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LIMIT CYCLES FROM A 4-DIMENSIONAL CENTER IN $\mathbb{R}^m$ IN RESONANCE $p : q$

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Abstract. Given positive coprime integers p and q, we consider the linear differential center $\dot{x} = Ax$ in $\mathbb{R}^m$ with eigenvalues $\pm pi$, $\pm qi$ and 0 with multiplicity $m - 4$. We perturb this linear center in the class of all polynomial differential systems of the form linear plus a homogeneous nonlinearity of degree $p + q - 1$, i.e., $\dot{x} = Ax + \varepsilon F(x)$ where every component of $F(x)$ is a linear polynomial plus a homogeneous polynomial of degree $p + q - 1$. When the displacement function of order ε of the perturbed system is not identically zero, we study the maximal number of limit cycles that can bifurcate from the periodic orbits of the linear differential center.

1. Introduction

In the qualitative theory of polynomial differential systems the study of their limit cycles and mainly the obtention of information on their number for a given polynomial differential system is one of the main topics. We recall that for a differential system a *limit cycle* is a periodic orbit isolated in the set of all its periodic orbits.

In dimension two, i.e. in the plane, the two main problems related with limit cycles are: First, the study of the number of limit cycles depending on the degree of the polynomial differential system. This is an old problem proposed by D. Hilbert in 1900, known as the 16–th Hilbert problem (see the surveys [5, 6] for details), and second the study of how many limit cycles emerge from the periodic orbits of a given center when we perturb it inside a certain class of differential systems (see the book [4]).

Since the study of limit cycles and mainly the obtention of information on their number for a given polynomial differential system is in general a very difficult problem (almost impossible), there are in the plane hundreds of papers trying to solve these questions for many particular families of polynomial systems, see the references quoted in the book [4] and in the surveys [5,6].

These problems have been studied intensively in dimension two, and unfortunately the results are far from be satisfactory. In fact, the Riemann conjecture and the 16-th Hilbert problem are the unique two problems of the famous list of Hilbert which are not solved.

Our main aim is to extend these studies from dimension two to higher dimension, and to observe the differences which appear due to the increasing of the dimension of the polynomial differential systems. Thus we take a linear resonant center $p : q$ of dimension 4 living inside dimension $m \geq 5$ and study how many of the periodic orbits of the center persist as limit cycles once this center is perturbed inside a

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class of polynomial differential systems of degree $p+q-1$. The interesting result obtained for this class of polynomial differential systems is that the number of limit cycles obtained are powers related with the dimension m having bases related with the degree of the perturbation $p+q-1$, for the precise result see Theorem 1.

Here we study how many limit cycles emerge from the periodic orbits of a center when we perturb it inside a given class of differential equations in dimension higher than four. More precisely given $m \geq 5$ we consider the linear differential center

$$\frac{dx}{dt} = \dot{x} = Ax \tag{1}$$

in $\mathbb{R}^m$, where

$$A = \begin{pmatrix} 0 & -p & 0 & 0 & 0 & \cdots & 0 \\ p & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -q & 0 & \cdots & 0 \\ 0 & 0 & q & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

for some positive coprime integers p and q. We perturb system (1) in the form

$$\dot{x} = Ax + \varepsilon F(x), \tag{2}$$

where ε is a small parameter, and where $F\colon \mathbb{R}^m \to \mathbb{R}^m$ is a polynomial of the form $F = (F_1^1 + F_N^1, \ldots, F_1^m + F_N^m)$ with F_1^k and F_N^k arbitrary homogeneous polynomials respectively of degrees 1 and $N = p+q-1$ in the variables $x = (x_1, \ldots, x_m)$ for $k = 1, \ldots, m$, with the exception that $F_1^k = \lambda_k x_k$ for $k = 5, \ldots, m$. We note that the polynomial perturbations $F(x)$ of this form cover all polynomial perturbations of system (2) of degree 2 and 3 (this follows from the theory of normal forms; see [3] for details).

For $\varepsilon = 0$ the differential system (2) has a 4-dimensional center in resonance $p : q$. Without loss of generality we can assume that $q > p$. We want to study how many limit cycles can bifurcate from the periodic orbits of this center when we perturb it inside the class of polynomial vector fields of the form linear plus a homogeneous nonlinearity of degree $p+q-1$. Our main result is the following.

Theorem 1. *Assume that $p, q \geq 1$ are coprime integers with $q > p$ and that $m \geq 5$. If $\varepsilon \neq 0$ is sufficiently small and the displacement function of order ε (see (5)) is not identically zero, then the maximum number of limit cycles of the differential system (2) bifurcating from the periodic orbits of the 4-dimensional linear differential center (1) is at most*

(a) $2^m + 2^{m-1}3^2 + 2^4 5^{m-4}$ *if* $q = 2$, $p = 1$, *and*
(b) $2pq(p+q-1)^{m-3}(p+q)^2 + 2pq(p+q+1)(p+q+2)^{m-4}$ *if* $q \geq 3$.

We refer to section 2 for the definition of the displacement function of order ε. Theorem 1 is proved in section 4 using the averaging theory described in section 2. Indeed, Theorem 1 depends heavily on the computation of the averaged system associated to the differential system (2), because its singular points with Jacobian nonzero provide the limit cycles of the differential system (2) when the displacement function of order ε is not identically zero. Theorem 1 improves and extends previous results for system (2) restricted to $\mathbb{R}^4$ (see [3, 7]) and in $\mathbb{R}^m$ for $p = 1$ (see [1]).

When p, q and m are relatively small the averaged system can be computed explicitly, thus allowing one to improve the upper bound for the number of limit cycles given by Theorem 1. In particular, we have established the following result in [1].

Theorem 2. *If $\varepsilon \neq 0$ is sufficiently small and the displacement function of order ε is not identically zero, then the maximum number of limit cycles of the differential system* (2) *bifurcating from the periodic orbits of the* 4*-dimensional linear differential center* (1) *is at most*

(a) 20 *if* $q = 2$, $p = 1$ *and* $m = 5$, *and*
(b) 46 *if* $q = 3$, $p = 1$ *and* $m = 5$.

We note that the corresponding upper bounds given by Theorem 1 are respectively 256 and 1044.

2. First-order averaging theory

The aim of this section is to present the first-order averaging method obtained in [2]. We first briefly recall the basic elements of averaging theory. Roughly speaking, the method gives a quantitative relation between the solutions of a nonautonomous periodic system and the solutions of its averaged system, which is autonomous. The following theorem provides a first order approximation for periodic solutions of the original system.

