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ON LIOUVILLIAN INTEGRABILITY OF THE FIRST–ORDER POLYNOMIAL ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Recently the authors provided an example of an integrable Liouvillian planar polynomial differential system that has no finite invariant algebraic curves, see [8]. In this note we prove that if a complex differential equation of the form $y' = a_0(x) + a_1(x)y + \cdots + a_n(x)y^n$ with $a_i(x)$ polynomials for $i = 0, 1, \ldots, n$, $a_n(x) \neq 0$ and $n \geq 2$ has a Liouvillian first integral, then it has a finite invariant algebraic curve. So, this result applies to the Riccati and Abel polynomial differential equations. We shall prove that in general this result is not true when $n = 1$, i.e. for linear polynomial differential equations.

1. Introduction and the main results

By definition a *complex planar polynomial differential system* or simply a *polynomial system* is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \tag{1}$$

where the dependent variables x and y are complex, and the independent one (the *time*) t can be real or complex, and $P, Q \in \mathbb{C}[x, y]$ where $\mathbb{C}[x, y]$ is the ring of all polynomials in the variables x and y with coefficients in $\mathbb{C}$. We denote by $m = \max\{\deg P, \deg Q\}$ the *degree* of the polynomial system.

Let $f = f(x, y) = 0$ be an algebraic curve in $\mathbb{C}^2$. We say that it is *invariant* or that it is a *finite invariant algebraic curve* by the polynomial system (1) if $P\,\partial f/\partial x + Q\,\partial f/\partial y = kf$, for some polynomial $k = k(x, y) \in \mathbb{C}[x, y]$, called the *cofactor* of the algebraic curve $f = 0$. Note that the degree of the polynomial k is at most $m - 1$.

Let $h, g \in \mathbb{C}[x, y]$ and assume that h and g are relatively prime in the ring $\mathbb{C}[x, y]$. Then the function $\exp(g/h)$ is called an *exponential factor* of the polynomial system (1) if for some polynomial $k \in \mathbb{C}[x, y]$ of degree at most $m - 1$ it satisfies equation $P\,\partial \exp(g/h)/\partial x + Q\,\partial \exp(g/h)/\partial y = k \exp(g/h)$. If $\exp(g/h)$ is an exponential factor it is easy to show that $h = 0$ is an invariant algebraic curve.

Let U be an open subset of $\mathbb{C}^2$. We say that a non-constant function $H : U \to \mathbb{C}$ is a *first integral* of the polynomial system (1) in U if H is constant on the trajectories of the polynomial system (1) contained in U.

We say that a non-constant function $R : U \to \mathbb{C}$ is an *integrating factor* of the polynomial system (1) in U if R satisfies that $\partial(RP)/\partial x + \partial(RQ)/\partial y = 0$, in the points $(x, y) \in U$.

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In 1992 Singer [14] showed that if a polynomial system has a Liouvillian first integral then the system has an integrating factor of the form

$$R(x,y) = \exp\left(\int_{(x_0,y_0)}^{(x,y)} U(x,y)\,dx + V(x,y)\,dy\right), \tag{2}$$

where U and V are rational functions which satisfy $\partial U/\partial y = \partial V/\partial x$. In 1999 Christopher [3] showed that the integrating factor (2) can be written into the form

$$R = \exp(g/h)\prod f_i^{\lambda_i}, \tag{3}$$

where g, h and f_i are polynomials and $\lambda_i \in \mathbb{C}$. This condition guarantees the existence of a first integral that can be expressed by quadratures of elementary functions (Liouvillian function). This type of integrability is known since then as *Liouvillian integrability theory.*

For more details on all these notions mentioned until here see the paper [8] and the references quoted there.

Non–algebraic invariant curves with polynomial cofactor can also be used in order to find a first integral for a system. This observation permits the generalization of the Liouvillian integrability theory given in [5, 6, 7, 10] where a new kind of first integrals, not only the Liouvillian ones, is described.

