(t_1))^2 \leq 0$. Consider the case $x_3(t_1) = 0$. Computing the second derivative of $x_3$ with respect to $t$ we get

$$\ddot{x}_3 = (\dot{x}_4 - \dot{x}_3)x_3 + (x_4 - x_3)\dot{x}_3 - 4x_3\dot{x}_3.$$

Evaluating in $t = t_1$, and using that $x_3(t_1) = x_5(t_1) = \dot{x}_3(t_1) = 0$ we get $\ddot{x}_3(t_1) = 0$. Now, computing the third derivative of $x_3$ with respect to $t$ we get

$$\dddot{x}_3 = (\dot{x}_4 - \dot{x}_3)x_3 + (x_4 - x_3)\ddot{x}_3 + (\dot{x}_4 - \dot{x}_3)\dot{x}_3 + (x_4 - x_3)x_3 + (x_4 - x_3)\ddot{x}_3 - 4\dot{x}_3\ddot{x}_3 - 4\dot{x}_3\dot{x}_3.$$

Evaluating in $t = t_1$, and using that $x_3(t_1) = x_5(t_1) = \dot{x}_3(t_1) = 0$ we get $\dddot{x}_3(t_1) = -4s^2(x_2(t_1))^2(x_0(t_1))^2$. Now we shall prove that $x_2(t_1) \neq 0$ and $x_0(t_1) \neq 0$.

Observe that the set $\{x \in \mathbb{R}^6 : x_2 = x_3 = x_5 = 0\}$ is an invariant manifold to system (2), i.e. if a solution of (2) has a point in $\{x \in \mathbb{R}^6 : x_2 = x_3 = x_5 = 0\}$ then the whole solution is contained in $\{x \in \mathbb{R}^6 : x_2 = x_3 = x_5 = 0\}$. So, if $x_2(t_1) = 0$ then $x_2(t) = x_3(t) = x_5(t) = 0$ for all $t \in \mathbb{R}$. From the first and sixth equation of (2), and using that $x_3(t) = x_5(t) = 0$, we get that there exist constants $b, c \in \mathbb{R}$ such that $x_1(t) = b$ and $x_0(t) = c$ for all $t \in \mathbb{R}$. The real function $x_4(t)$ is a periodic function that is solution of the equation $\dot{x}_4 = -x_2^2 + s(-ab^2 + c^2)$. It is known that any periodic solution of a differential equation in dimension one must be constant. So, there exists $d \in \mathbb{R}$ such that $x_4(t) = d$ for all $t \in \mathbb{R}$. In this case $\Gamma$ is constant and not a periodic solution. So we have proved that $x_2(t_1) \neq 0$.

Consider the case $x_0(t_1) = 0$. By using the fact that the set $\{x \in \mathbb{R}^6 : x_3 = x_5 = x_6 = 0\}$ is an invariant manifold to system (2) we get that $x_3(t) = x_5(t) = x_6(t) = 0$ for all $t \in \mathbb{R}$. From the first and second equation of (2) we get that $x_1(t)$ and $x_2(t)$ are constant. So, $x_4(t)$ also is constant and $\Gamma$ is constant. Hence we have proved that $x_0(t_1) \neq 0$.

In short, the claim that either (i) $\dot{x}_3(t_1) < 0$ or (ii) $\dot{x}_3(t_1) = 0$, $\ddot{x}_3(t_1) < 0$ and $\dddot{x}_3(t_1) < 0$ is proved. This implies that in all zeroes of $x_3(t)$, this function is decreasing. But this is a contradiction because $x_3(t)$ is a real periodic function.

Lemma 4. If there exists $\Gamma$ a periodic orbit for system (2) then there exists $c \in \mathbb{R} \setminus \{0\}$, such that the periodic orbit is contained in the set $\{x \in \mathbb{R}^6 : x_1 = 0$ and $x_2 = c\}$.

Proof. Since the hyperplane $\{x \in \mathbb{R}^6 : x_1 = 0\}$ is invariant for the system (2), if $\Gamma(t) = (x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t))$ is a periodic solution of system (2) then $x_1(t)$ does not change sign. From Lemma 3 we have that $x_3(t)$ also does not change sign. By the first equation of (2), using that $x_1(t)$ is a real periodic function and $x_1(t)x_3(t)$ does not change sign we get that $x_1(t) = 0$ for all $t \in \mathbb{R}$. Substituting $x_1(t) = 0$ in the second equation of (2) we get that there exists $c \in \mathbb{R}$ such that $x_2(t) = c$ for all $t \in \mathbb{R}$.

