

# On the Loewner Conjecture

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## Abstract

We construct a series of examples of vectorfields relevant to the conjectures of Loewner and Caratheodory.

## 1 Introduction

We will note a one-dimensional foliations on the plane by  $\mathcal{F}(\mathbb{R}^2)$ . In many situations we don't have a unique foliation, but the union of  $n$  distinct foliations. We will denote an  $n$ -foliation by  $\mathcal{F}^n(\mathbb{R}^2)$  and by  $\mathcal{F}_i^n(\mathbb{R}^2)$  each individual foliation of  $\mathcal{F}^n(\mathbb{R}^2)$ .

The following definition is taken from [7], with small modifications:

**Definition.** A smooth one-dimensional foliation  $\mathcal{F}^2(\mathcal{D}^2)$  with an isolated singularity at  $o$  is called a singular *Hessian foliation* if there exists a smooth real-valued function  $W$  on  $\mathcal{D}^2$  whose Hessian

$$\text{Hess}(W) = \begin{pmatrix} W_{xx} & W_{xy} \\ W_{xy} & W_{yy} \end{pmatrix}$$

has the following properties:

- 1)  $\text{Hess}(W)$  is not a multiple of the identity for any  $p \in \mathcal{D}^2 - o$ .
- 2) The eigenspace corresponding to the large ( small ) eigenvalue of  $\text{Hess}(W)$  is tangent to  $\mathcal{F}_1^2(\mathcal{D}^2)$ , (  $\mathcal{F}_2^2(\mathcal{D}^2)$  ) for each  $p \in \mathcal{D}^2 - o$ .

The Hessian foliations appears in the study of stagnation points in hydrodynamics and in differential geometry. In fact, ( see for instance [5] ), the directions of the *lines of curvature* in the Bonnet coordinates correspond to the leaves of the Hessian foliation of the Bonnet function. Let  $(dx, dy)$  be a principal direction corresponding to the eigenvalue  $\lambda$ :

$$W_{xx}dx + W_{xy}dy = \lambda dx, \quad W_{xy}dx + W_{yy}dy = \lambda dy \quad (1.1)$$

Therefore:

$$(W_{xx} - W_{yy})dxdy + W_{xy}(dy^2 - dx^2) = 0 \quad (1.2)$$

Consider two complex numbers  $dx + idy$  and  $\chi_1 + i\chi_2$  with:

$$\chi_1 = W_{xx} - W_{yy}, \quad \chi_2 = 2W_{xy} \quad (1.3)$$

The equation 1.2 shows that the radius-vector of  $(dx + idy)^2$  is orthogonal to the vector  $\chi_2 - i\chi_1$  and so it is collinear to the vector  $\chi_1 + i\chi_2$ . The argument

of  $(dx + \imath dy)^2$  is the double of the argument of the complex number  $(dx + \imath dy)^2$ . Therefore, the index of the field of principal directions is half of the index of the field  $(\chi_1, \chi_2)$ .

The Loewner conjecture about the index of an umbilic point will be proved if the index of the equilibrium point of:

$$\begin{aligned}\frac{dx}{dt} &= F_1(x, y) = W_{xx} - W_{yy}, \\ \frac{dy}{dt} &= F_2(x, y) = 2W_{xy}.\end{aligned}\tag{1.4}$$

is less or equal than two.

A vectorfield of the form 5.26 will be called a Loewner vector field.

This equation can be seen from a different point of view according to [7] and [9]. Consider the Cauchy-Riemann operator:

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + \imath \frac{\partial}{\partial y} \right)$$

Then:

$$\frac{\partial^2}{\partial \bar{z}^2} = \frac{1}{4} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + 2\imath \frac{\partial^2}{\partial x \partial y} \right)$$

A Loewner vectorfield can be identified with the square of the Cauchy-Riemann operator. In section 3 we state a generalization of this vectorfield.