We consider the differential system

$$\dot{x}(t) = \varepsilon H(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{3}$$

where $H\colon \mathbb{R} \times D \to \mathbb{R}^n$ and $R\colon \mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable, and where D is an open subset of $\mathbb{R}^n$. We define $h\colon D \to \mathbb{R}^n$ by

$$h(z) = \int_0^T H(s, z)\, ds, \tag{4}$$

and we denote by $d_B(h, V, a)$ the Brouwer degree of h at a (see [8] for the definition).

Theorem 3. *We assume that:*

(i) *H and R are locally Lipschitz with respect to x;*
(ii) *for $a \in D$ with $h(a) = 0$, there exists a neighborhood V of a such that $h(z) \neq 0$ for all $z \in V \setminus \{a\}$ and $d_B(h, V, a) \neq 0$.*

Then for $\varepsilon \neq 0$ sufficiently small there exists an isolated T-periodic solution $\phi(\cdot, \varepsilon)$ of system (3) *such that $\phi(a, 0) = a$.*

The system $\dot{x} = \varepsilon h(x)$, is called the *averaged system* associated to system (3).

Hypothesis (i) ensures the existence and uniqueness of the solution of each initial value problem on the interval $[0, T]$. Hence, for each $z \in D$, it is possible to denote by $x(\cdot, z, \varepsilon)$ the solution of system (3) with the initial value $x(0, z, \varepsilon) = z$. We also consider the function $\zeta\colon D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^n$ defined by

$$\zeta(z, \varepsilon) = \int_0^T \big(\varepsilon H(t, x(t, z, \varepsilon)) + \varepsilon^2 R(t, x(t, z, \varepsilon), \varepsilon)\big)\, dt. \tag{5}$$

This is called the *displacement function of order ε*. It follows from the proof of Theorem 3 that for every $z \in D$ the following relations hold:

$$x(T, z, \varepsilon) - x(0, z, \varepsilon) = \zeta(z, \varepsilon), \quad \text{and} \quad \zeta(z, \varepsilon) = \varepsilon h(z) + O(\varepsilon^2),$$

where h is given by (4) and where the symbol $O(\varepsilon^2)$ denotes a function bounded on every compact subset of $D \times (-\varepsilon_0, \varepsilon_0)$ multiplied by ε^2.

We note that in order to see that $d_B(h, V, a) \neq 0$ it is sufficient to check that the Jacobian of $D_z h(z)$ at $z = a$ is not zero, see for more details [8].

3. Averaged system

Writing

$$F_1 = (F_1^1, F_1^2, F_1^3, F_1^4, 0, \dots, 0), \quad F_N = (F_N^1, F_N^2, F_N^3, F_N^4, F_N^5, \dots, F_N^m),$$

system (2) becomes

$$\begin{aligned} \dot{x}_1 &= -px_2 + \varepsilon(F_1^1(x) + F_N^1(x)), \\ \dot{x}_2 &= px_1 + \varepsilon(F_1^2(x) + F_N^2(x)), \\ \dot{x}_3 &= -qx_4 + \varepsilon(F_1^3(x) + F_N^3(x)), \\ \dot{x}_4 &= qx_3 + \varepsilon(F_1^4(x) + F_N^4(x)), \\ \dot{x}_k &= \varepsilon(\lambda_k x_k + F_N^k(x)), \quad k = 5, \dots, m. \end{aligned} \tag{6}$$

Lemma 4. *Doing the change of variables from $(x_1, x_2, x_3, x_4, x_5, \dots, x_m)$ to the new variables $(\theta, r, \rho, s, y_5, \dots, y_m)$ given by*

$$x_1 = r\cos(p\theta),\ x_2 = r\sin(p\theta),\ x_3 = \rho\cos(q(\theta+s)),\ x_4 = \rho\sin(q(\theta+s)),\ x_k = y_k,$$

for $k = 5, \dots, m$, and taking θ as the new independent variable, system (6) is transformed into the system

$$\begin{aligned} \frac{dr}{d\theta} &= \varepsilon H_1(\theta, r, \rho, s, y_5, \dots, y_m) + O(\varepsilon^2), \\ \frac{d\rho}{d\theta} &= \varepsilon H_2(\theta, r, \rho, s, y_5, \dots, y_m) + O(\varepsilon^2), \\ \frac{ds}{d\theta} &= \varepsilon H_3(\theta, r, \rho, s, y_5, \dots, y_m) + O(\varepsilon^2), \\ \frac{dy_k}{d\theta} &= \varepsilon H_k(\theta, r, \rho, s, y_5, \dots, y_m) + O(\varepsilon^2), \quad k = 5, \dots, m, \end{aligned} \tag{7}$$

where

$$\begin{aligned} H_1 &= (F_1^1 + F_N^1)\cos(p\theta) + (F_1^2 + F_N^2)\sin(p\theta), \\ H_2 &= (F_1^3 + F_N^3)\cos(q(\theta+s)) + (F_1^4 + F_N^4)\sin(q(\theta+s)), \\ H_3 &= \frac{1}{q\rho}\big((F_1^4 + F_N^4)\cos(q(\theta+s)) - (F_1^3 + F_N^3)\sin(q(\theta+s))\big) \\ &\quad - \frac{1}{pr}\big((F_1^2 + F_N^2)\cos(p\theta) - (F_1^1 + F_N^1)\sin(p\theta)\big), \\ H_k &= \lambda_k y_k + F_N^k. \end{aligned}$$

Proof. In the variables $(\theta, r, \rho, s, y_5, \dots, y_m)$ system (6) becomes

$$\begin{aligned} \dot{\theta} &= 1 + \frac{\varepsilon}{r}\big(\cos(p\theta)(F_1^2 + F_N^2) - \sin(p\theta)(F_1^1 + F_N^1)\big), \\ \dot{r} &= \varepsilon H_1(\theta, r, \rho, s, y_5, \dots, y_m), \\ \dot{\rho} &= \varepsilon H_2(\theta, r, \rho, s, y_5, \dots, y_m), \\ \dot{s} &= \varepsilon H_3(\theta, r, \rho, s, y_5, \dots, y_m), \\ \dot{y}_k &= \varepsilon H_k(\theta, r, \rho, s, y_5, \dots, y_m), \qquad k = 5, \dots, m. \end{aligned} \tag{8}$$

For ε sufficiently small, $\dot{\theta}(t) > 0$ for each $(t, (\theta, r, \rho, s, y_5, \dots, y_m)) \in \mathbb{R} \times D$. Now we eliminate the variable t in the above system by considering θ as the new independent variable. It is clear that the right-hand side of the new system is well defined and continuous in $\mathbb{R} \times D \times (-\varepsilon_0, \varepsilon_0)$, 2π-periodic with respect to the independent variable θ, and locally Lipschitz with respect to $(r, \rho, s, y_5, \dots, y_m)$. From (8) equation (7) is obtained after an expansion with respect to the small parameter ε. □

We recall a technical result from [3] that we shall use later on.