There was the belief that a Liouvillian integrable system has always an invariant algebraic curve in $\mathbb{C}^2$. Moreover, this claim was proved under certain hypotheses, see [15]. However this claim was refuted in [8] where it is proved that there exist Liouvillian integrable polynomial systems without any finite invariant algebraic curve. For proving that result in [8] it is provided a Liouvillian integrable planar polynomial system of degree 2 in $\mathbb{C}^2$ without finite invariant algebraic curves.

The main result of this note is the following one.

Theorem 1. *If a complex differential equation of the form*

$$y' = \frac{dy}{dx} = a_0(x) + a_1(x)y + \cdots + a_n(x)y^n, \tag{4}$$

with $a_i(x)$ polynomials in the variable x, $a_n(x) \neq 0$ and $n \geq 2$, has a Liouvillian first integral, then it has a finite invariant algebraic curve.

Proof. For proving Theorem 1 we shall work with the autonomous planar polynomial differential system (1) associated to the non–autonomous differential equation (4); i.e.

$$P(x,y) = 1, \qquad Q(x,y) = a_0(x) + a_1(x)y + \cdots + a_n(x)y^n. \tag{5}$$

The proof of Theorem 1 is by contradiction. We assume that the differential equation (4) is Liouvillian integrable, i.e. has a Liouvillian first integral, and has not finite invariant algebraic curves. By the results of Singer in [14] (see also [3]) we know that if equation (5) is Liouvillian integrable then it has an integrating factor of the form (3). We recall that $f_i = 0$ and $h = 0$ in (3) are invariant algebraic curves and $\exp(g/h)$ is an exponential factor for system (5), for more details see [2]. Therefore if system (5) is a planar Liouvillian integrable polynomial differential system without finite invariant algebraic curves, then it must have an integrating factor of the form $R = \exp(g(x,y))$, where g is a polynomial. Note that $g = 0$ does not need to be an invariant algebraic curve of system (5). From [4] $\exp(g(x,y))$

is an exponential factor coming from the fact that the invariant straight line at infinity has multiplicity larger than one, but this fact is not relevant here.

Now we assume that the degree of g with respect to the variable y is m. Then we write g as a polynomial in the variable y with coefficients polynomials in the variable x, and we impose that $R = \exp\left(\sum_{j=0}^{m} g_j(x)y^j\right)$ is an integrating factor of system with $g_m(x) \neq 0$, i.e.

$$\frac{\partial R}{\partial x}P + \frac{\partial R}{\partial y}Q + \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right)R = 0.$$

After dividing R in the previous equality, the highest power is y^{m+n-1} and its coefficient is $ma_n(x)g_m(x)$. The vanish of this coefficient is a contradiction with the hypothesis of the theorem. Therefore, if equation (4) is Liouvillian integrable then it has a finite invariant algebraic curve. □

This kind of differential equations have been studied for several authors see Lloyd [11] and the references quoted there.

Corollary 2. *A Riccati polynomial differential equation* (*equation* (4) *with* $n = 2$) *is Liouvillian integrable if and only if has a finite invariant algebraic curve.*

Proof. The "if" part follows directly from Theorem 1. The "only if" is proved as follows. If a Riccati polynomial differential equation has a finite invariant algebraic curve, then by the classical method of resolution of the Riccati equations (see for instance [9]), we can transform it into a Bernoulli differential equation, and finally into a first order linear differential equation. Moreover any first order linear differential equation is integrable by quadratures, and consequently the Riccati equation is Liouvillian integrable. □

In any case Corollary 2 was obtained by first time by Ritt in [13], and later on by Singer in [14].

Corollary 2 has a close relationship with the classical result of Kolchin (see [1]) that a Riccati equation has an algebraic curve solution over certain differential field $\mathcal{K}$ if and only if is integrable in the Picard–Vessiot sense.

Corollary 3. *If an Abel polynomial differential equation* (*equation* (4) *with* $n = 3$) *is Liouvillian integrable, then it has a finite invariant algebraic curve.*

Corollary 3 follows directly from Theorem 1.