A necessary ( but not sufficient ) criterium for a system to be of this kind is:

$$\frac{\partial^3}{\partial x^3} \left( \frac{1}{2} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) = \frac{\partial^3}{\partial y^3} \left( \frac{1}{2} \frac{\partial F_2}{\partial y} + \frac{\partial F_1}{\partial x} \right)\tag{1.5}$$

since:

$$\frac{1}{2} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = W_{yyy}\tag{1.6}$$

$$\frac{1}{2} \frac{\partial F_2}{\partial y} + \frac{\partial F_1}{\partial x} = W_{xxx}\tag{1.7}$$

Another criterium is:

$$\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_2}{\partial y^2} = 2 \frac{\partial^2 F_1}{\partial x \partial y}\tag{1.8}$$

To prove it, try to obtain  $W$  from the system. Then:

$$W_{xy} = \frac{F_2}{2} \Rightarrow W = \frac{1}{2} \int \int F_2 dx dy + \alpha(x) + \beta(y)$$

Substituting this expression of  $W$  into:

$$W_{xx} - W_{yy} = F_1$$

we arrive at:

$$\int \frac{\partial F_2}{\partial x} dy - \int \frac{\partial F_2}{\partial y} dx = 2(F_1 + \beta'' - \alpha'')$$

If we derive with respect  $x$  and  $y$  this expression, we obtain 1.8.

A system in the form ( 5.26 ) will be called a *basic system*.

In ([4]) it is proved the following:

Let  $r = 3, 4, \dots, \infty, \omega$ . The next conjectures are equivalents.

$C^r$ -Loewner's Conjecture

The index of an umbilic, of a surface  $C^r$  embedded in  $\mathbb{R}^3$ , is at most one.

$C^r$ -Loewner's Conjecture\*

Let  $\beta : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  be a map of class  $C^r$  defined in a neighborhood  $U$  of  $(0, 0) \in \mathbb{R}^2$ . If  $(0, 0)$  is an isolated singularity of the vector field

$$X : (x, y) \rightarrow (\beta_{xx} - \beta_{yy}, 2\beta_{xy}),$$

then the index of  $X$  at  $(0, 0)$  is less or equal than 2.

## 2 Partial results on Carathéodory conjecture

The conjecture seems true for the analytic case with some proofs with more or less credibility. See [5]

Partial results are the following:

From [9].

Let:

$B$  be the open unit ball in  $\mathbb{R}^2$  centered at 0,

$T = \partial B$ ,

$f \in C^2(\overline{B})$ ,  $R$  be  $C^3$  near  $T$ ,

$\lambda, \mu, \lambda > \mu$  the eigenvalues of  $Hess(W)f = H_f$

$\Sigma_\lambda, \Sigma_\mu$  the eigenspaces associated to the eigenvalues,  $\Sigma = \Sigma_\lambda \cup \Sigma_\mu$

$\frac{\partial}{\partial r}$  the radial derivative.

Assume that the function  $\lambda - \mu - \frac{\partial \mu}{\partial r}$  ( resp.  $\lambda - \mu - \frac{\partial \lambda}{\partial r}$  ) has no zeros on  $\Sigma_\lambda$  ( resp.  $\Sigma_\mu$  )

Then:

$\Sigma$  is finite and  $\text{Ind} \left( \frac{\partial^2 f}{\partial \bar{z}^2}, 0 \right)$  is equal to:

$$\begin{aligned} 2 &+ \#(p \in T, H_f(p)p = \lambda(p)p, \lambda - \mu - \frac{\partial \mu}{\partial r}(p) < 0) \\ &- \#(p \in T, H_f(p)p = \lambda(p)p, \lambda - \mu - \frac{\partial \mu}{\partial r}(p) > 0) \end{aligned}$$

If  $\lambda - \mu - \frac{\partial \mu}{\partial r}(p) > 0$  on  $T$ , or  $\lambda - \mu - \frac{\partial \lambda}{\partial r}(p) < 0$  on  $T$ , then:

$$\text{Ind} \left( \frac{\partial^2 f}{\partial \bar{z}^2}, 0 \right) \leq 2.$$

In the same direction, in [8], the point 0 is a strong  $H_f$  umbilic if, for some  $\delta > 0$  and all  $r \in (0, \delta)$ , the eigenvalues  $\lambda, \mu$  satisfy on  $|z| = r$  that  $\min \lambda(z) > \max \mu(z)$  the, the index of the umbilic is at most 1.

Form [1]

” Let  $S$  be a surface in  $R^3$ . Then It is known that if  $S$  is a surface with constant mean curvature or special Weingarten, then the index of an isolated point on  $S$  is negative. In the paper we shall prove that the index of an isolated umbilical point on a Willmore surface does not exceed  $\frac{1}{2}$ .