Lemma 5. *Let α and β be real numbers. Given nonnegative integers i, j, k, l, there exist constants c_{uv} and d_{uv} such that $\cos^i \alpha \sin^j \alpha \cos^k \beta \sin^l \beta$ is equal to*

$$\sum_{u=0}^{[(i+j)/2]} \sum_{v=0}^{[(k+l)/2]} c_{uv} \cos\big((i+j-2u)\alpha \pm (k+l-2v)\beta\big)$$

if $j+l$ is even, and is equal to

$$\sum_{u=0}^{[(i+j)/2]} \sum_{v=0}^{[(k+l)/2]} d_{uv} \sin\big((i+j-2u)\alpha \pm (k+l-2v)\beta\big)$$

if $j+l$ is odd. Here $[x]$ denotes the integer part function of $x \in \mathbb{R}$.

Now we compute the corresponding averaged functions $h_j(r, \rho, s, y_5, \ldots, y_m)$ for $j = 1, \ldots, m$ of system (7) given in (4). We write

$$F_1^g = \sum_{j=1}^{m} a_j^g x_j \quad \text{and} \quad F_N^g = \sum_{i_1+i_2+\cdots+i_m=N} a_{i_1\cdots i_m}^g x_1^{i_1} x_2^{i_2} \cdots x_m^{i_m},$$

for $g = 1, \ldots, m$. We also write

$$h_j(r, \rho, s, y_5, \ldots, y_m) = \int_0^{2\pi} H_j(\theta, r, \rho, s, y_5, \ldots, y_m)\, d\theta$$

for $j = 1, 2, 3, 5, \ldots, m$.

Proposition 6. *We have*

$$\begin{aligned} h_1(r, \rho, s, y_5, \ldots, y_m) &= a_1 r + r^{q-1}\rho^p (b_1 \sin(pqs) + c_1 \cos(pqs)) \\ &\quad + \sum_{2l+i_5+\cdots+i_m=0}^{p+q-1} d_{li_5\cdots i_m}^1 r^{p+q-1-2l-i_5-\cdots-i_m} \rho^{2l} y_5^{i_5} \cdots y_m^{i_m}, \end{aligned}$$

for some constants a_1, b_1, c_1 and $d_{li_5\cdots i_m}^1$ depending on the coefficients of the perturbation.

Proof. We write the function H_1 as

$$H_1 = H_1^1 + H_1^N = (F_1^1 \cos(p\theta) + F_1^2 \sin(p\theta)) + (F_N^1 \cos(p\theta) + F_N^2 \sin(p\theta)).$$

Then

$$\begin{aligned} h_1^1(r, s, \rho, y_5, \ldots, y_m) &= \int_0^{2\pi} H_1^1(\theta, r, s, \rho, y_5, \ldots, y_m)\, d\theta \\ &= \sum_{j=1}^{m} \int_0^{2\pi} \big(a_j^1 \cos(p\theta) + a_j^2 \sin(p\theta)\big) x_j \, d\theta = \pi(a_1^1 + a_2^2) r, \end{aligned} \tag{9}$$

and

$$
\begin{aligned}
&h_1^N(r,s,\rho,y_5,\dots,y_m) = \int_0^{2\pi} H_1^N(\theta,r,s,\rho,y_5,\dots,y_m)\,d\theta\\
&= \sum_{i_1+\cdots+i_m=N} \int_0^{2\pi} \big(a^1_{i_1\cdots i_m} x_1^{i_1}\cdots x_m^{i_m}\cos(p\theta) + a^2_{i_1\cdots i_m} x_1^{i_1}\cdots x_m^{i_m}\sin(p\theta)\big)\,d\theta\\
&= \sum_{i_1+\cdots+i_m=N} \int_0^{2\pi} a^1_{i_1\cdots i_m} r^{i_1+i_2}\rho^{i_3+i_4}\cos^{i_1+1}(p\theta)\sin^{i_2}(p\theta)\\
&\quad\cdot \cos^{i_3}(q(\theta+s))\sin^{i_4}(q(\theta+s)) y_5^{i_5}\cdots y_m^{i_m}\,d\theta\\
&\quad+ \sum_{i_1+\cdots+i_m=N} \int_0^{2\pi} a^2_{i_1\cdots i_m} r^{i_1+i_2}\rho^{i_3+i_4}\cos^{i_1}(p\theta)\sin^{i_2+1}(p\theta)\\
&\quad\cdot \cos^{i_3}(q(\theta+s))\sin^{i_4}(q(\theta+s)) y_5^{i_5}\cdots y_m^{i_m}\,d\theta.
\end{aligned}
$$

By Lemma 5 we obtain

$$
\begin{aligned}
h_1^N(r,s,\rho,y_5,\dots,y_m) = &\sum_{i_1+\cdots+i_m=N} r^{i_1+i_2}\rho^{i_3+i_4} y_5^{i_5}\cdots y_m^{i_m}\\
&\cdot \int_0^{2\pi} \sum_{u=0}^{[(i_1+i_2+1)/2]} \sum_{v=0}^{[(i_3+i_4)/2]} C_{uv}^{i_1\cdots i_m}(\theta)\,d\theta,
\end{aligned}
$$

where

$$
\begin{aligned}
C_{uv}^{i_1\cdots i_m} = &\, c_{uv}^{i_1\cdots i_m}\cos\big((i_1+i_2+1-2u)p\theta \pm (i_3+i_4-2v)q(\theta+s)\big)\\
&+ d_{uv}^{i_1\cdots i_m}\sin\big((i_1+i_2+1-2u)p\theta \pm (i_3+i_4-2v)q(\theta+s)\big),
\end{aligned}
$$

for some constants $c_{uv}^{i_1\cdots i_m}$ and $d_{uv}^{i_1\cdots i_m}$. Therefore all the integrals with respect to θ are zero except possibly when

$$
p(i_1+i_2+1-2u) = q(i_3+i_4-2v). \tag{10}
$$

Without loss of generality we continue to assume that $p<q$. Since p and q are coprime, there exists a nonnegative integer n such that $i_1+i_2+1-2u = nq$ and $i_3+i_4-2v = np$. Furthermore, since

$$
0 \le i_1+i_2+1-2u \le N+1 = p+q,
$$

we have that $nq \le p+q$, and thus $n \le (p+q)/q < 2$. So either $n=1$ or $n=0$, i.e., either $i_3+i_4-2v=p$ or $i_3+i_4-2v=0$.