It remains to show that $c \neq 0$. Suppose that $c = 0$. From the sixth equation of (2) we get $\dot{x}_6(t) = -x_3(t)x_6(t)$. From Lemma 3 we have either $x_3(t) > 0$ for all $t$, or $x_3(t) < 0$ for all $t$. In the first case we have that the real function $x_6(t)$ is an increasing function in the set $\{t \in \mathbb{R} : x_6(t) < 0\}$,
and a decreasing function in the set \( \{ t \in \mathbb{R} : x_6(t) > 0 \} \). Therefore, this is impossible that \( c = 0 \), because \( x_6(t) \) is a periodic function except if \( x_6(t) \equiv 0 \). In the second case \( x_6(t) \) is an increasing function in the set \( \{ t \in \mathbb{R} : x_6(t) > 0 \} \) and a decreasing function in the set \( \{ t \in \mathbb{R} : x_6(t) < 0 \} \). Again it is impossible that \( c = 0 \) except if \( x_6(t) \equiv 0 \). Consequently, if \( c = 0 \) then \( x_6(t) = 0 \) for all \( t \in \mathbb{R} \). Substituting \( x_6(t) = 0 \) in the fourth equation of (2), and using that \( x_4(t) \) is periodic, we have that there exists \( d \in \mathbb{R} \) such that \( x_3(t) = x_4(t) = d \) for all \( t \). Now from the third equation of (2) we get that \( x_5(t) = 0 \) for all \( t \). This is a contradiction because \( \Gamma(t) \) is a constant solution instead of a periodic solution.

\[ (4) \]

\[ \dot{x}_3 = (x_4 - x_3)x_3 - 2x_2, \]
\[ \dot{x}_4 = -(x_4 - x_3)^2 + x_2, \]
\[ \dot{x}_5 = cx_6 + (x_3 - x_4)x_5, \]
\[ \dot{x}_6 = 2cx_5 - x_3x_6. \]

**Lemma 5.** For \( s = -1 \) system (2) has no periodic orbits.

**Proof.** In [6] the authors prove that for \( s = -1 \) system (2) restricted to the hyperplane \( \{ x \in \mathbb{R}^6 : x_1 = 0 \} \) has no periodic orbits. The proof that for \( s = -1 \) system (2) has no periodic orbits follows from this fact and from Lemma 4.

**Lemma 6.** If there exists a periodic orbit for system (2), with \( s = 1 \), restricted to the hyperplane \( \{ x \in \mathbb{R}^6 : x_1 = 0 \} \), then it is contained in the set \( \{ x \in \mathbb{R}^6 : x_3(x_4 - x_3) > 0 \} \).

**Proof.** Assume that \( \Gamma(t) = (0, x_2(t), x_3(t), x_4(t), x_5(t), x_6(t)) \) is a periodic solution of (2), with \( s = 1 \), restricted to the hyperplane \( \{ x \in \mathbb{R}^6 : x_1 = 0 \} \). From Lemma 3 we know that \( x_3(t) \) does not change sign. So either \( x_3(t) > 0 \) for all \( t \in \mathbb{R} \), or \( x_3(t) < 0 \) for all \( t \in \mathbb{R} \). Now we will prove that either \( x_3(t) - x_4(t) > 0 \) for all \( t \in \mathbb{R} \), or \( x_3(t) - x_4(t) < 0 \) for all \( t \in \mathbb{R} \).

Note that if \( x_3(t_0) - x_4(t_0) = 0 \), then from the third and fourth equation of (2) we get \( \dot{x}_3(t_0) - \dot{x}_4(t_0) = -2(x_5(t_0))^2 - (x_6(t_0))^2 \leq 0 \). Using that \( \dot{x}_3 - \dot{x}_4 \) is periodic we get that there exists at least \( t = t_1 \) such that \( x_5(t_1) = x_6(t_1) = 0 \). By using the fact that \( \{ x \in \mathbb{R}^6 : x_5 = x_6 = 0 \} \) is an invariant manifold for system (2), we get that \( x_5(t) = x_6(t) = 0 \) for all \( t \in \mathbb{R} \). Substituting \( x_1(t) = x_6(t) = 0 \) in the fourth equation of (2) and using the fact that \( x_4(t) \) is periodic we get that there exists \( b \in \mathbb{R} \) such that \( x_3(t) = x_4(t) = b \) for all \( t \in \mathbb{R} \). Substituting \( x_5(t) = 0 \) in the second equation of (2) we have that \( x_2(t) \) is constant. So, \( \Gamma \) is constant and this is a contradiction with the fact that \( \Gamma \) is a periodic solution. Hence, it is proved that either \( x_3(t) - x_4(t) > 0 \) for all \( t \in \mathbb{R} \), or \( x_3(t) - x_4(t) < 0 \) for all \( t \in \mathbb{R} \).

Now we prove that \( \Gamma(t) \) cannot be in \( \{ x \in \mathbb{R}^6 : x_3(x_4 - x_3) < 0 \} \). If the orbit associated to \( \Gamma(t) \) is contained in \( \{ x \in \mathbb{R}^6 : x_3(x_4 - x_3) \leq 0 \} \), then from the third equation of system (2) we have that \( \dot{x}_3(t) \leq 0 \) for all \( t \). It is impossible because \( x_3(t) \) is a real periodic function.