We say that  $S$  is a Willmore surface is it is a stationary surface of the Wilmore functional  $W$ , where the Willmore functional is define by the integral of the square of the mean curvature. ”

### 3 A generalization of Loewner’s vectorfields.

#### 3.1 Definition of $L_n(f)$

A natural generalization of a Loewner vectorfield is the vectorfield,  $L_n(f)$  defined as follows:

$$L_n(f) = (2^n) \left( \operatorname{Re} \frac{\partial^n}{\partial \bar{z}^n}, \operatorname{Im} \frac{\partial^n}{\partial \bar{z}^n} \right) (f) \quad (3.1)$$

$$= \left( \frac{\partial}{\partial x} + \iota \frac{\partial}{\partial y} \right)^n (f) \quad (3.2)$$

If  $n = 1$  we get a gradient vectorfield, if  $n = 2$  a Loewner vectorfield. For other values of  $n$ :

$$L_3(f) = (f_{xxx} - 3f_{xyy}, 3f_{xxy} - f_{yyy}).$$

$$L_4(f) = (f_{xxxx} - 6f_{xxyy} + f_{yyyy}, 4f_{xxxy} - 4f_{xyyy}).$$

The first component of  $L_n(f)$  is:

$$\sum_{k=0}^{k \leq \frac{n}{2}} (-1)^k \binom{n}{2k} \left( \frac{\partial}{\partial x} \right)^{n-2k} \left( \frac{\partial}{\partial y} \right)^{2k}$$

and the second one:

$$\sum_{k=0}^{k \leq \frac{n-1}{2}} (-1)^k \binom{n}{2k+1} \left( \frac{\partial}{\partial x} \right)^{n-2k-1} \left( \frac{\partial}{\partial y} \right)^{2k+1}$$

We can define also  $L_0(f)$  as the vectorfield:

$$\begin{aligned} \frac{dx}{dt} &= f(x, y) \\ \frac{dy}{dt} &= 0 \end{aligned}$$

It is interesting to remark that,  $L_1(\iota f)$  is the canonical system associated to the hamiltonian  $-f(x, y)$  and  $L_2(\iota f)$  is the Loewner vectorfield  $\frac{\pi}{2}$ -rotated.

In the definition of  $L_n$  we can avoid the use of complex derivation. Suppose that we have a vectorfield whose components are  $(f(x, y), g(x, y))$  then we can

generalize the previous definition saying that  $L_n(f, g)$  is a new vectorfield. The components are the real and imaginary parts of the Cauchy-Riemann operator applied to  $f + \imath g$ .

$$\begin{aligned}\frac{\partial}{\partial \bar{z}}(f + \imath g) &= \frac{1}{2}(f_x - g_y + \imath(g_x + f_y)) \\ L_1(f, g) &= (f_x - g_y, g_x + f_y).\end{aligned}$$

With this definition,  $L_1(f)$  corresponds to  $L_1(f, 0)$ .

We will use the notation:

$$\begin{aligned}\text{grad} f &= \begin{pmatrix} f_x \\ f_y \end{pmatrix} \\ \text{sgrad} f &= \begin{pmatrix} f_y \\ -f_x \end{pmatrix}\end{aligned}$$

Then:

$$L(f, g) = \text{grad} f - \text{sgrad} f. \quad (3.3)$$

This characterization of  $L$  make possible to consider higher-dimensional cases and phase space other than the plane.

### 3.2 Some properties

1.- From 3.3 we get a characterization of those maps  $h$  that preserves the structure of a Loewner vectorfield i.e:

$$h^* L(f, g) = L(f \circ h, g \circ h)$$

The map  $h$  must be a canonical transformation and an area preserving map ( $Dh = \pm 1$ ). Then:

$$h^* L(f, g) = h^*(\text{grad} f - \text{sgrad} f) = h^* \text{grad} f - h^* \text{sgrad} f = L(f \circ h, g \circ h)$$

2.- If in the expression of  $L_n f$  we substitute  $a f_{x^u y^v}$  by  $a(\cos(\theta))^u (\sin(\theta))^v$  we obtain the trigonometric expansion of  $\cos(n\theta)$  and  $\sin(n\theta)$  in terms of  $\cos \theta$  and  $\sin \theta$ . Let us prove this fact:

For shortness we call  $c = \cos(\theta)$  and  $s = \sin(\theta)$ . Then, the expression of:

$$L_n(f) = \left( \frac{\partial}{\partial x} + \imath \frac{\partial}{\partial y} \right)^n f$$

becomes:

$$(\text{Re}(c + \imath s)^n, \text{Im}(c + \imath s)^n)$$

But this expression is equal to:

$$(\text{Re}(e^{\imath n\theta}), \text{Im}(e^{\imath n\theta})) = (\cos(n\theta), \sin(n\theta))$$

Let  $B$  be the open unit ball in  $R^n$  centered at  $O$ , we state the following conjecture:

**2.- The Generalized Loewner Conjecture.** *If  $f$  is  $C^n(B)$  and  $L_n(f)$  has an isolated equilibrium point on  $O$ , then:*

$$\text{Ind}(L_n(f), O) \leq n \quad n \geq 0$$

It is easy to prove that there exists  $f(x, y)$  such that  $\text{Ind}(L_n(f), O) = n$ . One must take:

$$f = (x^2 + y^2)^n$$

Then:

$$L_n(f) = r^n(\cos \theta, \sin \theta)$$

The equilibrium point of this vectorfield has  $2n - 2$  sectors, all of which are elliptic.