If $i_3+i_4-2v=p$, then $i_1+i_2+1-2u=q$, and it follows from (10) that

$$
i_5+\cdots+i_m = N-(i_1+i_2+i_3+i_4) = -2(u+v).
$$

Therefore $u=v=0=i_5=\cdots=i_m=0$, and hence $i_1+i_2=q-1$ and $i_3+i_4=p$. This yields the term

$$
r^{q-1}\rho^p\big(b_1\sin(pqs)+c_1(\cos pqs)\big). \tag{11}
$$

If $i_3+i_4-2v=0$, then $2v+i_5+\cdots+i_m = N-i_1-i_2$, and $2v+i_5+\cdots+i_m$ runs from 0 to $N=p+q-1$. This yields the terms

$$
\sum_{2v+i_5+\cdots+i_m=0}^{p+q-1} d^1_{vi_5\cdots i_m} r^{p+q-1-2v-i_5-\cdots-i_m}\rho^{2v} y_5^{i_5}\cdots y_m^{i_m}. \tag{12}
$$

The proposition follows adding the terms from (9), (11) and (12). $\square$

Proposition 7. *We have*

$$h_2(r,\rho,s,y_5,\dots,y_m) = a_2\rho + r^q\rho^{p-1}(b_2\sin(pqs) + c_2\cos(pqs)) + \sum_{2v+i_5+\cdots+i_m=1}^{p+q} d^2_{vi_5\cdots i_m} r^{p+q-2v-i_5-\cdots-i_m}\rho^{2v-1}y_5^{i_5}\cdots y_m^{i_m},$$

for some constants a_2, b_2, c_2 and $d^2_{vi_5\cdots i_m}$ depending on the coefficients of the perturbation.

Proof. As in Proposition 6 we write the function H_2 as

$$H_2 = H_2^1 + H_2^N = \big(F_1^3\cos(q(\theta+s)) + F_1^4\sin(q(\theta+s)\big) + \big(F_N^3\cos(q(\theta+s)) + F_N^4\sin(q(\theta+s)\big).$$

Then

$$\begin{aligned} h_2^1(r,s,\rho,y_5,\dots,y_m) &= \int_0^{2\pi} H_2^1(\theta,r,s,\rho,y_5,\dots,y_m)\,d\theta \\ &= \sum_{j=1}^m \int_0^{2\pi} \big(a_j^3\cos(q(\theta+s)) + a_j^4\sin(q(\theta+s))\big)x_j\,d\theta \\ &= \pi(a_3^3 + a_4^4)\rho, \end{aligned} \tag{13}$$

and using Lemma 5 we obtain

$$\begin{aligned} h_2^N(r,s,\rho,y_5,\dots,y_m) &= \int_0^{2\pi} H_2^N(\theta,r,s,\rho,y_5,\dots,y_m)\,d\theta \\ &= \sum_{i_1+\cdots+i_m=N}\int_0^{2\pi} a^3_{i_1\cdots i_m} r^{i_1+i_2}\rho^{i_3+i_4}\cos^{i_1}(p\theta)\sin^{i_2}(p\theta) \\ &\qquad \cdot\cos^{i_3+1}(q(\theta+s))\sin^{i_4}(q(\theta+s))y_5^{i_5}\cdots y_m^{i_m}\,d\theta \\ &\quad + \sum_{i_1+\cdots+i_m=N}\int_0^{2\pi} a^4_{i_1\cdots i_m} r^{i_1+i_2}\rho^{i_3+i_4}\cos^{i_1}(p\theta)\sin^{i_2}(p\theta) \\ &\qquad \cdot\cos^{i_3}(q(\theta+s))\sin^{i_4+1}(q(\theta+s))y_5^{i_5}\cdots y_m^{i_m}\,d\theta \\ &= \sum_{i_1+\cdots+i_m=N} r^{i_1+i_2}\rho^{i_3+i_4}y_5^{i_5}\cdots y_m^{i_m} \\ &\qquad \cdot\int_0^{2\pi}\sum_{u=0}^{[(i_1+i_2)/2]}\sum_{v=0}^{[(i_3+i_4+1)/2]} D_{uv}^{i_1\cdots i_m}(\theta)\,d\theta, \end{aligned}$$

where

$$\begin{aligned} D_{uv}^{i_1\cdots i_m} &= c_{uv}^{i_1\cdots i_m}\cos\big((i_1+i_2-2u)p\theta \pm (i_3+i_4+1-2v)q(\theta+s)\big) \\ &\quad + d_{uv}^{i_1\cdots i_m}\sin\big((i_1+i_2-2u)p\theta \pm (i_3+i_4+1-2v)q(\theta+s)\big), \end{aligned}$$

for some constants $c_{uv}^{i_1\cdots i_m}$ and $d_{uv}^{i_1\cdots i_m}$. All the integrals with respect to θ are zero except possibly when

$$p(i_1+i_2-2u) = q(i_3+i_4+1-2v). \tag{14}$$

Since p and q are coprime, there exists a nonnegative integer u such that $i_1+i_2-2u = nq$ and $i_3+i_4+1-2v = np$. Furthermore, since

$$0 \le i_3+i_4+1-2v \le N+1 = p+q,$$

we have that $np \le p+q$, and thus $n \le (p+q)/q < 2$. So either $n=1$ or $n=0$, i.e., either $i_3+i_4+1-2v = p$ or $i_3+i_4+1-2v = 0$.

If $i_3+i_4+1-2v=p$, then by (14) we obtain that

$$p+q-1-i_3-i_4-i_5-\cdots-i_m-2u=i_1+i_2-2u=q,$$

and hence,

$$i_5+\cdots+i_m+2u+2v=p-1-i_3-i_4+2v=0.$$

This implies that $i_5=\cdots=i_m=0$ and $u=v=0$. Then $i_3+i_4=p-1$ and $i_1+i_2=q$, which yields the term

$$r^q\rho^{p-1}\big(b_2\sin(pqs)+c_2\cos(pqs)\big). \tag{15}$$

If $i_3+i_4+1-2v=0$, then

$$2v+i_5+\cdots+i_m-1=N-i_1-i_2.$$

Thus $2v+i_5+\cdots+i_m$ runs from 1 to $p+q$, yielding the terms

$$\sum_{2v+i_5+\cdots+i_m=1}^{p+q} d^2_{vi_5\cdots i_m} r^{p+q-2v-i_5-\cdots-i_m}\rho^{2v-1}y_5^{i_5}\cdots y_m^{i_m}. \tag{16}$$