For the case $n = 1$ there is no contradiction because the highest power is y^m and its coefficient is given by $ma_1(x)g_m(x) + g'_m(x)$. The vanish of this coefficient do not implies $g_m(x) = 0$ and allows to compute $g_m(x)$ in function of $a_1(x)$, i.e., $g_m(x) = \exp\left(-\int^x ma_1(s)ds\right)$. Hence, in the case that the differential equation (4) is a linear differential equation we cannot apply the arguments used when $n \geq 2$.

Proposition 4. *Theorem 1 does not hold for* $n = 1$.

Proof. In [8] it is proved that the quadratic polynomial differential system

$$\dot{x} = -1 - x(2x + y), \qquad \dot{y} = 2x(2x + y), \tag{6}$$

is Liouvillian integrable and has no finite invariant algebraic curves. In fact system (6) is Liouvillian integrable because it has the integrating factor $R = e^{-(2x+y)^2/4}$, and the Liouvillian first integral $H = 2e^{-(2x+y)^2/4}x - \sqrt{\pi}\,\mathrm{erf}\left((2x + y)/2\right)$, where $\mathrm{erf}(z)$ is the error function, i.e. the integral o f the Gaussian distribution, given by

$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$. Moreover system (6) can be transformed doing the change of variables $(x, y) \to (x, z)$ where $z = 2x + y$ into the linear polynomial differential equation $dx/dz = (1 + xz)/2$. Hence the proposition is proved. □

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References

[1] P.D. Acosta-Humánez, J.T. Lázaro, J.J. Morales-Ruiz and CH. Pantazi, *On the integrability of polynomials fiels in the plane by means of Picard-Vessiot theory*, preprint, 2012.

[2] J. Chavarriga, H. Giacomini, J. Giné and J. Llibre, *Darboux integrability and the inverse integrating factor*, J. Differential Equations **194** (2003), 116–139.

[3] C.J. Christopher, *Liouvillian first integrals of second order polynomials differential equations*, Electron J. Differential Equations **1999** (1999), 7pp (electronic).

[4] C. Christopher, J. Llibre and J.V. Pereira, *Multiplicity of invariant algebraic curves in polynomial vector fields*, Pacific J. of Math. **229** (2007), 63–117.

[5] I.A. García and J. Giné, *Generalized cofactors and nonlinear superposition principles*, Appl. Math. Lett. **16** (2003), 1137–1141.

[6] I.A. García and J. Giné, *Non-algebraic invariant curves for polynomial planar vector fields*, Discrete Contin. Dyn. Syst. **10** (2004), 755–768.

[7] J. Giné and M. Grau, *Weierstrass integrability of differential equations*, Appl. Math. Lett. **23** (2010) 523–526.

[8] J. Giné and J. Llibre, *A note on Liouvillian integrability*, J. Math. Anal. Appl. **387** (2012), 1044–1049.

[9] E.L. Ince, *Ordinary Differential Equations*, New York, Dover Publications, 1956.

[10] J. Llibre, S. Walcher and X. Zhang, *Local Darboux first integrals of analytic differential systems*, preprint 2011.

[11] N.G. Lloyd, *The number of periodic solutions of the equation* $z' = z^N + p_1(t)z^{N-1} + \ldots + p_N(t)$, Proc. London Math. Soc. **27** (1973), 667–700.

[12] M.J. Prelle and M.F. Singer, *Elementary first integrals of differential equations*, Trans. Amer. Math. Soc. **279** (1983), 215–229.

[13] J.F. Ritt, *On the integration in finite terms of linear differential equations of the second order*, Bull. Amer. Math. Soc. **33** (1927), 51–57.

[14] M.F. Singer, *Liouvillian first integrals of differential equations*, Trans. Amer. Math. Soc. **333** (1992), 673–688.

[15] A.E. Zernov and B. A. Scárdua, *Integration of polynomial ordinary differential equations in the real plane*, Aequationes Math. **61** (2001), 190–200.

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