If  $n = 0$  the lines  $y = C$  are invariant, therefore  $\text{Ind}(L_0(f), O) = 0$ .

If  $n = 1$ , we have a gradient vectorfield. Since  $f$  is increasing on all non equilibrium trajectories it is not possible any elliptic sector. Therefore  $(L_1(f), O) \leq 1$ .

For  $n > 2$  the numerical test where  $f$  is an homogeneous polynomial confirms the conjecture. An homogeneous polynomial has invariant rays. By means of a linear change of coordinates we can fix two rays on the axis, and by means of a scale we fix another ray as  $y = x$ . If the vectorfield is quadratic there are at most this three rays. The maximum number of elliptic sectors will be six. The test ( see for instance in subsection 6.1 one of the Mathematica programs ) prove that after a blow-up some of the new equilibrium points are of saddle type. Therefore the conjecture is not contradicted.

## 4 Index and derivability

To study the dependence of the index with respect the differentiability of the vectorfield we study the bifurcation of the Loewner vectorfield corresponding to:

$$f(x, y) = \left( \frac{x^4 + y^4}{x^2 + y^2} \right)^a$$

The Loewner vectorfield can be regularized by a temporal scaling:

$$\frac{d\tau}{dt} = \frac{4af}{(x^2 + y^2)^2(x^4 + y^4)^2}$$

It becomes:

$$\begin{aligned} \frac{dx}{dt} &= (x^2 - y^2) (ax^8 + (4a - 6)x^6y^2 + (10a - 8)x^4y^4 + (4a - 6)x^2y^6 + ay^8) \\ \frac{dy}{dt} &= -2xy ((a - 1)x^8 + 4x^6y^2 + 6(1 - a)x^4y^4 + 4x^2y^6 + (a - 1)y^8) \end{aligned}$$

In the next Figures (1, 2, 3, 4) we see the cases of

$C^0$ -differentiability :  $a < \frac{1}{2}$

and

$C^1$ -differentiability :  $a < \frac{3}{2}$

If the function is differentiable it doesn't exist elliptic sectors.

For surfaces it is easy to find examples of  $C^0$  differentiability and with umbilic points of index greater than two. See [3], [2].