**Lemma 7.** Let \( \Gamma(t) \) be a periodic solution of system (2). The function \( H = 2x_3x_4 - x_3^3 + x_5^2 - 2x_6^3 \) is a first integral of system (2) restricted to \( x_1 = 0, x_2 = c \) and \( s = 1 \), and there exists \( h \in \mathbb{R}, h > 0 \), such that \( H(\Gamma(t)) = h \) for all \( t \).

**Proof.** System (2) restricted to \( x_1 = 0, x_2 = c \) and \( s = 1 \) is given by

\[ \begin{align*}
\dot{x}_3 &= (x_4 - x_3)x_3 - 2x_2, \\
\dot{x}_4 &= -(x_4 - x_3)^2 + x_2, \\
\dot{x}_5 &= cx_6 + (x_3 - x_4)x_5, \\
\dot{x}_6 &= 2cx_5 - x_3x_6.
\end{align*} \]
Clearly $H$ is a first integral of (4), because it satisfies

$$\dot{H} = \sum_{i=3}^{6} \frac{\partial H}{\partial x_i} \dot{x}_i = 0.$$

This means that $H$ is constant along the solutions of (4). So, there exists $h \in \mathbb{R}$ such that $H(\Gamma(t)) = h$ for all $t$. It remains to show that $h > 0$. From $2x_3x_4 - x_3^2 + x_6^2 - 2x_5^2 = h$ we get

$$x_4 = \frac{1}{2x_3}(h - x_3^2 + 2x_5^2 - x_6^2).$$

Substituting this expression in the first equation of (4) we obtain $\dot{x}_3 = (h - x_3^2 - 2x_5^2 - x_6^2)/2$. The fact that function $x_3(t)$ is periodic implies that $\dot{x}_3$ must be zero at some point. So $h > 0$ because $x_3(t) \neq 0$ for all $t$. □

**Lemma 8.** Let $\Gamma(t) = (0, c, x_3(t), x_4(t), x_5(t), x_6(t))$ be a periodic solution of system (2), and $h = H(\Gamma(t))$, where $H$ is given in Lemma 7. The coodinates of $\Gamma(t)$ satisfy $c > 0$, $x_3(t) < 0$, $x_4(t) - x_3(t) < 0$, $x_5(t)x_6(t) < 0$, $x_4(t)$ is given by (5), and $(x_3(t), x_5(t), x_6(t))$ is a periodic solution of

$$\begin{align*}
\ddot{x}_3 &= \frac{1}{2}(h - x_3^2 - 2x_5^2 - x_6^2), \\
\ddot{x}_5 &= \frac{1}{2x_3}(-hx_5 + 2cx_3x_6 + x_3^2x_5 - 2x_5^3 + x_5x_6^2), \\
\dot{x}_6 &= 2cx_5 - x_3x_6.
\end{align*}$$

**Proof.** Since $x_2 = c$, due to the fact that the symmetry

$$(x_1, x_2, x_3, x_4, x_5, x_6, t) \mapsto (-x_1, -x_2, -x_3, -x_4, -x_5, -x_6, -t)$$

leaves the differential system (2) invariant, we can assume that $c > 0$.

From the proof of Lemma 7 it is clear that $x_4(t)$ is given by (5). Substituting (5) in system (4) and eliminating the second equation we get system (6). So, it is clear that $(x_3(t), x_5(t), x_6(t))$ is a periodic solution of system (6).

We observe that system (6) is symmetric with respect to $(x_3, x_5, x_6, t) \mapsto (-x_3, x_5, -x_6, -t)$, and from Lemma 3 we have that $x_3(t)$ does not change sign. So, we can assume that the periodic orbit lives in $x_3 < 0$. By Lemma 6 we get $x_4(t) - x_3(t) < 0$ for all $t$. So, $x_4(t) < 0$ for all $t$.

From system (2) we get

$$\frac{d}{dt}(x_5x_6) = c(x_5^2 + 2x_6^2) - x_4x_5x_6.$$  

It means that in all points $t = t_0$ where $x_5(t_0)x_6(t_0) = 0$ we have that $\frac{d}{dt}(x_5x_6)|_{t=t_0}$ has the same sign of $c$, i.e., positive sign. But it is impossible because $x_5(t)x_6(t)$ is a periodic real function. This implies that $x_5(t)$ and $x_6(t)$ never change sign. From (7), and since the function $x_5(t)x_6(t)$ is periodic and $x_4(t) < 0$ for all $t$, we get $x_5(t)x_6(t) < 0$ for all $t$. □

**Proof of Theorem 1.** Statements (a), (b), (c) and (d) follow from lemmas 4, 5, 7 and 8 respectively. □
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