The proposition follows adding the terms of (13), (15) and (16). □

Proposition 8. *We have*

$$\begin{aligned}h_3(r,\rho,s,y_5,\ldots,y_m)=\;&a_3+r^{q-2}\rho^p(b_3\sin(pqs)+c_3\cos(pqs))\\&+r^q\rho^{p-2}\big(d_3\sin(pqs)+e_3\cos(pqs)\big)\\&+\sum_{2v+i_5+\cdots+i_m=0}^{p+q-1} d^3_{vi_5\cdots i_m} r^{p+q-2-2v-i_5-\cdots-i_m}\rho^{2v}y_5^{i_5}\cdots y_m^{i_m}\\&+\sum_{2v+i_5+\cdots+i_m=1}^{p+q} d^4_{vi_5\cdots i_m} r^{p+q-2v-i_5-\cdots-i_m}\rho^{2v-2}y_5^{i_5}\cdots y_m^{i_m},\end{aligned}$$

for some constants a_3,b_3,c_3,d_3,e_3, $d^3_{vi_5\cdots i_m}$ and $d^4_{vi_5\cdots i_m}$ depending on the coefficients of the perturbation.

Proof. We have $H_3=H_3^1+H_3^N$ where

$$\begin{aligned}H_3^1&=\frac{1}{q\rho}\Big(F_1^4\cos\big(q(\theta+s)\big)-F_1^3\sin\big(q(\theta+s)\big)\Big)-\frac{1}{pr}\Big(F_1^2\cos(p\theta)-F_1^1\sin(p\theta)\Big),\\H_3^N&=\frac{1}{q\rho}\Big(F_N^4\cos\big(q(\theta+s)\big)-F_N^3\sin\big(q(\theta+s)\big)\Big)-\frac{1}{pr}\Big(F_N^2\cos(p\theta)-F_N^1\sin(p\theta)\Big).\end{aligned}$$

Proceeding in a similar manner to the proofs of Propositions 6 and 7 we get

$$\begin{aligned}h_3^1(r,\rho,s,y_5,\cdots,y_m)&=\int_0^{2\pi}H_3^1(\theta,r,\rho,s,y_5,\ldots,y_m)\,d\theta\\&=\frac{\pi(a_3^4-a_4^3)}{q}-\frac{\pi(a_1^2-a_1^1)}{p}.\end{aligned} \tag{17}$$

Now we calculate

$$h_3^N(r,\rho,s,y_5,\cdots,y_m)=\int_0^{2\pi}H_3^N(\theta,r,\rho,s,y_5,\ldots,y_m)\,d\theta.$$

In a similar manner to the proofs of Propositions 6 and 7 we get

$$
\begin{aligned}
h_3^N(r,\rho,s,y_5,\cdots,y_m) = \frac{1}{q} \sum_{i_1+\cdots+i_m=N} & r^{i_1+i_2}\rho^{i_3+i_4-1}y_5^{i_5}\cdots y_m^{i_m} \\
& \cdot \int_0^{2\pi} \sum_{u=0}^{[(i_1+i_2)/2]} \sum_{v=0}^{[(i_3+i_4+1)/2]} E_{uv}^{i_1\cdots i_m}(\theta)\, d\theta \\
- \frac{1}{p} \sum_{i_1+\cdots+i_m=N} & r^{i_1+i_2-1}\rho^{i_3+i_4}y_5^{i_5}\cdots y_m^{i_m} \\
& \cdot \int_0^{2\pi} \sum_{u=0}^{[(i_1+i_2+1)/2]} \sum_{v=0}^{[(i_3+i_4)/2]} F_{uv}^{i_1\cdots i_m}(\theta)\, d\theta,
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
E_{uv}^{i_1\cdots i_m} &= c_{uv}^{i_1\cdots i_m}\cos\big((i_1+i_2-2u)p\theta \pm (i_3+i_4+1-2v)q(\theta+s)\big) \\
&\quad + d_{uv}^{i_1\cdots i_m}\sin\big((i_1+i_2-2u)p\theta \pm (i_3+i_4+1-2v)q(\theta+s)\big),
\end{aligned} \tag{19}
$$

and

$$
\begin{aligned}
F_{uv}^{i_1\cdots i_m} &= f_{uv}^{i_1\cdots i_m}\cos\big((i_1+i_2+1-2u)p\theta \pm (i_3+i_4-2v)q(\theta+s)\big) \\
&\quad + g_{uv}^{i_1\cdots i_m}\sin\big((i_1+i_2+1-2u)p\theta \pm (i_3+i_4-2v)q(\theta+s)\big).
\end{aligned} \tag{20}
$$

The terms whose integrals need not be zero satisfy

$$p(i_1+i_2-2u) = q(i_3+i_4+1-2v)$$

in equation (19), and

$$p(i_1+i_2+1-2u) = q(i_3+i_4-2v)$$

in equation (20).

The arguments in the proof of Proposition 7 show that in (18) the terms that may remain in the first sum are

$$
\begin{aligned}
& r^q\rho^{p-2}\big(d_3\sin(pqs)+e_3\cos(pqs)\big) \\
& + \sum_{2v+i_5+\cdots+i_m=1}^{p+q} d^4_{vi_5\cdots i_m} r^{p+q-2v-i_5-\cdots-i_m}\rho^{2v-2}y_5^{i_5}\cdots y_m^{i_m},
\end{aligned} \tag{21}
$$

and the arguments in the proof of Proposition 6 show that the terms that may remain in the second sum are

$$
\begin{aligned}
& r^{q-2}\rho^{p}\big(b_3\sin(pqs)+c_3\cos(pqs)\big) \\
& + \sum_{2v+i_5+\cdots+i_m=0}^{p+q-1} d^3_{vi_5\cdots i_m} r^{p+q-2-2v-i_5-\cdots-i_m}\rho^{2v}y_5^{i_5}\cdots y_m^{i_m}.
\end{aligned} \tag{22}
$$

The proposition follows adding the terms in (17), (21) and (22). $\square$

Proposition 9. *For $k=5,\ldots,m$, we have*

$$
\begin{aligned}
h_k(r,\rho,s,y_5,\ldots,y_m) = \lambda_k y_k & \\
& + \sum_{2v+i_5+\cdots+i_m=0}^{p+q-1} d^5_{vi_5\cdots i_m} r^{p+q-1-2v-i_5-\cdots-i_m}\rho^{2v}y_5^{i_5}\cdots y_m^{i_m},
\end{aligned}
$$

for some constants $d^5_{vi_5\cdots i_m}$ depending on the coefficients of the perturbation.