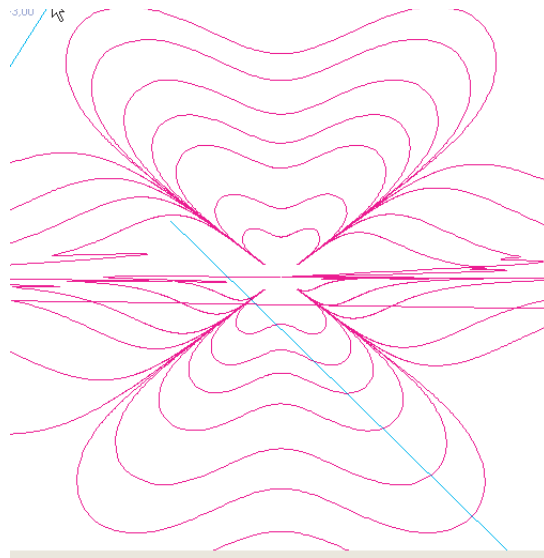


Figure 1:  $a=-3$

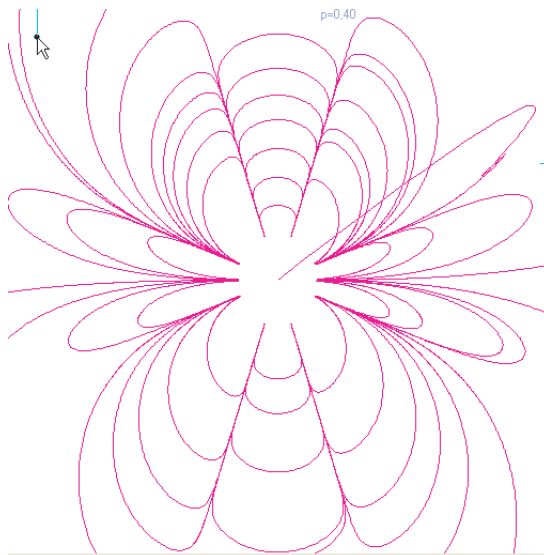


Figure 2:  $a=0.4$

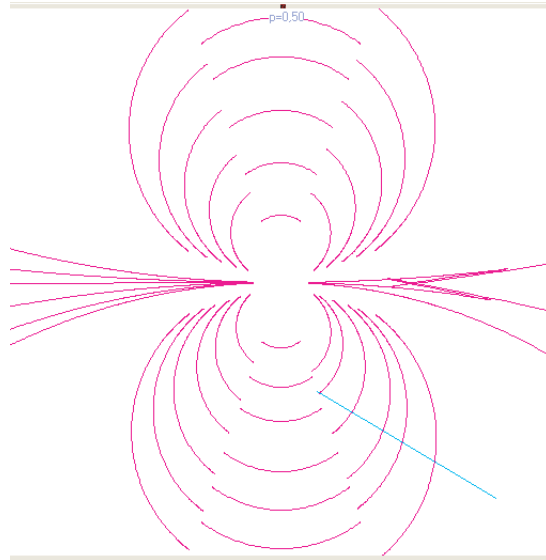


Figure 3:  $a=0.5$

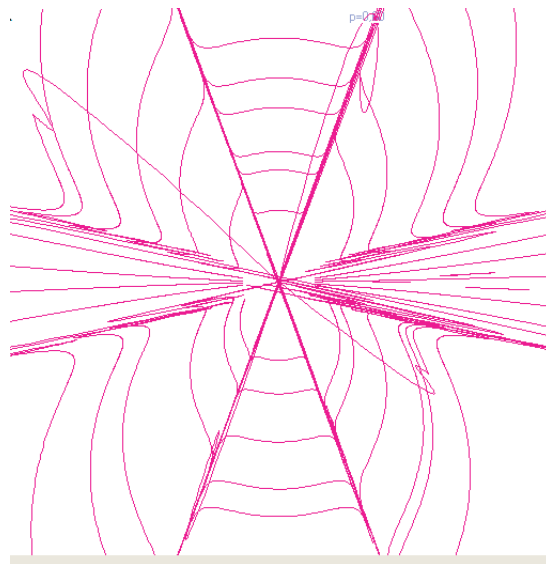


Figure 4:  $a=0.6$



## 5 First integrals

In this section we want to study basic systems (5.26) with a given first integral.

### 5.1 Hamiltonian case

If we assume that the system 5.26 is hamiltonian, putting the condition:

$$F_1 = \frac{\partial H}{\partial y}, \quad F_2 = -\frac{\partial H}{\partial x}$$

into the condition (1.6) one arrives to

$$\frac{\partial^4 H}{\partial x^4} + \frac{\partial^4 H}{\partial x^2 \partial y^2} + \frac{\partial^4 H}{\partial y^4} = G(y) \quad (5.1)$$

### 5.2 General case

Assume now that  $K(x, y)$  is a first integral. Then:

$$K_x(W_{xx} - W_{yy}) + 2K_y W_{xy} = 0 \quad (5.2)$$

Or

$$K_x W_{xx} + 2K_y W_{xy} - K_x W_{yy} = 0 \quad (5.3)$$

If we solve this equation, we find a Hessian foliation which leaves are the level sets of  $K(x, y)$ . If the first integral is, for instance:

$$\frac{-2x(x^2 - 3y^2)}{3(x^2 + y^2)^3} \quad (5.4)$$

whose level sets are in the Figure (5), we have a counterexample of the Loewner Conjecture.

In ([2]) it is constructed a closed surface with a single topological umbilic of index two. But the differentiability conditions do not seem clear.

As a guide to solve (5.3) we follow ([6]). We want to find  $W(x, y)$  from the given first integral  $K(x, y)$ . The equation (5.2) is a particular case of a second order linear partial differential equation:

$$L(u) = A \frac{\partial^2 u}{\partial x^2} + 2B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} = 0 \quad (5.5)$$

To write (5.3) in normal form we consider, as usual, the equation of the characteristics:

$$A \left( \frac{\partial u}{\partial x} \right)^2 + 2B \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + C \left( \frac{\partial u}{\partial y} \right)^2 = 0 \quad (5.6)$$

In our particular case:

$$K_x \left( \left( \frac{\partial u}{\partial x} \right)^2 - \left( \frac{\partial u}{\partial y} \right)^2 \right) + 2K_y \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} = 0 \quad (5.7)$$

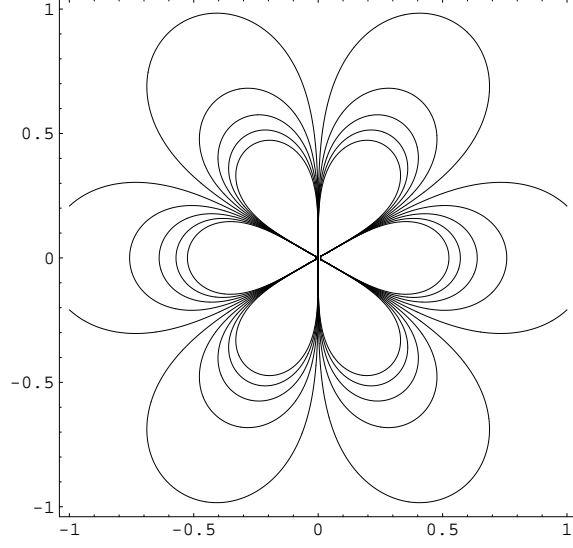


Figure 5: Level sets of (5.4)

The discriminant of (5.7):

$$K_y^2 + K_x^2$$

is positive. Therefore we have an equation of hyperbolic type.

The equation of (5.7) can be broken into two equations with real coefficients:

$$\frac{\partial u}{\partial x} - \alpha_1 \frac{\partial u}{\partial y} = 0, \quad \frac{\partial u}{\partial x} - \alpha_2 \frac{\partial u}{\partial y} = 0 \quad (5.8)$$

where  $\alpha_1, \alpha_2$  are roots of the equation

$$K_x \alpha^2 + 2K_y \alpha - K_x = 0 \quad (5.9)$$

That is to say:

$$\alpha = \frac{-K_y \pm \sqrt{K_x^2 + K_y^2}}{K_x} \quad (5.10)$$

Assuming that  $K_x(x, y) \neq 0$  we are lead to solve the equation:

$$K_x \frac{\partial u}{\partial x} + \left( K_y \pm \sqrt{K_x^2 + K_y^2} \right) \frac{\partial u}{\partial y} = 0 \quad (5.11)$$

Suppose that  $u_1, u_2$  are independent solutions of (5.11). Then, the new variables:

$$u_1(x, y) = \xi, \quad u_2(x, y) = \eta \quad (5.12)$$

transform the equation (5.5) into its canonical form:

$$2B^* \frac{\partial^2 u}{\partial x \partial y} + D^* \frac{\partial u}{\partial x} + E^* \frac{\partial u}{\partial y} = 0 \quad (5.13)$$

where:

$$\begin{aligned} B^* &= A\chi_x\eta_x + B(\chi_x\eta_y + \chi_y\eta_x) + C\chi_y\eta_y \\ D^* &= L(\chi) \\ E^* &= L(\eta) \end{aligned}$$

Let's apply this process to the function (5.4).

$$\frac{2(x^4 - 6x^2y^2 + y^4)}{(x^2 + y^2)^4}(W_{xx} - W_{yy}) + 2\frac{8x(x-y)y(x+y)}{(x^2 + y^2)^4}W_{xy} = 0 \quad (5.14)$$

Or equivalently:

$$(x^4 - 6x^2y^2 + y^4)(W_{xx} - W_{yy}) + 8x(x-y)y(x+y)W_{xy} = 0 \quad (5.15)$$

Then,  $\alpha_1, \alpha_2$  in (5.8) are:

$$\begin{aligned} \alpha_1 &= -1 - \frac{4xy}{x^2 - 2xy - y^2} \\ \alpha_2 &= 1 - \frac{4xy}{x^2 + 2xy - y^2} \end{aligned} \quad (5.16)$$

We must solve the associated partial differential equation:

$$\begin{aligned} u_x + (1 + \frac{4xy}{x^2 - 2xy - y^2})u_y &= 0 \\ u_x + (-1 + \frac{4xy}{x^2 + 2xy - y^2})u_y &= 0 \end{aligned} \quad (5.17)$$

They admit the two independent solutions:

$$\begin{aligned} u_1 &= \frac{x^2 + y^2}{-x + y} \\ u_2 &= \frac{x^2 + y^2}{x + y} \end{aligned} \quad (5.18)$$