Proof. As in the former proofs, we write $H_k = H_k^1 + H_k^N$ where $H_k^1 = \lambda_k y_k$ and $H_k^N = F_N^k$, and we compute the function

$$h_k^N(r, s, \rho, y_5, \dots, y_m) = \int_0^{2\pi} H_k^N(\theta, r, s, \rho, y_5, \dots, y_m)\, d\theta.$$

Proceeding as in the proofs of Propositions 6 or 7 we obtain

$$\begin{aligned} h_k^N(r, \rho, s, y_5, \cdots, y_m) &= \sum_{i_1+\cdots+i_m=N} \int_0^{2\pi} a^k_{i_1\cdots i_m} r^{i_1+i_2} \rho^{i_3+i_4} \cos^{i_1}(p\theta) \sin^{i_2}(p\theta) \\ &\qquad \cdot \cos^{i_3}(q(\theta+s)) \sin^{i_4}(q(\theta+s)) y_5^{i_5} \cdots y_m^{i_m}\, d\theta \\ &= \sum_{i_1+\cdots+i_m=N} r^{i_1+i_2} \rho^{i_3+i_4} y_5^{i_5} \cdots y_m^{i_m} \\ &\qquad \cdot \int_0^{2\pi} \sum_{u=0}^{[(i_1+i_2)/2]} \sum_{v=0}^{[(i_3+i_4)/2]} G_{uv}^{i_1\cdots i_m}(\theta)\, d\theta, \end{aligned}$$

where

$$\begin{aligned} G_{uv}^{i_1\cdots i_m} &= g_{uv}^{i_1\cdots i_m} \cos\big((i_1+i_2-2u)p\theta \pm (i_3+i_4-2v)q(\theta+s)\big) \\ &\quad + h_{uv}^{i_1\cdots i_m} \sin\big((i_1+i_2-2u)p\theta \pm (i_3+i_4-2v)q(\theta+s)\big). \end{aligned}$$

All the integrals with respect to θ are zero except possibly when

$$p(i_1+i_2-2u) = q(i_3+i_4-2v). \tag{23}$$

Proceeding as in the proof of Proposition 6, we find that either $i_3+i_4-2v = p$ or $i_3+i_4-2v = 0$.

If $i_3+i_4-2v = p$, then by (23) we obtain

$$q-1-i_5-\cdots-i_m-2v-2u = q, \quad \text{that is,} \quad -1-i_5-\cdots-i_m-2v-2u = 0,$$

which yields a contradiction. Therefore, this case does not occur.

If $i_3+i_4-2v = p$, then

$$2v+i_5+\cdots+i_m = p+q-1-i_1-i_2.$$

Hence $2v+i_5+\cdots+i_m$ runs from 0 to $p+q-1$, and we obtain the terms

$$\sum_{2v+i_5+\cdots+i_m=0}^{p+q-1} d^5_{vi_5\cdots i_m} r^{p+q-1-2v-i_5-\cdots-i_m} \rho^{2v} y_5^{i_5} \cdots y_m^{i_m}.$$

This yields the desired statement. $\square$

4. Proof of Theorem 1

We recall a technical result proved in [7].

Lemma 10. *If p, q, α and β are nonnegative integers with $\alpha+\beta=q-1$ and $\gamma+\delta=p$, then*

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^\alpha(p\theta)\sin^\beta(p\theta)\cos^\gamma(q(\theta+s))\sin^\delta(q(\theta+s))\,d\theta$$
$$=\begin{cases}\dfrac{(-1)^{(\beta+\delta)/2}}{2^{p+q-1}}\cos(pqs) & \text{if } \beta,\ \delta \text{ are even},\\[2mm] \dfrac{(-1)^{(\beta+\delta-1)/2}}{2^{p+q-1}}\sin(pqs) & \text{if } \beta \text{ is even and } \delta \text{ is odd},\\[2mm] \dfrac{-(-1)^{(\beta+\delta-1)/2}}{2^{p+q-1}}\sin(pqs) & \text{if } \beta \text{ is odd and } \delta \text{ is even},\\[2mm] \dfrac{-(-1)^{(\beta+\delta)/2}}{2^{p+q-1}}\cos(pqs) & \text{if } \beta,\ \delta \text{ are odd}.\end{cases}$$

We will use the following proposition.

Proposition 11. *The function $h_3(r,\rho,s,y_5,\dots,y_m)$ is given by*

$$\begin{aligned}h_3(r,\rho,s,y_5,\dots,y_m)=a_3&+\frac{1}{p}r^{q-2}\rho^p\big(-c_1\sin(pqs)+b_1\cos(pqs)\big)\\&+\frac{1}{q}r^q\rho^{p-2}\big(-c_2\sin(pqs)+b_2\cos(pqs)\big)\\&+\sum_{2v+i_5+\cdots+i_m=0}^{p+q-1}d^3_{vi_5\cdots i_m}r^{p+q-2-2v-i_5-\cdots-i_m}\rho^{2v}y_5^{i_5}\cdots y_m^{i_m}\\&+\sum_{2v+i_5+\cdots+i_m=1}^{p+q}d^4_{vi_5\cdots i_m}r^{p+q-2v-i_5-\cdots-i_m}\rho^{2v-2}y_5^{i_5}\cdots y_m^{i_m},\end{aligned}$$

where b_1,c_1 are the constants in Proposition 6, and b_2,c_2 are the constants in Proposition 7.

Proof. Using the notation of Proposition 8 we shall prove that $b_3=-c_1/p$, $c_3=b_1/p$, $d_3=-c_2/q$ and $e_3=b_2/q$. To simplify the proof, let $a^1_{i_1i_2\cdots i_m}x_1^{i_1}x_2^{i_2}\cdots x_m^{i_m}$ be a monomial in F^1_N such that $i_1+i_2=q-1$, $i_3=0$, $i_4=p$ and $i_5=\cdots=i_m=0$. When we compute h_1 and h_3, this monomial appears in h_1 as

$$\int_0^{2\pi}a^1_{i_1\cdots i_m}\cos^{i_1+1}(p\theta)\sin^{i_2}(p\theta)\sin(q(\theta+s))\,d\theta, \tag{24}$$

and in h_3 as

$$\frac{1}{p}\int_0^{2\pi}a^1_{i_1\cdots i_m}\cos^{i_1}(p\theta)\sin^{i_2+1}(p\theta)\sin(q(\theta+s))\,d\theta. \tag{25}$$