After some simplifications, the new variables:

$$\begin{aligned} x &= \frac{\eta\chi(-\eta + \chi)}{\eta^2 + \chi^2} \\ y &= \frac{\eta(\eta + \chi)\chi}{\eta^2 + \chi^2} \end{aligned} \quad (5.19)$$

convert the equation (5.15) in:

$$\chi\eta(\eta^2 + \chi^2)V_{\chi\eta} = 2V_\eta\eta^3 + 2V_\chi\chi^3 \quad (5.20)$$

This equation has the solution:

$$\frac{1}{\chi^2} - \frac{1}{\eta^2} \quad (5.21)$$

In the original variables:

$$W(x, y) = -\frac{4xy}{(x^2 + y^2)^2} \quad (5.22)$$

The differential equation is:

$$\begin{aligned}\frac{dx}{dt} &= 96xy \frac{y^2 - x^2}{(x^2 + y^2)^4}, \\ \frac{dy}{dt} &= 24 \frac{x^4 - 6x^2y^2 + y^4}{(x^2 + y^2)^4}.\end{aligned}\tag{5.23}$$

This is not a counterexample to the Loewner Conjecture since  $W$  is not continuous at the origin.

Another solution is:

$$\chi^2 \eta^2 \tag{5.24}$$

For this solution:

$$W(x, y) = \frac{(x^2 + y^2)^4}{(x^2 - y^2)^2} \tag{5.25}$$

The differential equation is:

$$\begin{aligned}\frac{dx}{dt} &= -192x^2y^2 \frac{(y^2 + x^2)^2}{(x^2 - y^2)^3}, \\ \frac{dy}{dt} &= 48xy \frac{(y^2 + x^2)^2(x^4 - 6x^2y^2 + y^4)}{(x^2 - y^2)^4}.\end{aligned}\tag{5.26}$$

Since the basic differential equation is linear a linear combination of the solutions is again a solution. Therefore:

$$\frac{a(x^2 + y^2)^6 + b(x^2 - y^2)^2}{(x^2 - y^2)^2(x^2 + y^2)^2} \tag{5.27}$$

is a family of solutions with the same problems with the continuity.

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## 6 Appendixes

### 6.1 $L_3(f)$ , $f$ homogeneous polynomial

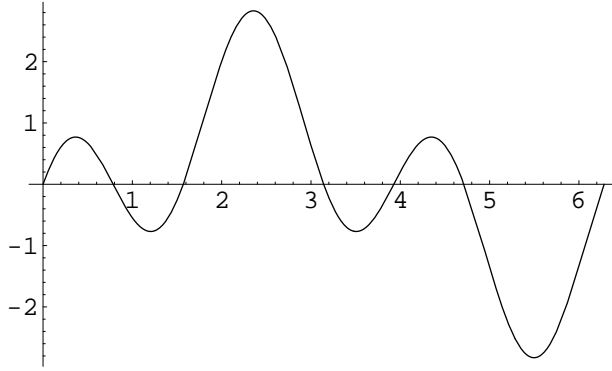
Cas homogeni, grau cinc amb 3 rectes invariants.

**a14** =  $-\frac{1}{192}(-60a^2 - b^2)$ ;  
**a50** =  $-\frac{1}{192}(-108a^2 - 5b^2)$ ; **a41** =  $-\frac{1}{192}(-b^2 - 12c^2)$ ;  
**a32** = **6a14**;  
**a23** = **6a41**;  
**a05** = **5a14** + **5a41** - **a50**;  
**f**[x-, y-] = **a50** \*  $x^5$  + **a41** \*  $x^4 * y$  + **a32** \*  $x^3 * y^2$  + **a23** \*  $x^2 * y^3$  + **a14** \*  $x * y^4$  + **a05** \*  $y^5$   
 $\frac{1}{192}(108a^2 + 5b^2)x^5 + \frac{1}{192}(b^2 + 12c^2)x^4y + \frac{1}{32}(60a^2 + b^2)x^3y^2 + \frac{1}{32}(b^2 + 12c^2)x^2y^3 +$   
 $\frac{1}{192}(60a^2 + b^2)xy^4 + (\frac{1}{192}(-108a^2 - 5b^2) + \frac{5}{192}(60a^2 + b^2) + \frac{5}{192}(b^2 + 12c^2))y^5$   
**xp** = FullSimplify[D[D[D[f[x, y], x], x], x] - 3 \* D[D[D[f[x, y], y], y], x]  
**yp** = FullSimplify[3 \* D[D[D[f[x, y], x], y], x] - D[D[D[f[x, y], y], y], y]  
 $x(b^2(x - y) - 12c^2y)$   
 $y(60a^2(x - y) + b^2(x - y) - 12c^2y)$   
**Solve**[-6 \* **a14** + **a32** == 0, **a32**]  
**Solve**[-6 \* **a41** + **a23** == 0, **a23**]  
{{**a32** → 6**a14**}}  
{{**a23** → 6**a41**}}  
**tn** = FullSimplify[(**yp**/**xp**)/.y → **m** \* **x**]  
 $\frac{m(-16a14+25a14m+16a41m-5a50m)}{9a14-5a50+16a41m}$   
**pol** = Normal[Series[Denominator[**tn**] \* **m** - Numerator[**tn**], {**m**, 0, 6}]]  
 $(25a14 - 5a50)m + (-25a14 + 5a50)m^2$   
**vpol** = **pol**/.**m** → 1  
0  
**Solve**[**vpol** == 0, **a05**]  
{{**a05** → 5**a14** + 5**a41** - **a50**}}  
FullSimplify[Solve[**pol** == 0, **m**]]  
{{**m** → 0}, {**m** → 1}}  
**f1** = **xp**;  
**f2** = **yp**;