By Lemma 10 the term in (24) is equal to

$$\begin{cases}\dfrac{(-1)^{i_2/2}\pi}{2^{p+q}}a^1_{i_1\cdots i_m}\sin(pqs), & \text{if } i_2 \text{ is even},\\[2mm] -\dfrac{(-1)^{(i_2+1)/2}\pi}{2^{p+q}}a^1_{i_1\cdots i_m}\cos(pqs), & \text{if } i_2 \text{ is odd},\end{cases}$$

and the term in (25) is equal to

$$\begin{cases}\dfrac{(-1)^{(i_2+1)/2}\pi}{2^{p+q}p}a^1_{i_1\cdots i_m}\sin(pqs), & \text{if } i_2+1 \text{ is even},\\[2mm] \dfrac{(-1)^{i_2/2}\pi}{2^{p+q}p}a^1_{i_1\cdots i_m}\cos(pqs), & \text{if } i_2+1 \text{ is odd}.\end{cases}$$

For i_2 odd the coefficient of the monomial appears in a sum determining the coefficient of $r^{q-1}\rho^p\cos(pqs)$ in h_1, and also appears in a sum determining the coefficient of $r^{q-2}\rho^p\sin(pqs)$ in h_3 with the opposite sign. In a similar way for i_2 even the coefficient of the monomial appears in a sum determining the coefficient of $r^{q-1}\rho^p\sin(pqs)$ in h_1, and appears in a sum determining the coefficient of $r^{q-2}\rho^p\cos(pqs)$ in h_3 with the same sign.

We can do the same for all monomials in F_N^2, F_N^3 and F_N^4, and thus we conclude that $b_3 = -c_1/p$, $c_3 = b_1/p$, $d_3 = -c_2/q$ and $e_3 = b_2/q$. □

Now we have all the ingredients to prove Theorem 1.

Proof of Theorem 1. It follows from Propositions 6, 7, 9 and 11 that

$$\begin{aligned}
h_1 = {} & a_1 r + r^{q-1}\rho^p\big(b_1\sin(pqs) + c_1\cos(pqs)\big)\\
& + \sum_{2v+i_5+\cdots+i_m=0}^{p+q-1} d^1_{vi_5\cdots i_m} r^{p+q-1-2v-i_5-\cdots-i_m}\rho^{2v} y_5^{i_5}\cdots y_m^{i_m},\\
h_2 = {} & a_2\rho + r^q\rho^{p-1}\big(b_2\sin(pqs) + c_2\cos(pqs)\big)\\
& + \sum_{2v+i_5+\cdots+i_m=1}^{p+q} d^2_{vi_5\cdots i_m} r^{p+q-2v-i_5-\cdots-i_m}\rho^{2v-1} y_5^{i_5}\cdots y_m^{i_m},\\
h_3 = {} & a_3 + \frac{1}{p} r^{q-2}\rho^p\big(-c_1\sin(pqs) + b_1\cos(pqs)\big)\\
& + \frac{1}{q} r^q\rho^{p-2}\big(-c_2\sin(pqs) + b_2\cos(pqs)\big)\\
& + \sum_{2v+i_5+\cdots+i_m=0}^{p+q-1} d^3_{vi_5\cdots i_m} r^{p+q-2-2v-i_5-\cdots-i_m}\rho^{2v} y_5^{i_5}\cdots y_m^{i_m}\\
& + \sum_{2v+i_5+\cdots+i_m=1}^{p+q} d^4_{vi_5\cdots i_m} r^{p+q-2v-i_5-\cdots-i_m}\rho^{2v-2} y_5^{i_5}\cdots y_m^{i_m},\\
h_k = {} & \lambda_k y_k + \sum_{2v+i_5+\cdots+i_m=0}^{p+q-1} d^5_{vi_5\cdots i_m} r^{p+q-1-2v-i_5-\cdots-i_m}\rho^{2v} y_5^{i_5}\cdots y_m^{i_m},
\end{aligned}$$

where $h_j = h_j(r,\rho,s,y_5,\ldots,y_m)$.

According to the results of section 2 we must study the real solutions of the system

$$h_k(r,\rho,s,y_5,\ldots,y_m) = 0 \quad \text{for } k = 1,2,3,5,\ldots,m \tag{26}$$

that have nonzero Jacobian. In order that these solutions can provide limit cycles of system (2) we must look for those such that $r^2+\rho^2 \neq 0$. We distinguish three cases.

Case 1: $r = 0$ and $\rho \neq 0$. If $q > 2$ then in system (26) the variable s does not appear. So the Jacobian of the system is always zero, and consequently the number of limit cycles of system (2) provided by the averaging theory is zero in this case.

In this case if $q = 2$, then $p = 1$, and it is easy to check that all the equations of system (26) (except the first one which is identically zero) are polynomial equations of degree two in the variables r, ρ, $y_5,\ldots,y_m$, $\cos(2s)$ and $\sin(2s)$. Therefore, adding to system (26) the equation $\cos^2(2s)+\sin^2(2s) = 1$ by the Bézout Theorem (see [9]) the maximum number of limit cycles that can appear in this subcase is 2^{m-1}. Since for each solution $w_0 = \cos(2s)$ and $z_0 = \sin(2s)$ of $\cos^2(2s)+\sin^2(2s) = 1$ we can find $s_1, s_2 \in [0,2\pi)$ such that $\sin(2s_i) = z_0$ and $\cos(2s_i) = w_0$ for $i = 1,2$, we get that the total number of solutions of system (26) is at most 2^m.

Case 2: $b_2 = c_2 = 0$, $\rho = 0$ and $r \neq 0$. Then the degree of the polynomial equations of system (26) in the variables r, ρ, $y_5, \ldots, y_m$, $\cos(pqs)$ and $\sin(pqs)$ are $p+q-1$, $p+q$, $p+q$, $p+q-1, \ldots, p+q-1$ respectively. Therefore, adding to system (26) the equation $\cos^2(pqs) + \sin^2(pqs) = 1$ by the Bézout Theorem the maximum number of limit cycles that can appear in this case is $2(p+q-1)^{m-3}(p+q)^2$. Since for each solution $w_0 = \cos(pqs)$ and $z_0 = \sin(pqs)$ of $\cos^2(pqs) + \sin^2(pqs) = 1$ we can find $s_1, \ldots, s_{pq} \in [0, 2\pi)$ such that $\sin(pqs_i) = z_0$ and $\cos(pqs_i) = w_0$ for $i = 1, \ldots, pq$, we obtain that the total number of solutions of system (26) is at most $2pq(p+q-1)^{m-3}(p+q)^2$.

Case 3: $r\rho \neq 0$. Now we perform the change of variables

$$r^{q-1}\rho^{p-1} = B, \quad \rho/r = A, \quad \sin(pqs) = z, \quad \cos(pqs) = w, \quad y_k/r = C_k$$

for $k = 5, \ldots, m$. In the new variables the functions

$$\tilde{h}_1 = h_1/r, \quad \tilde{h}_2 = h_2/r, \quad \tilde{h}_3 = \rho h_3/r, \quad \tilde{h}_4 = z^2 + w^2 - 1, \quad \tilde{h}_k = h_k/r$$

for $k = 5, \ldots, m$ are given by

$$\begin{aligned}
\tilde{h}_1 &= a_1 + AB(b_1 z + c_1 w) + A^{1-p} B P_1(A^2, C_5, \ldots, C_m),\\
\tilde{h}_2 &= a_2 A + B(b_2 z + c_2 w) + A^{-p} B P_2(A^2, C_5, \ldots, C_m),\\
\tilde{h}_3 &= a_3 A + \frac{1}{p} A^2 B(-c_1 z + b_1 w) + \frac{1}{q} B(-c_2 z + b_2 w)\\
&\quad + A^{2-p} B P_3(A^2, C_5, \ldots, C_m) + A^{-p} B P_4(A^2, C_5, \ldots, C_m),\\
\tilde{h}_4 &= z^2 + w^2 - 1,\\
\tilde{h}_k &= \lambda_k C_k + A^{1-p} B P_k(A^2, C_5, \ldots, C_m),
\end{aligned}$$

for $k = 5, \ldots, m$, where

$$P_i(A^2, C_5, \cdots, C_m) = \sum_{2l+i_5+\cdots+i_m=0}^{p+q-1} d^i_{li_5\cdots i_m} A^{2l} C_5^{i_5} \cdots C_m^{i_m}$$

for $i = 1, 3, k$ and

$$P_i(A^2, C_5, \cdots, C_m) = \sum_{2l+i_5+\cdots+i_m=1}^{p+q} d^i_{li_5\cdots i_m} A^{2l} C_5^{i_5} \cdots C_m^{i_m}$$

for $i = 2, 4$.

Solving $(\tilde{h}_1, \tilde{h}_2, \tilde{h}_3) = (0, 0, 0)$ we find the solution

$$\begin{aligned}
&z = A^{-p} Z(A^2, C_5, \cdots, C_m), \quad w = A^{-p} W(A^2, C_5, \cdots, C_m),\\
&B = A^{p-1} \overline{B}(A^2, C_5, \cdots, C_m),
\end{aligned}$$

where

$$Z = \frac{Z_1}{Z_2}, \quad W = \frac{W_1}{Z_2}, \quad \text{and} \quad \overline{B} = \frac{B_1}{B_2},$$

with

$$
\begin{aligned}
Z_1 = \;& A^4 a_2(b_1P_1 - c_1pP_3)q + a_1p(-b_2P_2 + c_2P_4q) + A^2(a_2b_2pP_1 \\
& -(a_1b_1P_2 + p(a_3c_2P_1 - a_3c_1P_2 - a_1c_2P_3 + a_2c_1P_4))q), \\
Z_2 = \;& a_1(b_2^2 + c_2^2)p - A^4a_2(b_1^2 + c_1^2)q + A^2(-a_2(b_1b_2 + c_1c_2)p \\
& +(a_1b_1b_2 + a_1c_1c_2 - a_3b_2c_1p + a_3b_1c_2p)q), \\
W_1 = \;& A^4a_2(c_1P_1 + b_1pP_3)q - a_1p(c_2P_2 + b_2P_4q) \\
& -A^2((-a_3b_2pP_1 + a_1c_1P_2 + a_3b_1pP_2 + a_1b_2pP_3)q - a_2p(c_2P_1 + b_1P_4q)), \\
B_1 = \;& a_1(b_2^2 + c_2^2)p - A^4a_2(b_1^2 + c_1^2)q + A^2(-a_2(b_1b_2 + c_1c_2)p \\
& +(a_1b_1b_2 + a_1c_1c_2 - a_3b_2c_1p + a_3b_1c_2p)q), \\
B_2 = \;& (b_1b_2 + c_1c_2)pP_2 - (b_2^2 + c_2^2)pP_1 + (A^2(b_1^2P_2 + c_1(-c_2P_1 + c_1P_2 + b_2pP_3) \\
& -b_1(b_2P_1 + c_2pP_3)) + (b_2c_1 - b_1c_2)pP_4)q.
\end{aligned}
$$

Therefore in the variables $(A^2, C_5, \ldots, C_m)$, $\overline{B}$ is a quotient of a polynomial of degree 2 by a polynomial of degree $p + q + 1$, Z is a quotient of a polynomial of degree $p + q + 1$ by a polynomial of degree 2, and W is a quotient of a polynomial of degree $p + q + 1$ by a polynomial of degree 2.

Substituting z and w in the equation $\tilde{h}_4 = 0$, we obtain a quotient of a polynomial of degree $2(p+q+1)$ by a polynomial of degree $4+p$ in the variables $(A^2, C_5, \ldots, C_m)$.

Substituting $\overline{B}$ in the equations $\tilde{h}_k = 0$ we obtain a quotient of a polynomial of degree $p+q+2$ by a polynomial of degree $p+q+1$ in the variables $(A^2, C_5, \ldots, C_m)$. Therefore, by applying Bézout's theorem we have that the maximum number of possible roots $(A^2, C_5, \ldots, C_m)$ of the numerator of $(\tilde{h}_4, \tilde{h}_5, \ldots, \tilde{h}_m) = 0$ is given by $2(p + q + 1)(p + q + 2)^{m-4}$. For each solution $(A_0^2, C_{50}, \ldots, C_{m0})$ we have at most one $B_0 = B(A_0^2, C_{50}, \ldots, C_{m0})$ and one pair

$$(z_0, w_0) = (z(A_0^2, C_{50}, \ldots, C_{m0}), w(A_0^2, C_{50}, \ldots, C_{m0})).$$

For each pair (z_0, w_0) we can find $s_1, \ldots, s_{pq} \in [0, 2\pi)$ such that $\sin(pqs_i) = z_0$ and $\cos(pqs_i) = w_0$ for $i = 1, \ldots, pq$. So in this case the maximum number of zeros of system (26) is at most $2pq(p + q + 1)(p + q + 2)^{m-4}$.

Now we put together the results of the three cases. By Theorem 3 the maximum number of limit cycles obtained via averaging theory for system (2) is

$$2^m + 2pq(p+q-1)^{m-3}(p+q)^2 + 2pq(p+q+1)(p+q+2)^{m-4} = 2^m + 2^{m-1}3^2 + 2^4 5^{m-4}$$

if $q = 2$, $p = 1$, or

$$2pq(p + q - 1)^{m-3}(p + q)^2 + 2pq(p + q + 1)(p + q + 2)^{m-4},$$

if $p + q > 3$. This completes the proof of the theorem. □

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