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f1p = Normal[FullSimplify[((x * f1 + y * f2)/.{x → r * Cos[a], y → r * Sin[a]})/r]]
r2 ((15a2 + b2) Cos[a] - 15a2 Cos[3a] - (45a2 + b2 + 12c2) Sin[a] + 15a2 Sin[3a])
f2p = Normal[FullSimplify[((x * f2 - y * f1)/.{x → r * Cos[a], y → r * Sin[a]})/r2]]
15a2 r (-Cos[a] + Cos[3a] + Sin[a] + Sin[3a])
Series[f2p, {r, 0, 3}]
15a2 (-Cos[a] + Cos[3a] + Sin[a] + Sin[3a]) r + O[r]4
g1 = FullSimplify[f1p/r]
g2 = FullSimplify[f2p/r]
r ((15a2 + b2) Cos[a] - 15a2 Cos[3a] - (45a2 + b2 + 12c2) Sin[a] + 15a2 Sin[3a])
15a2 (-Cos[a] + Cos[3a] + Sin[a] + Sin[3a])
sg2 = FullSimplify[g2/.r → 0]
ss = Solve[sg2 == 0, a]
15a2 (-Cos[a] + Cos[3a] + Sin[a] + Sin[3a])
Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solutions
{{a → 0}, {a → -π}, {a → -3π/4}, {a → -π/2}, {a → π/4}, {a → π/2}, {a → π}}
Plot[-Cos[a] + Cos[3a] + Sin[a] + Sin[3a], {a, 0, 2 * Pi}]

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–Graphics–

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m11 = FullSimplify[D[g1, r]];
m12 = FullSimplify[D[g1, a]];
m21 = FullSimplify[D[g2, r]];
m22 = FullSimplify[D[g2, a]];
A = {{m11, m12}, {m21, m22}};
A0 = FullSimplify[A/.r → 0];
MatrixForm[A0]
( (15a2 + b2) Cos[a] - 15a2 Cos[3a] - (45a2 + b2 + 12c2) Sin[a] + 15a2 Sin[3a]
0
15a((-2 + a) Cos[a] + (2 + 3a) Sin[a])
FullSimplify[Eigenvalues[A0/.a → 0]]
{0, b2}
FullSimplify[Eigenvalues[A0/.a → Pi/4]]
{-15π2/4√2, -6√2c2}
FullSimplify[Eigenvalues[A0/.a → Pi/2]]
{15π2, -b2 - 12c2 - 15π2}
FullSimplify[Eigenvalues[A0/.a → -3π/4]]
{135π2/4√2, 6√2c2}
FullSimplify[Eigenvalues[A0/.a → -Pi/2]]

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{-15π2, b2 + 12c2 + 15π2}
FullSimplify[Eigenvalues[A0/.a → Pi]]
{-b2, -60π2}
Solve[{5a14 - a50 == a^2, -108a14 + 60a50 == b^2, 9a14 + 16a41 - 5a50 == c^2}, {a14, a50, a41}]
{{a14 → - $\frac{1}{192}(-60a^2 - b^2)$ , a50 → - $\frac{1}{192}(-108a^2 - 5b^2)$ , a41 → - $\frac{1}{192}(-b^2 - 12c^2)$ }}

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